10-8-25

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EXCISION AXIOM

**FIVE LEMMA** SUPPOSE WE HAVE A COMM DIAGRAM OF ABELIAN GPS WITH EXACT ROWS

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha"'] & A_2 \arrow[r] \arrow[d, "\beta"'] & A_3 \arrow[r] \arrow[d, "\gamma"'] & A_4 \arrow[r] \arrow[d, "\delta"'] & A_5 \arrow[d, "\varepsilon"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

IF $\alpha$, $\beta$, $\delta$ AND $\varepsilon$ ARE ISOMORPHISMS, SO IS $\gamma$.

PROOF ONLINE OR IN ANY BOOK ON HOMOLOGICAL ALGEBRA.

COR SUPPOSE WE HAVE A MAP $(X, A) \xrightarrow{f} (Y, B)$. THEN WE HAVE $H_*(X) \to H_*(Y)$, $H_*(A) \to H_*(B)$, AND $H_*(X, A) \to H_*(Y, B)$. IF ANY TWO OF THESE ARE ISOMORPHISMS, SO IS THE THIRD.

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**EXCISION AXIOM:**

WHEN $Z \subset A \subset X$ WITH

EXCISION AXIOM.

GIVEN $Z \subset A \subset X$ WITH

CLOSURE$(Z) \subset$ INTERIOR$(A)$,

$$H_*(X-Z, A-Z) \xrightarrow{\cong} H_*(X, A)$$

EQUIVALENT FORMULATION

$A, B \subset X$ WITH $\mathrm{INT}(A) \cup \mathrm{INT}(B) = X$

(LET $Z = X - B$). THEN

$$H_*(A, A\cap B) \xrightarrow{\cong} H_*(X, A)$$

CONSIDER THE FOLLOWING SUB-CHAIN COMPLEX OF $S(X) \xleftarrow{\lambda} S(A+B)$

$S(A+B) :=$ SUB GP GENERATED BY

MAPS $\Delta^n_{TOP} \to A \to X$

AND $\Delta^n_{TOP} \to B \to X$

CLAIM: $\lambda$ IS A CHAIN HOMOTOPY EQUIVALENCE. PROOF LATER

$$\begin{tikzcd} & A \arrow[dr, "j_1"] & \\ A\cap B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X = A\cup B \\ & B \arrow[ur, "j_2"'] & \end{tikzcd}$$

THERE IS A SES OF CHAIN CXS

$S(X)$

THERE IS A SES OF CHAIN CX'S

$$0 \to S(A\cap B) \xrightarrow{i_1 \oplus i_2} S(A)\oplus S(B) \xrightarrow{j_1 - j_2} \overset{\substack{S(X)\\ \cup|}}{S(A+B)} \to 0$$

IT LEADS TO A LES IN $H_{*}$.
THE CLAIM IMPLIES IT IS THE MAYER-VIETORIS SEQUENCE.

<u>PROOF OF CLAIM</u>: LET

$\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ WHERE $U_i \subset X$

WITH $\bigcup_i \mathrm{INT}(U_i) = X$. LET

$S^{\mathcal{U}}(X) \subset S(X)$ BE THE SUBCX GENERATED BY SIMPLICES OF THE FORM $\Delta^n_{TOP} \to U_i \to X$. CALL SUCH SIMPLICES <u>$\mathcal{U}$-SMALL</u>.

TO SHOW $S^{\mathcal{U}}(X) \xrightarrow{\lambda} S(X)$ IS A CHE, WE NEED A SUITABLE MAP $S(X) \xrightarrow{P} S^{\mathcal{U}}(X)$ WITH

1. $P\lambda$ = IDENTITY ON $S^{\mathcal{U}}(X)$ AND
2. $\lambda P \simeq$ IDENTITY ON $S(X)$.

TO DEFINE $P(\sigma)$ FOR $\sigma : \Delta^n_{TOP} \to X$

TO DEFINE $\rho(\sigma)$ FOR $\sigma : \Delta^n_{TOP} \to X$

BY COMPACTNESS OF $\Delta^n_{TOP}$, ITS IMAGE LIES IN A FINITE UNION OF THE $U_i$'S. IF WE SUBDIVIDE $\Delta^n_{TOP}$ ENOUGH TIMES, EACH LITTLE SIMPLEX WILL BE $\mathcal{U}$-SMALL.

SUPPOSE WE NEED $k$ SUBDIVISIONS. THIS MAKES $\Delta^n_{TOP}$ THE UNION OF $((n+1)!)^k$ LITTLE SIMPLICES.

LET $\rho(\sigma)$ = SUM OF ALL THESE.

$\rho\lambda$ IS THE IDENTITY BY OUR ASSUMPTION THAT $k$ IS AS SMALL AS POSSIBLE.

FOR THE REQUIRED CHAIN HOMOTOPY BETWEEN $\lambda\rho$ AND $\mathbb{1}_{S(X)}$, SEE HATCHER pp 121-124.

COR. THE CASE $\mathcal{U} = \{A, B\}$ IS THE CLAIM ABOVE.

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WE HAVE ALL ... EILENBERG...

WE HAVE ALL 4 EILENBERG-STEENROD AXIOMS.

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TWO VARIATIONS OF HOMOLOGY

(1) LET $G$ AN ABELIAN GP, E.G. $\mathbb{Q}$ AND $\mathbb{Z}/p$ FOR A PRIME $p$

CONSIDER THE CHAIN COMPLEX $S(X) \otimes_{\mathbb{Z}} G$. DEFINE

$$H_*(X;G) := H_*(S(X) \otimes G)$$

= HOMOLOGY WITH COEFFICIENTS IN $G$.

<u>QUESTION</u>: HOW IS IT RELATED TO $H_*(X)$? NAIVE BUT INCORRECT GUESS: $H_n(X;G) = H_n(X) \otimes G$.

(2) CONSIDER $\mathrm{Hom}(S(X), G)$

FOR ABELIAN GPS $G_1, G_2$,

$\mathrm{Hom}(G_1, G_2) =:$ SET OF HOMOMORPHISMS $G_1 \to G_2$

IT IS AN ABELIAN GP SINCE $G_2$ IS ONE.

$G_2$ IS ONE.

$$\begin{tikzcd}
\cdots & S_{n-1}(X) \arrow[l] \arrow[dr, dashed] & S_n(X) \arrow[l, "d_n"'] \arrow[d, "\alpha"] & S_{n+1}(X) \arrow[l, "d_{n+1}"'] \arrow[dl, "\alpha d_{n+1}"] & \cdots \arrow[l] \\
 & & G & &
\end{tikzcd}$$

$d_{n+1}$ INDUCES A MAP

$$\mathrm{Hom}(S_n(X), G) \xrightarrow{\delta_n} \mathrm{Hom}(S_{n+1}(X), G)$$

THE GROUPS $\mathrm{Hom}(S_n(X), G)$ FORM A <u>COCHAIN COMPLEX</u>

TERMINOLOGY: $\alpha$ IS A <u>COCHAIN</u>

IF $\alpha d_{n+1} = 0$, IT IS A <u>COCYCLE</u>

IF IT FACTORS THRU $S_{n-1}(X)$, IT IS A <u>COBOUNDARY</u>.

COCYCLES/COBOUNDARIES = COHOMOLOGY $H^*$

COHOMOLOGY OF $X$ WITH COEFFICIENTS

$$:= H^*(\mathrm{Hom}(S(X), G)) = H^*(X; G)$$

QUESTION (HOW IS IT RELATED TO $H_*(X)$ ???

$$H^n(X;G) \neq \mathrm{Hom}\big(H_n(X), G\big)$$

IN GENERAL.

THESE TWO QUESTIONS, WITH $S(X)$ REPLACED BY A GENERAL CHAIN COMPLEX, ARE ALGEBRAIC, NOT TOPOLOGY.

2 BASIC DIFFICULTIES.

SUPPOSE WE HAVE A SHORT EXACT SEQUENCE

$$0 \to A' \xrightarrow{1-1} A \xrightarrow{\text{ONTO}} A'' \to 0$$

OF ABELIAN GPS. ① APPLY THE FUNCTOR $- \otimes G$

$$A' \otimes G \xrightarrow[\text{BE } 1-1]{\text{MAY NOT}} A \otimes G \xrightarrow{\text{ONTO}} A'' \otimes G \to 0$$

② APPLY $\mathrm{Hom}(-, G)$

$$0 \leftarrow \mathrm{Hom}(A', G) \leftarrow \mathrm{Hom}(A, G) \leftarrow \mathrm{Hom}(A'', G)$$

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EXAMPLE SES IS

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

AND $G = \mathbb{Z}/2$

TENSORING GIVES

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{=} \mathbb{Z}/2 \to 0$$

APPLYING $\mathrm{Hom}(-, \mathbb{Z}/2)$

$$\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{=} \mathbb{Z}/2 \leftarrow 0$$

HOMOLOGICAL ALGEBRA