10-1-25

Wednesday, October 1, 2025 2:12 PM

APPLICATIONS OF THE EILENBERG-STEENROD AXIOMS

STRATEGY

1) ASSOCIATE TO EACH SPACE X A FUNCTORIAL CHAIN COMPLEX $S(X)$ TO BE NAMED LATER. $S(\emptyset) = 0$

2) DEFINE $H_*(X) := H_*(S(X))$

   FOR $A \subset X$, $S(A) \hookrightarrow S(X)$

   $$H_*(X, A) := H_*(S(X)/S(A))$$

3) THERE IS A NATURAL TRANSFORMATION

   $$H_n(X, A) \longrightarrow H_{n-1}(A)$$

   TO BE NAMED LATER.

IT TURNS OUT THAT THIS FUNCTOR SATISFIES 4 AXIOMS

1. EXACTNESS. THERE IS A LONG EXACT SE

$$\cdots \to H_n(A) \xrightarrow{i} H_n(X) \xrightarrow{j} H_n(X, A) \to H_{n-1}(A) \to \cdots$$

i IS INDUCED BY THE INCLUSION MAP $A \hookrightarrow X$. FOR j

$$\underbrace{H_n(X) \cong H_n(X, \emptyset) \longrightarrow H_n(X, A)}_{j}$$

$$\emptyset \longrightarrow A$$

(2) HOMOTOPY. IF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ ARE HOMOTOPIC, THEY INDUCE THE MAP IN H_*

(3) EXCISION GIVEN $B \subset A \subset X$ WITH CLOSURE$(B) \subseteq$ INTERIOR(A), THEN

$$H(X-B, A-B) \xrightarrow{\cong} H_*(X, A)$$

(4) DIMENSION

$$H_i(\text{point}) = \begin{cases} \mathbb{Z} & \text{FOR } i=0 \\ 0 & \text{FOR } i>0 \end{cases}$$

<u>THEOREM</u> IF THE ES AXIOMS HOLD, THEN FOR $n>0$

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=n \\ 0 & \text{ELSE} \end{cases}$$

$$H_i(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$$

PROOF BY INDUCTION ON n

S^0 = 2 POINTS WITH DISCRETE TOPOLOGY

EACH PT IS OPEN AND CLOSED (CLOPEN). WE WILL SEE $S(X_1 \amalg X_2) \cong S(X_1) \oplus S(X_2)$

$$H_*(X_1 \amalg X_2) \cong H_*(X_1) \oplus H_*(X_2)$$

IT FOLLOWS THAT $H_*(S^0)$ IS AS

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<u>INDUCTIVE STEP</u>: ASSUME $H_*(S^{n-1})$ IS AS CLAIMED. CONSIDER THE PAIR

$A = D^n$ = NORTHERN HEMISPHERE

$X = S^n$.

THE HOMOTOPY AXIOM IMPLIES THAT A HTY EQUIV INDUCES AN ISO IN H_*. HENCE

$H_*(D^n) \cong H_*(pt.)$.

USE EXCISION AXIOM

$$\begin{array}{l} B = D^n_{SMALL} = \text{NORTHERN TEMPERATE REGION} \\ \quad \cap \\ A = D_n = \text{NORTHERN HEMISPHERE} \\ \quad \cap \\ X = S^n \end{array}$$

$A - B \simeq S^{n-1}$

$X - B \simeq D^n$

EXCISION AXIOM SAYS

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$$H_*(X-B, A-B) \cong H_*(X, A)$$

$$H_*(D^n, S^{n-1}) \cong H_*(S^n, D^n)$$

THE PAIR ON THE LEFT LEADS TO

$$\cdots \to H_i(D^n - D^n_{SMALL}) \to H_i(S^n - D^n_{SMALL}) \to H_i(-,-) \to \cdots$$

$$H_i(D^n - D^n_{SMALL}) = H_i(S^{n-1}) \qquad H_i(S^n - D^n_{SMALL}) = H_i(pt)$$

SO $H_i(S^n - D^n_{SMALL}, D^n - D^n_{SMALL}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

$$H_i(S^n - D^n_{SMALL}, D^n - D^n_{SMALL}) = H_i(S^n, D^n)$$

THIS IMPLIES $H_*(S^n)$ IS AS CLAIMED. QED

MAYER-VIETORIS SEQUENCE

WALTER MAYER 1887-1948

LEOPOLD VIETORIS 1891-2002

THEOREM LET $X = A \cup B$ WITH

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$$\text{CLOSURE}(A\cap B) \subset \text{INT}(A)$$
$$\subset \text{INT}(B)$$

$$\begin{tikzcd}
 & A \arrow[dr, "j_1"] & \\
A\cap B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X \\
 & B \arrow[ur, "j_2"'] &
\end{tikzcd}$$

PUSHOUT

THERE IS A LES

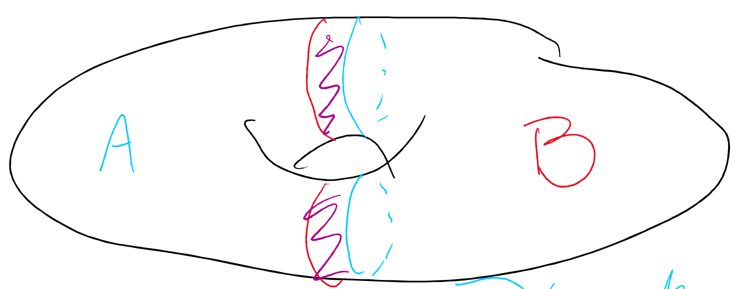
$$\cdots \to H_n(A\cap B) \xrightarrow{i_1 \oplus i_2} H_n(A)\oplus H_n(B) \xrightarrow{j_1 - j_2} H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

PROOF LATER

EXAMPLE 1 $X =$ TORUS

$$A, B \approx S^1 \times I \simeq S^1$$

$$A \cap B \approx S^0 \times S^1 \times I \simeq S^1 \amalg S^1$$



$\mathbb{Z} = \ker \alpha_1$

2 $\quad 0 \longrightarrow 0 \longrightarrow H_2(X) \to$

1 $\quad \overset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(S^1)\oplus H_1(S^1)} \xrightarrow{\alpha_1} \overset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(S^1)\oplus H_1(S^1)} \longrightarrow H_1(X) \to$

$$1 \quad \to H_1(S^1) \oplus H_1(S^1) \xrightarrow{\alpha_1} H_1(S^1) \oplus H_1(S^1) \to H_1(X) \to$$

$$0 \quad \to H_0(S^1) \oplus H_0(S^1) \xrightarrow{\alpha_0} H_0(S^0) \oplus H_0(S^1) \to H_0(X) \to 0$$

$\mathbb{Z} \oplus \mathbb{Z}$ $\qquad$ $\mathbb{Z} \oplus \mathbb{Z}$ $\qquad$ coker α_0 $= \mathbb{Z}$

EACH MAP $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$

CORRESPONDS $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

ITS KERNEL AND COKERNEL ARE EACH $\mathbb{Z}$, SO $H_2(X) = \mathbb{Z}$ AND $H_0(X) = \mathbb{Z}$

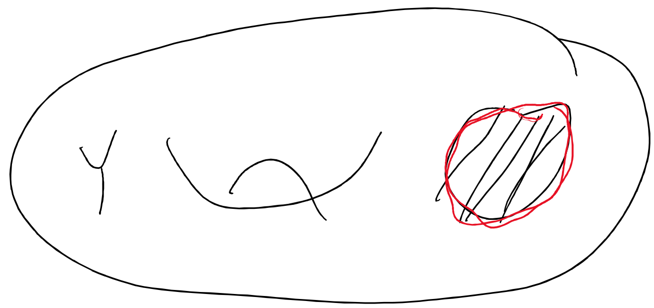
BY EXACTNESS $\exists$ SES

$$0 \to \operatorname{coker} \alpha_1 \to H_1(X) \to \ker \alpha_0 \to 0$$

$$\operatorname{coker} \alpha_1 = \mathbb{Z}, \quad H_1(X) = \mathbb{Z} \oplus \mathbb{Z}, \quad \ker \alpha_0 = \mathbb{Z}$$

$$H_i(\text{TORUS}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 0 \end{cases}$$

EXAMPLE 2



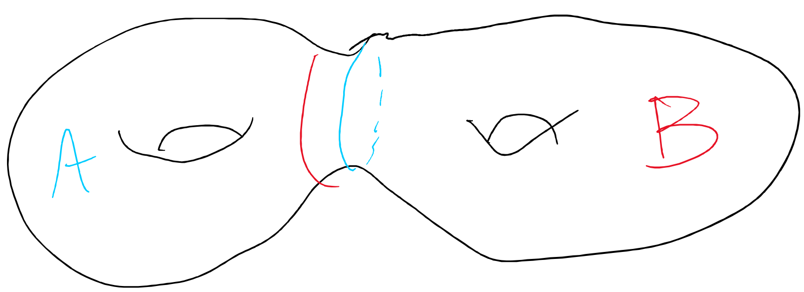
$$\text{TORUS} = Y \cup D^2 \qquad Y \cap D^2 = S^1$$

MVS SHOWS

$$H_i(Y) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 0 \\ 0 & i \geq 2 \end{cases}$$

EXAMPLE 3

X = SURFACE OF GENUS 2

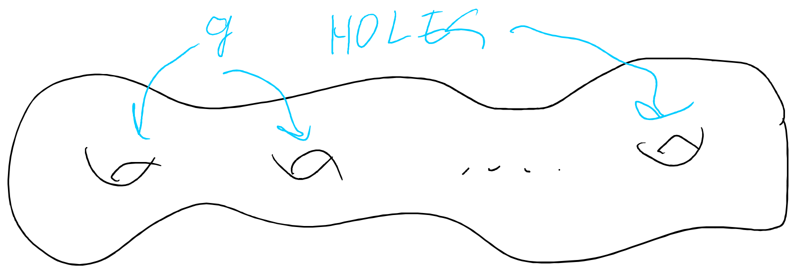


$$A \simeq Y \simeq B \qquad A \cap B \simeq S^1 \times I \simeq S^1$$

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^4 & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

EXAMPLE 4

X_g = SURFACE OF GENUS g



BY INDUCTION ON g.

$$H_i X_g = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{2g} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>0 \end{cases}$$