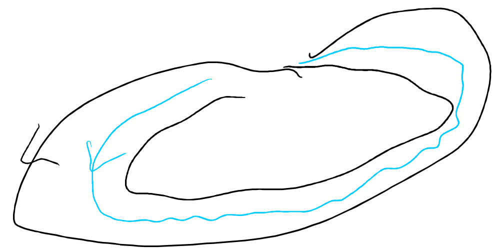
10-6-25

Monday, October 6, 2025 1:18 PM

USE MVS TO COMPUTE

$H_* \mathbb{R}P^2$: $\quad \mathbb{R}P^2 = A \cup B$

A = MÖBIUS STRIP $\simeq S^1$



$B = D^2$

$A \cap B = S^1$ NOTE: THE MAP

$$\begin{tikzcd} S^1 \simeq A \cap B = \partial A \arrow[r] \arrow[rr, bend right, "\text{DEGREE } 2"'] & A & \simeq S^1 \end{tikzcd}$$

$$\begin{tikzcd} 2 & & 0 \arrow[r] & H_2\mathbb{R}P^2 \arrow[dll] & \\ 1 & H_1 S^1 \arrow[r] & H_1(A)\oplus H_1(B) \arrow[r] & H_1\mathbb{R}P^2 \arrow[dll] & \\ 0 & H_0 S^1 \arrow[r] & H_0(A)+H_0(B) \arrow[r] & H_0\mathbb{R}P^2 \arrow[r] & 0 \end{tikzcd}$$

$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \quad 0 \quad \mathbb{Z}/2$

$\mathbb{Z} \xrightarrow{1\ 1} \mathbb{Z}\oplus\mathbb{Z} \quad \mathbb{Z}$

$$H_i\,\mathbb{R}P^2 = \begin{cases} \mathbb{Z} & \text{FOR } i=0 \\ \mathbb{Z}/2 & i=1 \\ 0 & i\geq 2 \end{cases}$$

WHAT IS $S(X)$, THE SINGULAR CHAIN COMPLEX OF THE SPACE X?

DEF THE STANDARD n-SIMPLEX

FOR $n \geq 0$ IS THE SUBSPACE OF $\mathbb{R}^{n+1}$

$$\Delta^n_{TOP} := \left\{ (x_0, x_1, \dots x_n) : x_i \geq 0 , \sum_{i=0}^{n} x_i = 1 \right\}$$

EXAMPLE $\Delta^0_{TOP} = pt.$

$\Delta^1_{TOP} \cong I^1$

$\Delta^2_{TOP} \cong$ TRIANGLE

$\Delta^3_{TOP} \cong$ SOLID TETRAHEDRON

FOR $0 \leq i \leq n$, THE iTH FACE MAP

$$\Delta^{n-1}_{TOP} \xrightarrow{f_i} \Delta^n_{TOP}$$

iTH CO-ORDINATE

$$(x_0, \dots, x_{n-1}) \longmapsto (x_0, \dots x_{i-1}, 0, x_i, \dots x_n)$$

DEF FOR A SPACE X, LET $S_n(X)$ BE THE FREE ABELIAN GP GENERATED BY THE SET OF ALL CONTINUOUS MAPS $\Delta^n_{TOP} \xrightarrow{\sigma} X$

WE DEFINE $S_n(X) \xrightarrow{d_n} S_{n-1}(X)$ AS FOLLOWS

$$\Delta^{n-1} \xrightarrow{f_i} \Delta^n_{TOP} \xrightarrow{\sigma} X$$

$$\Delta^{n-1}_{TOP} \xrightarrow{f_i} \Delta^n_{TOP} \xrightarrow{\sigma} X$$

$$S_{n-1}(X) \xleftarrow{f_i^*} S_n(X)$$

$$d_n = \sum_{0 \le i \le n} (-1)^i f_i^*$$

<u>EXERCISE</u> $S_{n-1}(X) \xleftarrow{d_n} S_n(X) \xleftarrow{d_{n+1}} S_{n+1}(X)$

$d_n d_{n+1} = 0$ FOR $n \ge 1$.

HENCE WE HAVE A CHAIN COMPLEX $S(X)$, AND WE DEFINE $H_*(X) := H_*(S(X))$.

<u>EXAMPLE</u> $X = pt$. THERE IS ONLY ONE MAP $\Delta^n_{TOP} \longrightarrow X$

$$\begin{array}{ccccccc} S_0(pt) & \xleftarrow{d_1} & S_1(pt) & \xleftarrow{d_2} & S_2(pt) & \xleftarrow{d_3} & \cdots \\ \| & & & & & & \\ \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} & \xleftarrow{0} & \end{array}$$

WE HAVE SEEN THIS CHAIN COMPLEX BEFORE. WE FIND

$$H_i(pt) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

DIMENSION AXIOM

$$H_i(pt) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$ DIMENSION AXIOM

WE ALSO DEFINE FOR $A \subset X$, $S(X,A) := S(X)/S(A)$. IT FOLLOWS THAT WE HAVE A SES OF CHAIN COMPLEXES

$$0 \to S(A) \longrightarrow S(X) \longrightarrow \underbrace{S(X)/S(A)}_{=:\, S(X,A)} \to 0$$

AND WE GET A LES

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

THIS IS THE EXACTNESS AXIOM.

NOTE IF WE HAVE $X \xrightarrow{f} Y$ INDUCES A CHAIN MAP

$$S(X) \xrightarrow{S(f)} S(Y)$$

$$H_*(X) \xrightarrow{H_*(f)} H_*(Y)$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \Delta^n_{TOP} \arrow[u, "\sigma"] \arrow[ur] & \end{tikzcd}$$

Δ^{n}_{TOP}

THM (HOMOTOPY AXIOM) IF $f_0, f_1 : X \longrightarrow Y$ ARE HOMOTOPIC, THEN $H_*(f_0) = H_*(f_1)$. RECALL $H_*(f_0) = H_*(f_1)$ IF $S(f_0)$ AND $S(f_1)$ ARE CHAIN HOMOTOPIC, i.e. $\exists$ MAPS

$$D_n : S_n(X) \longrightarrow S_{n+1}(Y) \quad \text{FOR } n \geq 0$$

SUCH THAT

$$\begin{tikzcd}
S_{n-1}(X) \arrow[drr, "D_{n-1}"] & & S_n(X) \arrow[ll, "d_n^X"'] \arrow[d, shift right, "S_n(f_0)"'] \arrow[d, shift left, "S_n(f_1)"] \arrow[drr, "D_n"] & & S_{n+1}(X) \arrow[ll, "d_{n+1}^X"'] \\
S_{n-1}(Y) & & S_n(Y) \arrow[ll, "d_n^Y"] & & S_{n+1}(Y) \arrow[ll, "d_{n+1}^Y"]
\end{tikzcd}$$

$$D_{n-1} d_n^X + d_{n+1}^Y D_n = S_n(f_0) - S_n(f_1)$$

WE A HOMOTOPY $h : X \times I \longrightarrow Y$ S.T $h(x, 0) = f_0(x)$ AND $h(x, 1) = f_1(x)$

WILL USE IT TO PRODUCE THE DESIRED CHAIN HOMOTOPY.

DESIRED CHAIN HOMOTOPY.

GIVEN $\sigma : \Delta^n_{TOP} \to X$, WE GET

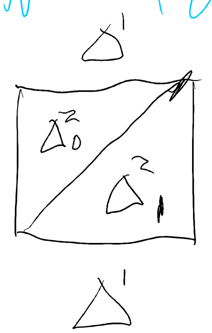
PRISM

$$\begin{tikzcd}
\coprod_{n+1} \Delta^{n+1}_{TOP} \arrow[r, phantom, "\cong"] & \Delta^n_{TOP} \times I \arrow[r, "\sigma \times I"'] & X \times I \arrow[r, "h"] & Y \\
(\Delta^{n+1}_{TOP})_i \arrow[u] \arrow[urrr, bend right, "\alpha_i"] & & &
\end{tikzcd}$$

iTH $\qquad 0 \le i \le n$

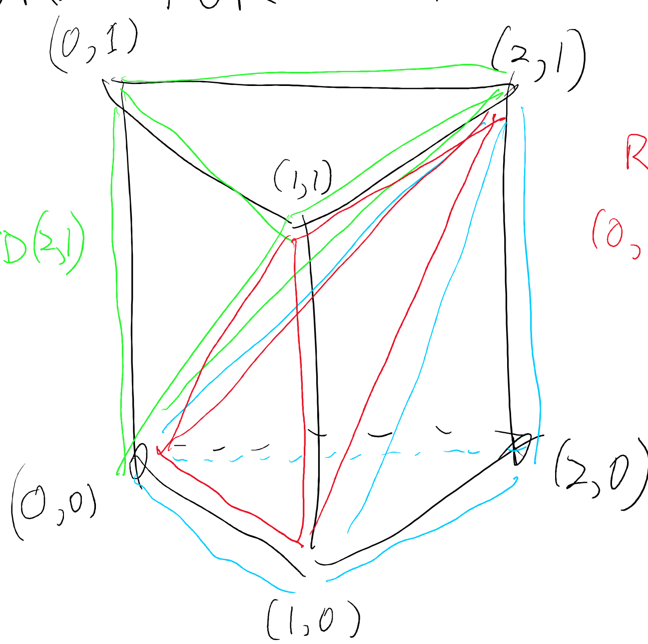
WILL DEFINE $D_n(\sigma) = \sum_{0 \le i \le n} \alpha_i$

ILLUSTRATION FOR $n = 1$



$\Delta^1 \times I \cong \Delta^2_0 \amalg \Delta^2_1$

PICTURE FOR $n = 2$



GREEN HAS VERTICES (0,0), (0,1), (1,1) AND (2,1)

BLUE HAS VERTICES (0,0), (1,0), (2,0) AND (2,1)

RED HAS VERTICES (0,0), (1,0), (1,1) AND (2,1)

IN GENERAL WE HAVE $(2n+2)$ VERTICES $(i, 0)$ AND $(i, 1)$ FOR

VERTICES $(i,0)$ AND $(i,1)$ FOR $0 \le i \le n$. THE VERTICES OF $(\Delta^{n+1}_{TOP})_i$ FOR $0 \le i \le n$ ARE

$$(0,0),\ (1,0) \ldots,\ (i,0),\ (i,1),\ (i+1,1) \ldots (n,1)$$

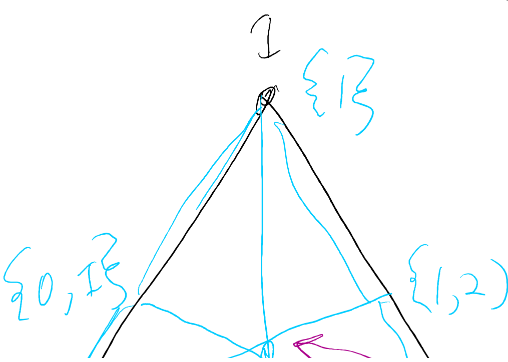
MORE DETAILS ARE IN HATCHER. QED.

THEOREM (EXCISION AXIOM)
FOR $B \subset A \subset X$ WITH CLOSURE$(B) \subseteq$ INTERIOR (A)

$$H_*(X-B,\ A-B) \xrightarrow{\cong} H_*(X, A)$$

WE NEED BARYCENTRIC SUBDIVISION $\Delta^n_{TOP} \cong \coprod_{(n+1)!} (\Delta^n_{TOP})$

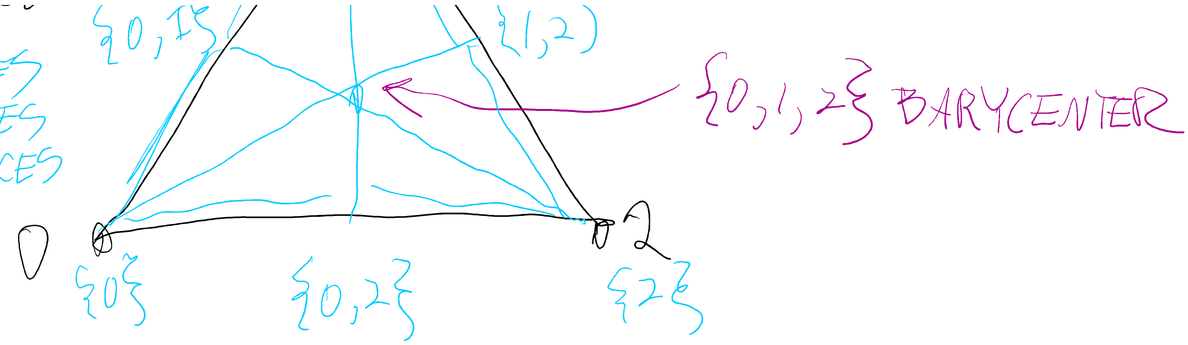
3 VERTICES
3 EDGES
1 2-FACE
7 VERTICES



EACH VERTEX IS A SUBSET OF $\{0,1,2\}$

$\{0,1,2\}$ BARYCENTER

7 VERTICES
12 EDGES
6 2-FACES



IN GENERAL WE GET A VERTEX
FOR NONEMPTY SUBSET OF $\{0, \ldots, n\} =: [n]$
THERE $2^{n+1}-1$ OF THEM

THERE A k-FACE FOR ASCENDING
SEQUENCE OF $(k+1)$ SUCH SUBSETS.