11-19-25

Wednesday, November 19, 2025 1:25 PM

HOW TO COMPUTE

$H_*(X \times Y)$ IN TERMS $H_*(X)$ AND $H_*(Y)$

RELATED PROBLEM:

GIVEN TWO CHAIN CXS OF FREE ABELIAN GROUPS C' AND C'', HOW TO EXPRESS $H_*(C)$ IN TERMS OF $H_*(C')$ AND $H_*(C'')$. $C = C' \otimes C''$

DEF $C' \otimes C'' = C$ IS DEFINED BY

$$(C' \otimes C'')_n = \bigoplus_{0 \le i \le n} C'_i \otimes C''_{n-i}$$

LET $x \in C'_i$, $y \in C''_j$ SO $x \otimes y \in C_{i+j}$.

$$d_{i+j}(x \otimes y) \overset{?}{=} \underbrace{d'_i(x)}_{C'_{i-1}} \otimes y + x \otimes \underbrace{d''_j(y)}_{C''_{j-1}}$$

NOTE

$$d_{i+j-1}\, d_{i+j}(x \otimes y) = d_{i+j+1}(SUM)$$

$$= \cancel{d'_{i-1} d'_i(x) \otimes y} + d'_i(x) \otimes d''_j(y) + d'_i(x) \otimes d''_j(y)$$

$$+ x \otimes \cancel{d''_{j-1} d''_j(y)}$$

$$= 2\, d'_i(x) \otimes d''_j(y) \neq 0$$

INSTEAD WE DEFINE

$$d\ (\ \ ) \quad \dots \quad \overbrace{(-1)^i \ \dots}^{SIGN}$$

INSTEAD WE DEFINE SIGN

$$d_{i+j}(x\otimes y) = d_i'(x)\otimes y + \boxed{(-1)^i x \otimes d_j''(y)}$$

THEN $d_{i+j-1} d_{i+j}(x\otimes y) = 0$

KÜNNETH THEOREM. THERE IS A SHORT EXACT SEQUENCE

$$0 \to \bigoplus_{0\le i\le n} H_i(C')\otimes H_{n-i}(C'') \to H_n(C) \to \bigoplus_{0\le i\le n-1} \mathrm{Tor}^1(H_i C', H_{n-i-1} C'') \to 0$$

PROOF IS SIMILAR TO THAT OF THE UCT.

FOR SPACES X AND Y WITH SINGULAR CHAIN COMPLEXES $S(X)$ AND $S(Y)$, THERE A SIMILAR STATEMENT ABOUT

$$H_*(S(X)\otimes S(Y)),$$

BUT

$$H_*(X\times Y) = H_*(S(X\times Y))$$

THERE IS A MAP

$$S(X)\otimes S(Y) \xrightarrow{f} S(X\times Y)$$

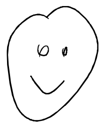
RELATED TO

$$|\Delta^{i+j}| \longleftrightarrow |\Delta^i| \times |\Delta^j|$$

EILENBERG-ZILBER THM (~1950)

f IS A CHAIN HOMOTOPY EQUIVALENCE, SO

$$H_*(S(X)\otimes S(Y)) \cong H_*(S(X\times Y))$$



PROOF  METHOD OF ACYCLIC MODELS VERY BORING

ALTENATIVE SUPPOSE X AND Y ARE CW-COMPLEXES. THEN EACH HAS A CELLULAR CHAIN COMPEX $C(X)$ AND $C(Y)$.

USING THE CELLULAR STRUCTURES ON X AND Y, ONE CAN DEFINE A CELLULAR STRUCTURE ON $X\times Y$ SUCH THAT

$$C(X\times Y) \cong C(X)\otimes C(Y).$$

HOW TO DEFINE $C(X)$

RECALL A CW CX X IS A SEQUENCE OF SPACES

SEQUENCE OF SPACES

$$X^0 \to X^1 \to X^2 \to \cdots \to X = \operatorname{colim} X^n$$

$X^n :=$ n-SKELETON, WHERE

$K_0 = X^0$ IS DISCRETE, AND X^n IS OBTAINED FROM X^{n-1} AS FOLLOWS. THERE IS A DISCRETE SET K_n AND AN ATTACHING MAP

$$\begin{tikzcd}
K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, "K_n \times i"'] & X^{n-1} \arrow[d] \\
K_n \times D^n \arrow[r] & X^n
\end{tikzcd} \quad \text{PUSHOUT} \qquad (1)$$

$$S^{n-1} = \partial D^{n-1} \hookrightarrow D^n$$

EQUIVARIANT DIGRESSION

G = FINITE GROUP

EACH K_n IS A G-SET

THEN THE THREE SPACES IN (1) HAVE G-ACTIONS, AND f_n IS EQUIVARIANT.

THEN X^n IS ALSO A G-SPACE

THIS IS A G-CW COMPEX.

THEN CHAIN CX $C(X)$ IS DEFINED BY

DEFINED BY

$C(X)_n$ = FREE ABELIAN GP ON THE SET K_n

WE NEED A BOUNDARY OPERATOR

NOTE $X^n/X^{n-1} = \bigvee_{K_n} S^n$ = ONE POINT OF n-SPERES

SO $H_n(X^n/X^{n-1}) = C(X)_n$,

$H_{n-1}(X^{n-1}/X^{n-2}) = C(X)_{n-1}$

AND X^n/X^{n-2} IS A CW-CX WITH CELLS IN DIMENSIONS n AND $n-1$. CONSIDER

$$X^{n-1}/X^{n-2} \longrightarrow X^n/X^{n-2} \longrightarrow X^n/X^{n-1}$$

IT LEADS TO A LES IN H_* WITH CONNECTING HOMOMORPHISM

$$\begin{array}{ccc} H_n(X^n/X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}/X^{n-2}) \\ \| & & \| \\ C(X)_n & \xrightarrow[\text{OPERATOR}]{\text{BOUNDARY}} & C(X)_{n-1} \end{array}$$

ONE CAN SHOW BY SKELETAL INDUCTION THAT

$H_*(C(X)) = H_*(X)$

$$H_*(C(X)) = H_*(X)$$

THM $X \times Y$ HAS A CW-STRUCTURE SUCH $C(X \times Y) \cong C(X) \otimes C(Y)$

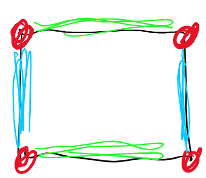
LEMMA LET M AND N BE MANIFOLD WITH BOUNDARIES ∂M AND ∂N. THEN

$$\partial(M \times N) = \partial M \times N \bigcup_{\partial M \times \partial N} M \times \partial N$$

COR $\partial(D^m \times D^n) = \partial(D^m) \times D^n \cup D^m \times \partial D^n$

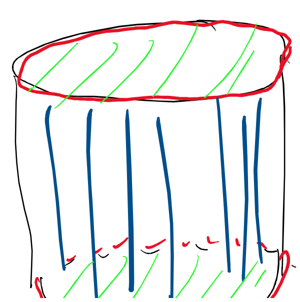
$$S^{m+n-1} \cong \partial(D^m \times D^n) = S^{m-1} \times D^n \bigcup_{S^{m-1} \times S^{n-1}} D^m \times \partial D^n$$

ILLUSTRATION FOR $m = n = 1$



$D^1 \times D^1$ $\quad \partial D^1 \times \partial D^1 = S^0 \times S^0$ $\quad D^1 \times S^0$ $\quad S^0 \times D^1$

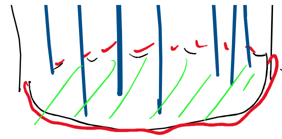
$m=1$ $\quad n=2$



$S^0 \times S^1$

$S^0 \times D^1$

$D^1 \times S^1$



$m = n = 2$

$$S^3 = (S^1 \times D^2) \underset{S^1 \times S^1}{\times} (D^2 \times S^1)$$

LET X AND Y BE CW-CXS WITH PUSHOUT DIAGRAMS

$$\begin{array}{ccc} K_i \times S^{i-1} & \xrightarrow{f_i} & X^{i-1} \\ \downarrow & & \downarrow \\ K_i \times D^i & \xrightarrow[\varphi_i]{} & X^i \end{array} \quad \text{AND} \quad \begin{array}{ccc} L_j \times S^{j-1} & \xrightarrow{g_j} & Y^{j-1} \\ \downarrow & & \downarrow \\ L_j \times D^j & \xrightarrow[\gamma_j]{} & Y^j \end{array}$$

FOR $X \times Y$, LET

$$M_n := \coprod_{0 \leq i \leq n} K_i \times L_{n-i}$$

WE NEED A DIAGRAM

$$\begin{array}{ccc} M_n \times S^{n-1} & \xrightarrow{h_n} & (X \times Y)^{n-1} \\ \downarrow & & \downarrow \\ M_n \times D^n & \longrightarrow & (X \times Y)^n \end{array}$$

$$(X \times Y)^0 = X^0 \times Y^0 = K_0 \times L_0 = M_0$$

BY THE LEMMA FOR $0 \leq i \leq n$

$$S^{n-1} = (S^{i-1} \times D^{n-i}) \cup (D^i \times S^{n-1-i})$$

WE NEED ATTACHING MAPS

$$K_i \times L_{n-i} \times S^{n-1} \xrightarrow{h_n} (X \times Y)$$

$$K_i \times L_{n-i} \times S^{n-1} \xrightarrow{h_n} (X \times Y)_{n-1}$$

$$\|$$

$$K_i \times L_{n-i} \times \left(S^{i-1} \times D^{n-i} \cup D^i \times S^{n-i-1} \right)$$

$$\|$$

$$\left(K_i \times S^{i-1} \times L_{n-i} \times D^{n-i}\right) \cup \left(K_i \times D^i \times L_{n-i} \times S^{n-i-1}\right)$$

$$\begin{tikzcd}
K_i \times S^{i-1} \times L_{n-i} \times D^{n-i} \arrow[d, "f_i \times g_{n-i}"'] & K_i \times D^i \times L_{n-i} \times S^{n-i-1} \arrow[d, "\varphi_i \times g_{n-i}"] \\
X^{i-1} \times Y^{n-i} \arrow[dr] & X^i \times Y^{n-i-1} \arrow[d] \\
& (X \times Y)^{n-1}
\end{tikzcd}$$

THIS DEFINES THE ATTACHING MAP h_n

IN $C(X)$, $C_i(X) = \mathbb{Z}^{K_i}$

$C(Y)$, $C_{n-i}(Y) = \mathbb{Z}^{L_{n-i}}$

$$C_n(X \times Y) = \mathbb{Z}^{M_n} = \bigoplus_{0 \le i \le n} \mathbb{Z}^{L_i \times K_{n-i}}$$

$$= \bigoplus_{0 \le i \le n} C_i(X) \otimes C_{n-i}(Y)$$

HENCE $C(X \times Y) = C(X) \otimes C(Y)$

AS CLAIMED. QED

THE EULER CHARACTERISTIC

LET X BE A FINITE CW-CX

IT HAS FINITELY MANY CELLS

$$\chi(X) = \sum_{n\geq 0} (-1)^n \# K_n$$

THM $\chi(X)$ IS A TOPOLOGICAL INVARIANT AND IS EQUAL TO

(LET k BE EITHER $\mathbb{Q}$ OR $\mathbb{Z}/p$)

$$\sum_{n\geq 0} (-1)^n \text{ RANK } H_n(X;k)$$

EXAMPLES: PLATONIC SOLIDS

SOLID	$\# K_0$ = # VERTICES	$\# K_1$ = # EDGES	$\# K_2$ = # FACES	χ
TETRAHEDRON	4	6	4	2
CUBE	8	12	6	2
OCTAHEDRON	6	12	8	2
DODECAHEDRON	20	30	12	2
ICOSAHEDRON	12	30	20	2

FUTURE MTG

M 11/24 M 12/1 W 12/3 . M 12/8

M 11/24, M 12/1, M 12/3, M 12/8