11-14-25

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ENDS + COENDS

$\mathcal{C}$ IS A SMALL CATEGORY

$\mathcal{D}$ IS A COMPLETE (HAS LIMITS) CATEGORY

$T : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{D}$

FOR EACH MORPHISM $f : x \to y$ IN $\mathcal{C}$,

WE HAVE A DIAGRAM

$$\begin{tikzcd} & T(x,x) \arrow[d, "f_*"] \\ T(y,y) \arrow[r, "f^*"] & T(x,y) \end{tikzcd} \xrightarrow[\text{ALL } f : x \to ?]{\text{PRODUCT OVER}} \begin{tikzcd} T(x,x) \arrow[d, "\phi_*"] \\ \prod\limits_{f : x \to ?} T(x,?) \end{tikzcd} \xrightarrow[x \in \mathcal{C}]{\text{PRODUCT OVER ALL}} \begin{tikzcd} \prod\limits_{x} T(x,x) \arrow[d, "\phi_*"] \\ \prod\limits_{f} T(\mathrm{dom}\, f, \mathrm{cod}\, f) \end{tikzcd}$$

SIMILARLY f^* LEADS TO A MAP

$$\prod_{y \in \mathcal{C}} T(y,y) \xrightarrow{\phi^*} \prod_{f \in \mathcal{C}} T(\mathrm{dom}\, f, \mathrm{cod}\, f)$$

THUS WE HAVE

$$\prod_{x \in \mathrm{Obj}(\mathcal{C})} T(x,x) \underset{\phi^*}{\overset{\phi_*}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\, \mathcal{C}} T(-,-)$$

THE END $\int^{\mathcal{C}} T(x,x)$ IS THE EQUALIZER OF ϕ^* AND ϕ_*.

COENDS: ASSUME $\mathcal{D}$ IS COCOMPLETE

FOR EACH $f : x \to y$ IN $\mathcal{C}$ WE HAVE

$$T(y,x) \xrightarrow{f_*} T(y,y)$$

$$\begin{tikzcd} T(y,x) \arrow[r, "f_*"] \arrow[d, "f^*"] & T(y,y) \\ T(x,x) & \end{tikzcd}$$

$$\begin{tikzcd} \coprod_{f:x\to y} T(y,x) \arrow[r, "\varphi_*"] \arrow[d, "\varphi^*"] & \coprod_{y\in\mathcal{C}} T(y,y) \\ \coprod_{x\in\mathcal{C}} T(x,x) & \end{tikzcd}$$

THE COEND $\int^{\mathcal{C}} T(x,x)$ IS THE COEQUALIZER OF φ^* AND φ_*.

<u>REMARK</u> IF $T:\mathcal{C}^{op}\times\mathcal{C}\to\mathcal{D}$ IS CONSTANT ON THE FIRST VARIABLE, THEN WE CAN REGARD IT AS A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$. THE END AND COEND ARE THE ORDINARY LIMIT AND COEND OF F.

BACK TO HOMOLOGICAL ALGEBRA.

ASSUME R IS A COMM. RING

FOR R-MODULES M AND N, WE HAVE DEFINED (i) $\mathrm{Tor}_*^R(M,N)$ AND (ii) $\mathrm{Ext}_R^*(M,N)$ IN TERMS OF

(i) PROJECTIVE RESOLUTIONS OF

(i) PROJECTIVE RESOLUTIONS OF M ON N. (Tor IS COVARIANT IN BOTH VARS AND SYMMETRIC WITH $Tor_0^R(M, N) = M \otimes_R N$

ii) PROJECTIVE RESOLUTION OF M OR INJECTIVE " N.

$Ext^*(M, N)$ IS CONTRAVARIANT IN M AND COVARIANT IN N.

$Ext_R^0(M, N) = Hom_R(M, N)$.

* SHORT EXACT SEQUENCES IN EITHER VARIABLE LEAD TO LONG EXACT SEQUENCES IN Tor AND Ext

* IF R IS A PID, THEN

$Tor_i^R(M, N) = 0 = Ext_i^R(M, N)$ FOR $i > 1$.

UNIVERSAL COEFFICIENT THEOREM

LET R BE A PID

C IS A CHAIN COMPLEX OF FREE R-MODULES

N IS ANY R MODULE.

THEN THERE IS A SHORT EXACT SEQ.

NAIVE GUESS — DESTINATION — ERROR TERM

$$0 \to H_i(C) \otimes_R N \longrightarrow H_i(C \otimes_R N) \longrightarrow Tor_1^R(H_{i-1}C, N) \to 0$$

REMARKS

① FOR $R = \mathbb{Z}$, $C = S(X) =$ SINGULAR CHAIN CX OF A SPACE X

$N =$ ABELIAN GP A

$$H_*(X;A) := H_*(S(X) \otimes A)$$

= HOMOLOGY OF X WITH COEFFICIENTS IN A

② LET C BE

$$\begin{array}{ccccccc} & & 0 & & 1 & & 2 \\ 0 & \leftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} \end{array}$$

AND $A = \mathbb{Z}/2$

$$H_i(C) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & i = 2 \end{cases} \qquad H_i(C) \otimes \mathbb{Z}/2 = \begin{cases} \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \end{cases}$$

$C \otimes A$ IS

$$\begin{array}{ccccccc} & & 0 & & 1 & & 2 \\ 0 & \leftarrow & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 \end{array}$$

$$H_i(C \otimes \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = 0, 1, 2 \\ 0 & i > 2 \end{cases}$$

WE KNOW

EXERCISE

$$\mathrm{Tor}_1(0, \mathbb{Z}/2) = 0$$

$$\mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/2) = 0$$

$$\mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$$

$$\mathrm{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m,n)$$

$$\mathrm{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m,n)$$

PROOF: IN ANY COMPLEX C

$$\cdots \leftarrow C_{i-1} \xleftarrow{d_i} C_i \xleftarrow{d_{i+1}} C_{i+1} \leftarrow \cdots$$

$$\ker d_i =: Z_i = \text{GROUP OF } i\text{-CYCLES}$$

$$\mathrm{im}\, d_{i+1} =: B_i = \text{GROUP OF } i\text{-BOUNDARIES}$$

$$Z_i \supseteq B_i$$

$$H_i(C) := Z_i / B_i$$

FOR EACH i THERE IS A SES

$$\begin{tikzcd} 0 \arrow[r] & Z_i \arrow[r] \arrow[d, "0"'] & C_i \arrow[r] \arrow[d, "d_i"] & B_{i-1} \arrow[r] \arrow[d, "0"] \arrow[dll, dashed] & 0 \\ 0 \arrow[r] & Z_{i-1} \arrow[r] & C_i \arrow[r] & B_{i-2} \arrow[r] & 0 \end{tikzcd}$$

1. DEFINE CHAIN COMPLEXES Z AND B, WHERE THE iTH CHAIN GP OF Z IS Z_i, B IS B_{i-1}

BOTH BOUNDARY OPERATORS ARE TRIVIAL. WE HAVE A SES OF CHAIN CXS

$$0 \to Z \to C \to B \to 0 \quad \text{I}$$

$$H_i(Z) = Z_i \qquad H_i(B) = B_{i-1}$$

THE LES IN HOMOLOGY READS

B_{i-1}

THE LES IN HOMOLOGY READS

$$\cdots \to H_i(Z) \xrightarrow{\text{ONTO}} H_i(C) \xrightarrow{0} \underset{\substack{\| \\ B_{i-1}}}{H_i(B)} \xrightarrow{1-1} \underset{\substack{\| \\ Z_{i-1}}}{H_{i-1}(Z)} \xrightarrow{\text{ONTO}} \underset{\substack{\| \\ Z_{i-1}/B_{i-1}}}{H_{i-1}(C)} \xrightarrow{0} \cdots$$

THE LES BREAKS INTO SHORT EXACT SEQS

$$0 \to \underset{\substack{\cap \\ C_i}}{B_i} \to \underset{\substack{\cap \\ C_i}}{Z_i} \to \underset{\substack{\| \\ Z_i/B_i}}{H_i(C)} \to 0 \qquad (2)$$

THIS IS A PROJECTIVE RESOLUTION OF $H_i(C)$

IF WE TENSOR (2) WITH N WE GET A SIX TERM EXACT SEQ

$$0 \to \overset{\substack{0 \\ \|}}{\mathrm{Tor}_1(B_i, N)} \to \overset{\substack{0 \\ \|}}{\mathrm{Tor}_1(Z_i, N)} \to \mathrm{Tor}_1(H_i(C), N) \to$$

$$\to \underset{\substack{\| \\ B_i \otimes_R N}}{\mathrm{Tor}_0(B_i, N)} \to \underset{\substack{\| \\ Z_i \otimes_R N}}{\mathrm{Tor}_0(Z_i, N)} \to \underset{\substack{\| \\ H_i(C) \otimes_R N}}{\mathrm{Tor}_0(H_i C, N)} \to 0$$

SINCE B, Z AND C ARE ALL PROJECTIVE, TENSORING WITH N PRESERVES EXACTNESS

$$0 \to Z\otimes_R N \to C\otimes_R N \to B\otimes_R N \to 0 \qquad (3)$$

SINCE Z AND B HAVE TRIVIAL BOUNDARY,

$$H_i(Z\otimes_R N) = Z_i \otimes N$$

$$H_i(B\otimes_R N) = B_{i-1}\otimes N$$

THE LES IN H_* FOR (3) IS

$$\xrightarrow{d_{i+1}\otimes N} \overset{Z_i\otimes_R N}{H_i(Z\otimes_R N)} \to \overset{\text{MYSTERY}}{H_i(C\otimes_R N)} \to \overset{B_{i-1}\otimes_R N}{H_i(B\otimes_R N)} \xrightarrow{d_i\otimes N} \underset{Z_{i-1}\otimes_R N}{H_{i-1}(Z\otimes_R N)}$$

THIS GIVES A SES

$$0 \to \underset{\substack{\| \\ H_i(C)\otimes_R N}}{\operatorname{coker}(d_{i+1}\otimes N)} \to H_i(C\otimes_R N) \to \underset{\substack{\| \\ \mathrm{Tor}_1^R(H_{i-1}(C),N)}}{\ker(d_i\otimes N)} \to 0$$

BY PREVIOUS CALCULATION

QED