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FROM RATIONALITY TO TRANSCENDENCE IN FINITE CHARACTERISTIC

by

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Abstract. — In this expository article, we survey progress, and mention open questions, speculations and conjectures regarding the study of various degrees of irrationality, algebraicity, and transcendence from various angles, for general as well as special quantities in function field arithmetic. Though we often mention the number field case and characteristic zero function field case, we mostly concentrate on function fields in finite characteristic, and often specialize to those over finite fields.

Contents

1. Basic problems 21
2. Complexity of algebraic quantities 23
3. Exponent of approximation 23
4. Differential exponents in function fields 27
5. Continued fractions 28
6. Distribution of higher degree exponents 33
7. Deformation and Exponents hierarchies 34
8. Diophantine classification 34
9. Computational classification for function fields over a finite field 35
10. Nature of and relations between special values 37
References 44

1. Basic problems

Starting with the counting numbers, our concept of numbers evolved in various directions: using the usual operations of addition, subtraction, multiplication, division leading to rational numbers; using the notion of distances and length leading to real

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numbers; solving equations with unknowns formed by the operations above leading to algebraic numbers. The basic questions of diophantine approximation study how general or special real numbers are distributed distance-wise in relation to rational or algebraic numbers with respect to their complexity. In other words, we compare the error in approximation by a simpler quantity with the measure of its simplicity.

Modern algebra led to more abstract fields F, leading to more general number systems such as those modulo a prime, finite fields, and function fields $F(x)$ obtained by introducing a variable x and applying the usual operations above, thus leading to rational functions (over F) versus rational numbers, algebraic functions versus algebraic numbers, and (finite tailed) laurent series (obtained by completions) versus real numbers (obtained analogously). In other words, we consider $F[x]$, $F(x)$, $F((1/x))$ as analogs of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ respectively. So depending on the context, by integers, rationals, algebraics, real, we may mean corresponding analogs in the function field case. We can ask the basic questions mentioned above in this function field setting also.

Despite the basic differences between the characteristics, non-archimedean absolute versus archimedean absolute value, the lack of order or positivity, infinite versus finite degree over the prime field, and despite different tools like differentiation, specialization, stronger properties of p-th power maps; amazingly, many results in the number field and the function field situations are often parallel. There are some stark contrasts also, and despite function fields being often simpler to deal with, we will find situations where the number fields case is fully proved and the function field case is not even conjecturally understood!

In this survey paper, we discuss the progress and open questions about some themes of this study, focusing mainly in finite characteristic function fields $F(x)$, sometimes specializing to a finite F, which often gives best analogies to the number field situations. We often bring in the number fields or characteristic zero function field situation for comparison of analogies and contrasts.

We restrict to real numbers and do not go into complex or p-adic directions, except for a couple of remarks. First we deal with general numbers, with results for all, almost all or algebraic numbers. In the last section, we deal with special values in function field arithmetic [**G96, T04**], mostly dealing with Carlitz-Drinfeld theory. A quick look at the headings of the sections and subsections should give a good idea about the organization of the material.

We refer to [**S80, B04, BG06, W?**], which are some excellent references for the basic material, as well as for surveys and detailed references on recent progress, mostly in the number fields case. When there are no specific references given, the results or references can be found in these basic references. Often, we give only convenient rather than original references. In the last two sections, we will mention a few excellent surveys written at different time periods on the related material. Since there are hardly any proofs in this paper, reader looking for sketches or ideas of proofs should look at these surveys.

I will appreciate any comments, suggestions and references to improve this survey. I hope to incorporate them in an updated version on my homepage.

2. Complexity of algebraic quantities

2.1. Absolute values. — By $|\alpha|$, we mean the usual absolute value or size for a real number α, and $c^{\deg(\alpha)}$, for some fixed $c > 1$ for a nonzero $\alpha \in F(x)$. In the case of a finite field F with q elements, we choose c to be q, so that in an analogy with the case of integers, the number of remainders (which is finite in this case) is $|\alpha|$, when you divide by a nonzero polynomial α.

2.2. Heights. — A crude complexity measure of an algebraic number or function is its algebraic degree. The complexity of an algebraic number (of a given degree) is traditionally measured by notion of height which roughly relates to the space it takes to describe the number by the traditional method.

2.3. Definition of Absolute and Field height. — For β a nonzero algebraic number, define $H(\beta)$ to be the maximum of the absolute values of the coefficients of a non-trivial irreducible polynomial with co-prime integral coefficients that it satisfies. For β lying in a number field L, define $H_L(\beta)$ similarly by replacing the irreducible polynomial by $c\prod(x - \beta^{(i)})$, where $\beta^{(i)}$ are its field conjugates and multiple c makes the coefficients co-prime integers.

We get similar definition for algebraic functions β by replacing 'integers' by their analogs 'polynomials'.

Note that for p/q, a reduced rational number or function, the height is the maximum of $|p|, |q|$ and it does not make any difference in the definition of exponents below if we replace it with the more traditional choice of $|q|$.

For number fields and function fields over finite fields, it follows easily from definitions that there are only finitely many elements of bounded height and degree.

See [**S80, BG06**] for other definitions of variants of heights and relations between different variants.

3. Exponent of approximation

In comparison of errors and heights, we only focus on the simplest traditional irrationality measure, namely that of the exponent, by relying only on the power functions and ignoring logarithms or other lower growth functions.

3.1. Definition of exponents. — Let L be a number field inside $\mathbb{R}$ and d be a positive integer. For α an irrational real number not algebraic of degree $\leq d$, define $E_d(\alpha)$ as $\limsup(-\log|\alpha - \beta|/\log H(\beta))$, where β varies through all algebraic real numbers of degree d, with height tending to infinity.

For $\alpha \in \mathbb{R} - L$, define $E_L(\alpha)$ by the same formula, but with H replaced by H_L and with with β varying through elements of L.

Similar definitions are clear for the function field case, and for $E_{\leq d}$, $E_{<d}$ etc.

The usual exponent $E(\alpha) := \limsup(-\log|\alpha - P/Q|/\log|Q|)$ is just $E_1(\alpha) = E_{\mathbb{Q}}(\alpha)$.

Some other common measures such as $w_d^* := E_{\leq d} - 1$, and $w_d(\alpha)$ defined to be the supremum of w such that $|P(\alpha)| \leq H(P)^{-w}$ for infinitely many polynomials P of degree at most d and with integral coefficients.

3.2. Bounds and values of exponents. — (1) For an irrational real number or function α, we have $E(\alpha) \geq 2$. This is seen immediately by applying the definition to the approximations provided by truncation of continued fractions. Another proof, in number field or function field over finite field case when there are only finitely many remainders, follows by applying the Dirichlet box principle to fractional parts of $m\alpha$'s, for integral m's bounded in some range.

(2) For almost all (in terms of measure theory) real numbers α, we have Khintichine's theorem $E(\alpha) = 2$ and for almost all real numbers, we have [**Sp69**] $w_d(\alpha) = w_d^*(\alpha) = d$, so that $E_d(\alpha) = d + 1$. For function field over finite field case, w_d statement is proved [**Sp69**] and and w_d^* statement stated in [**Gu96**] follows from it and inequalities stated in 3.7 below (answering the question of a possible gap raised in [**B04**, Sec. 9.4], as I have confirmed with Bugeaud). See also [**Kh64, dM70, B04**] for more metrical results.

(3) For algebraic real number or function α of degree $> d$, we have $E_{\leq d}(\alpha) \leq \deg \alpha$. For $d = 1$, this is Liouville's theorem for real case, obtained by applying the mean value theorem to the minimal polynomial of α between α and the approximation, and adapted by Mahler [**M49**] to function field case. The general case [**BG06**] follows similarly by resultants in general.

(4) For real algebraic number α, we have $E(\alpha) = 2$ by Roth's celebrated deep theorem improving Liouville, Thue, Siegel, Dyson bounds, using several variable polynomials. For function fields of characteristic zero, the equality was proved by Fena and Uchiyama.

(5) For real number α not in a number field L, we have $E_L(\alpha) \geq 2$ and for real algebraic number α not in L, we have $E_L(\alpha) = 2$ by Leveque's generalization of Roth's theorem.

(6) For real α not algebraic of degree $\leq d$, Wirsing [**W60**] (generalizing Dirichlet result) conjectured (See also [**R03**] for different than then expected answer in the approximation by algebraic integers case, raising questions on what should be the correct conjecture) $E_{\leq d}(\alpha) \geq d + 1$ and proved slightly better lower bound (and K.

Tishchenko improved) than $(d+3)/2$, whereas Davenport and Schmidt (see [**S80**]) proved his conjecture for $d=2$. Sprindzuk 1963 ([**Gu96**] respectively) proved Wirsing analog for function fields of zero (any respectively) characteristic zero. Analog of the Davenport-Schmidt bound $E_{\leq 2}(\alpha) \geq 3$ is proved for real functions of any characteristic is also proved in [**Gu96**].

(7) As a corollary to Schmidt's subspace theorem generalizing Roth's result, Schmidt (see [**S80**]) proved that for real algebraic numbers α of degree greater than d, $E_{\leq d}(\alpha) \leq d+1$. The same inequality was proved in [**R78**] for the case of function fields of characteristic zero. Combining Schmidt's theorem with Wirsing's results (Section 3.7) we know [**BM86**] that for real algebraic numbers (or functions of characteristic zero) of degree $> d$, we have $E_{\leq d}(\alpha) = d+1$.

(8) In case, α is algebraic of degree $d+1$, we can replace Schmidt's subspace theorem in (7) by much simpler Liouville's inequality (in numbers or functions case) [**W60, BM86**] to show $E_{\leq d}(\alpha) = d+1$ then. There is no 'epsilon' in this case, just as $E(\alpha)=2$ follows easily from theory of continued fractions and their periodicity for real quadratic without using Roth's deeper theorem and there is no 'epsilon' then.

While (3) is 'effective', most of its improvements are not.

Interestingly, when $d > 1$, by comparing (6) and (7), we see that the Dirichlet direction has turned out to be harder than Roth's direction!

We will discuss omitted cases of finite characteristic below.

3.3. What is a small set or a rare event?— Liouville numbers are those numbers (e.g., $\sum 10^{-n!}$) whose exponent is infinite. By (2), or directly from the definitions, we see that the set L of Liouville numbers is of measure zero. It is uncountable, dense (countable intersection of open dense sets, so G_δ) set, which is of Hausdorff measure zero in all dimensions. But its complement is small in the sense that it is of first category, countable union of nowhere dense sets. See [**Ox80**] and for more on its Hausdorff dimension with more general measures than power functions, see [**OR06**].

On the other hand, the set of algebraic numbers is countable, thus of measure zero and of first category. Baker-Schmidt [**BS70**] (Jarnik-Besicovitch for $d=1$) theorem shows that for $\lambda > 1$, the set of real number α with $E_d(\alpha) > (d+1)\lambda$, which, by (2), is of measure zero, has Hausdorff dimension $1/\lambda$.

In the case of a function field over finite field, [**K03**] proves analog of $d=1$ case.

3.4. Failure of naive analog of Roth's theorem in finite characteristic. — While (1)-(3) of Section 3.2 work in both function fields and number fields with essentially the same proofs, the naive analog of Roth's theorem fails as shown by the following example of Mahler [**M49**].

Let F be of characteristic $p > 0$, and q be a power of p. Then, as Mahler observed, $E(\alpha) = q$ for $\alpha = \sum t^{-q^i}$, by a straight estimate of approximation by truncation of this series. Now $\alpha^q - \alpha - t^{-1} = 0$, so that α is algebraic of degree q over $F(t)$, and hence the Liouville upper bound is best possible in this case.

Mahler suggested (and it was claimed to have been proved in a published paper and believed for a while) that such phenomena may be special to the degrees divisible by the characteristic, but Osgood [**Os75**], and Baum and Sweet [**BS76**] gave examples in each degree for which Liouville exponent is the best possible. A few more isolated examples by Buck, Robbins, Mills (see [**L09**] for extensive references) were proved after extensive computer searches. See below for more.

3.5. Good approximations and Computation of exponents. — Sequence of good approximations, for example, those coming from truncation of series or product expansions if they converge rapidly, usually leads to only to a lower bound on exponents, and unlike Mahler's example, where it coincided with Liouville's upper bound, we do not get exact exponent in general. But under certain conditions listed below, we can [V88] (see also [**T11**, Pa. 15]) calculate the exact exponent.

If β_n are algebraic of degree d, $\beta_n \to \alpha$, satisfying

$$\limsup \log H(\beta_{n+1})/\log H(\beta_n) = b, -\log|\alpha - \beta_n|/\log H(\beta_n) \to a, a > d(b^{1/2}+1),$$

then $E_d(\alpha) = a$, for α not algebraic of degree $\leq d$.

If you know the continued fraction of α, you can usually get its exponent (see below). See [**T04**, Thm. 9.3.4] and [**V95**] for another general useful application by Voloch of his lemma above.

3.6. Mikowski's successive minima theorem. — Mahler [**M41**] developed an analog to Minkowski's geometry of numbers, including his theorem on successive minima, where the usual bounds of $2^n/n!$ and 2^n for the volume times product of successive minima are replaced by 1 (hence equality), and similarly for inequalities for the polar body. These results imply analogs of Minkowski's theorems on sizes of linear forms. On the other hand, naive analog of Schmidt's subspace theorem generalizing Roth's theorem fails in the function field case.

3.7. Relations between w_d and w_d^*. — Let α be real (number or function) not algebraic of degree $\leq d$, and write $w_d := w_d(\alpha)$, $w_d^* := w_d^*(\alpha)$, temporarily. Then in the function field case, Mahler's Minkowski result implies $w_d \geq d$, whereas [**Gu96**] proved that $w_d^* \geq (d+1)/2$ and for $d = 2$, that $w_2^* \geq 2$. We have [**Sp69**, Pa. 150] $w_d \geq w_d^*$, $w_d^* \geq w_d/(w_d - d + 1)$, and if further, the characteristic is zero, we also have $w_d^* \geq w_d - d + 1$ and $w_d^* \geq (w_d + 1)/2$. All these hold in real number case [**W60**] (see also [**BM86, B04**] for much more).

Note that Schmidt's subspace theorem (so for real numbers and functions in characteristic zero) implies that for algebraic α, not of degree $\leq d$, we have $w_d \leq d$.

4. Differential exponents in function fields

In a function field $F(t)$ of any characteristic, we can differentiate with respect to t, in contrast to the number field situation. Maillet, Kolchin and Osgood [**Os73, Os75**] used this to get better and/or effective bounds for diophantine approximation. Kolchin's idea [**Kol59**] was to use the Liouville argument replacing the minimal polynomial of α with a 'small' differential polynomial that kills α. Frequently, this gives a smaller exponent and we get good effective bounds by more refined work of Osgood. See original references or surveys in [**T04**, Cha. 9] and [**T09**] for more.

4.1. Definition of denomination. — We denote the m-th derivative of y with respect to t by $y^{(m)}$ and also write y' for $y^{(1)}$ following the usual practice. For a vector $\overline{e} = (e_0, \cdots, e_k)$ of non-negative integers, let us write $y^{\overline{e}}$ as a short-form for the differential monomial $y^{e_0}(y')^{e_1} \cdots (y^{(k)})^{e_k}$.

Consider a differential polynomial $P(y) = \sum p_{\overline{e}} y^{\overline{e}}$. Note that the j-th derivative of a/b has the power b^{j+1} in the denominator. So define the denomination $\overline{d}(P)$ to be the maximum of $\sum_{j=0}^{k}(j+1)e_j$ corresponding to $\overline{e}$ such that $p_{\overline{e}} \neq 0$.

So, if $P(a/b) \neq 0$, then $|P(a/b)| \geq 1/|b|^{\overline{d}(P)}$, hence $\overline{d}(P)$ replaces the degree in the Liouville argument. Define the differential exponent $\overline{d}(\alpha)$ to be the smallest possible $\overline{d}(P)$ for P satisfying $P(\alpha) = 0$.

Here α can be differentially algebraic. If, in fact, it is also algebraic, then $\overline{d}(\alpha) \leq \max(\deg(\alpha) - 1, 2)$: Note that differentiating the minimal polynomial $P(x)$ for α we get the equation $\alpha' P_x(\alpha) + P'(\alpha) = 0$. Simplifying, we get $\alpha' = \sum_{j=0}^{w} a_j(t)\alpha^j$, with $w < \deg \alpha$.

4.2. Kolchin's analog of Liouville's theorem. — Given an irrational α which is differentially algebraic over a characteristic zero function field, there is a constant $c > 0$ such that $|\alpha - a/b| > c/|b|^{\overline{d}(\alpha)}$.

The proof is by the Liouville argument, except the catch is that the differential minimal polynomial P has, in general, infinitely many zeros and a priori, some approximations a/b can be among those. Kolchin shows that this is impossible in characteristic zero.

We get smallest denomination 2 for irrational α satisfying the Riccati equation $y' = ay^2 + by + c$ with rational functions as coefficients. So for such elements (for example, any element of degree 3 or any irrational n-th root of a rational) we have an (effective) Roth estimate (there is no ϵ even) in characteristic zero.

In contrast, in characteristic p, α which is (called element of Class I below) a rational Möbius transformation of its p^n-th power, satisfies the Riccati equation, and we will see below that in this case the Riccati examples can have any rational exponent within Dirichlet and Liouville bounds, at least for some degrees.

4.3. Riccati equation-Thue bound connection. — Osgood proved [**Os73, Os75**] the following very interesting theorem

In the situation of function fields of finite characteristic, the exponent bound can be reduced from the Liouville bound to the Thue bound $E(\alpha) \leq \lfloor \deg(\alpha)/2 \rfloor + 1$ for all non-Riccati α's.

4.4. Why Riccati?— The relevance of the Riccati equation to this question is clearly brought out by the theorem of Osgood and Schmidt [**S76**]

If $y'B(y) + A(y) = 0$, where A and B are coprime polynomials with integral (i.e., polynomial in t) coefficients, then all its rational solutions have height bounded in terms of those of A and B, as long as the equation is not Riccati (i.e., we do not have $\deg(B) = 0$ and $\deg(A) \leq 2$).

This theorem implies that the close enough rational approximations will not be the roots and hence the Liouville-Thue type argument goes through when applied to $y'B(y) + A(y)$.

4.5. Separability. — Note that since differentiation of p-th powers is zero, it is easy to see that algebraic Laurent series are in fact separable over rational function field over finite field.

5. Continued fractions

Continued fractions are natural tools of the theory of diophantine approximation. See [**dM70, BS76, S00, BN00, T04**] for their basics (and several references) in the case of function field over a field. Artin introduced them and proved first few theorems in this case.

5.1. Basic notation. — We write $\alpha = a_0 + 1/(a_1 + 1/(a_2 + \cdots))$ in the short-form $[a_0, a_1, \cdots]$. We write $\alpha_n := [a_n, a_{n+1}, \cdots]$. Let us define p_n and q_n as usual in terms of the partial quotients a_i's, so that p_n/q_n is the n-th convergent $[a_0, \cdots, a_n]$ to α. Hence $\deg q_n = \sum_{i=1}^{n} \deg a_i$.

To generate the continued fraction in the function field case, we use the 'polynomial part' in place of the 'integral part' of the 'real' number $\alpha \in K_\infty$. In the function field case, for $i > 0$, a_i can be any non-constant polynomial and so the degree of q_i increases with i, but a_i or q_i need not be monic. (We refer to [**BN00**] for good discussion of signs, and variants of the continued fraction algorithms with minus sign etc.).

5.2. Basic formulas. — We have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, \qquad \alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}} \tag{1}$$

implying the usual basic approximation formula

$$\alpha - p_n/q_n = (-1)^n/((\alpha_{n+1} + q_{n-1}/q_n) q_n^2), \tag{2}$$

which, in the function field case, because of non-archimedean nature of the absolute value, simplifies to the fundamental formula giving the error

$$|\alpha - p_n/q_n| = 1/(|a_{n+1}||q_n|^2). \tag{3}$$

If we know the continued fraction for α, the equation allows us to calculate the exponent, using $\deg q_n = \sum_1^n \deg a_i$, as

$$E(\alpha) = 2 + \limsup \frac{\deg a_{n+1}}{\sum_{i=1}^n \deg a_i}. \tag{4}$$

5.3. Statistical distribution of a_n and q_n. — Since the diophantine approximation properties of α are closely related to sizes of corresponding a_n's and q_n's, let us record some theorems [**Kh64**] about these quantities in the real number case.

(1) For almost all real α, $a_n > n \log n$ ($a_n > n(\log n)^2$ respectively) has infinitely (finitely respectively) many solutions n. There are also more refined results modeling zero-one law (Borel-Cantelli lemma) of probability theory.

(2) For almost all real α, we have $q_n^{1/n} \to c$, where $\ln c = \pi^2/(12 \ln 2)$.

(3) (Gauss-Kuzmin law) For x between 0 and 1, the measure of the set of numbers α between 0 and 1, that satisfy $\alpha_n > 1/x$ approaches $\log_2(1+x)$, as n tends to ∞. Equivalently, the measure of the set of α between 0 and 1 that satisfy $a_n = k$, for a fixed integer $k > 1$, approaches $\log_2(1 + 1/(k(k+2)))$. Note that this distribution has mode at $k = 1$, median at $k = 2$, but the mean is infinite (as can also be seen using (1)).

In the function field case (over finite fields), various metrical results are known. For example, [**H79**] shows (see also [**N88, BN00**])

(1) For almost all α, we have $\limsup \deg a_n/(\log n) = 1/(\log q)$.

(2) For almost all α, we have $\lim(\deg q_n)/n = q/(q-1)$. In words, for almost all α, the average degree of the first n partial quotients tends to $q/(q-1)$ as n tends to infinity.

(3) If $P \in \mathbb{F}_q[t]$ is a polynomial of degree $d > 0$, then for almost all α, we have $\lim s_n/n = q^{-2d}$, where s_n is the number of i between 1 and n for which $a_i = P$.

5.4. Basic patterns. — It is immediate that finite continued fractions correspond to rational numbers or functions and that eventually periodic sequence of a_n's gives real quadratic numbers or functions. The converse of the last statement is also true for function fields over finite fields. Also, Artin proved that purely (starting from a_1) periodic continued fraction for quadratic real irrational α exactly corresponds to $\deg \alpha < 0$ and $\deg \alpha' > 0$, where α' is the algebraic conjugate of α. (For real quadratic irrational numbers, the corresponding conditions are $\alpha > 1$ and $-1 < \alpha' < 0$, (for period starting from a_0). For this and the situation over general fields, see [**S00, BN00**] and references there.

On the other hand, it is not known for a single (explicit or not) algebraic real number of degree more than two, whether the corresponding sequence of a_n's is bounded or unbounded. In the function field case, [**BS76**] produced first examples of both kinds. See below for many more examples.

5.5. Connection with good approximations. — By (3) in the Section 5.2, we see that convergents p_n/q_n always approximate by error $< 1/q_n^2$ (i.e., $\leq 1/(|t|q_n^2)$). Conversely, any such good approximation is a convergent. (For real numbers one needs $< 1/(2q^2)$.)

In fact, due to discreteness of absolute values, there are a priori many more variants of notions of good approximations than in the real number case, and they are [**T04**, Thm. 9.2.3] connected with convergents and intermediate convergents, generalizing results of [**dM70**].

Discreteness of absolute values also makes analogs [**T04**, Remarks 9.2.1] of Markoff spectrum and Lehmer problem easier and much less interesting.

5.6. Explicit Continued Fractions and Exponents. — We now present [**S00, T99**] explicit continued fraction families of algebraic quantities in finite characteristic, with exponents ranging through all Dirichlet-Liouville range.

Let F be a field of characteristic p, and q be a power of p. Let $A_i(t) \in F[t]$ be of degree $d_i > 0$. Then

$$\alpha := [A_1, \cdots, A_k, A_1^q, \cdots, A_k^q, A_1^{q^2}, \cdots] \tag{5}$$

is algebraic over $F(t)$ because it satisfies the algebraic equation

$$\alpha = [A_1, \cdots, A_k, \alpha^q] = \frac{A\alpha^q + b}{C\alpha^q + D}. \tag{6}$$

We have [**S00, T99**]

$$E(\alpha) = 2 + (q-1)MAX_{1 \leq i \leq k}(d_i/((d_1 + \cdots + d_{i-1})q + d_i + \cdots + d_k)), \tag{7}$$

and given any rational μ between $q^{1/k}+1$ (which tends to 2 as k tends to infinity) and $q+1$, we can construct a family of α's as above with $E(\alpha) = \mu$ and $\deg(\alpha) \leq q+1$.

5.7. Class I and Class IA. — Let F be a finite field of characteristic p and q be a power of p. If α satisfies $\alpha = (A\alpha^q + B)/(C\alpha^q + D)$, for $A, B, C, D \in F[t]$ with determinant $AD-BC$ nonzero, α is said to be of Class I, and if further $AD-BC \in F^*$, then it is said to be of Class IA. Since, for $f \in F^*$, we have $f[a_0, a_1, a_2, \cdots] = [fa_0, f^{-1}a_1, fa_2, \cdots]$, the examples above take care of continued fractions of all α of class IA.

The pattern of continued fractions for general α of class I is an interesting open question, with interesting isolated examples and results given by e.g., Baum, Sweet, Mills, Robbins, Buck, de Mathan, Lasjaunias, Ruch, Schmidt, Firicel. See [**MR86, L00, L09, S00, T08, T09, F10?**] and references there for some interesting explicit continued fractions of class I, but not of class IA.

The exponent is known [**dM92**] to be rational for any element of class I. It is not known whether for general algebraic function the exponent can ever be irrational.

All elements of Class I are easily seen to satisfy Riccati equation with rational (function) coefficients which has a rational solution. The easiest way to show something is not of Class I is to show that it does not satisfy such an equation. But since the converse is not true, and q in the definition of Class I is not fixed, it is hard to show that something satisfying rational Riccati equation with rational solution is not Class I.

5.7.1. Class I-Thue bound connection. — In [**LdM96**], generalizing Osgood's theorem 4.3, and establishing conjecture of [**V88**], it was shown that if algebraic α over function field over field F of finite characteristic is not of Class I, then Liouville bound can be improved to Thue bound: $E(\alpha) \leq \lceil (\deg \alpha)/2 \rceil + 1$. They also have [**LdM99**] a slight refinement (of 'epsilon') and different proof when F is finite.

Is there an hierarchy of differential (or difference-Frobenius) equations and exponent bounds, perhaps Riccati generalizing to Weierstrass and Painleve? How about higher exponents? We will mention below deformation hierarchy in this spirit.

5.8. Statistical distribution of exponents in Class IA. — In her Ph. D. thesis of 2011 from National Central University, Taiwan, Huei Jeng Chen used the exponent formula (7) in Section 5.6 to get statistical distribution, in case when F is a finite field so that we can count, as follows.

Let F be a finite field of characteristic p and fix q a power of p. For α as in Section 5.6 (or of Class IA which can be handled exactly similarly using multiplication by constants as explained above), we put $H(\alpha)$ to be the maximum of the degrees of A, B, C, D (in the reduced form). This is the usual height of α, if the equation there is irreducible. (Chen also shows that in the result of Section 5.6, we can further stipulate that the degree is $q + 1$).

Let $2 < m < q + 1$. Let N_d be the number of such α's of height at most d, and let $N_{d,m}$ be the cardinality of the subset where $E(\alpha) > m$. Then $N_{d,m}/N_d^{1-(m-2)/(2(q-1))}$ is bounded between two positive numbers independent of d.

This shows a nice power law with the power going down from 1 to 1/2 as you move the exponent bound from Roth value 2 to $q + 1$, which is the (maximal) degree here.

Does similar distribution work in Class I or more generally for all algebraic elements of any given degree?

Here we only note that every element of degree 3 is of Class I and that if $q = 2$, we can get Class I elements of all determinants (but not all Class I elements unfortunately) by applying Mobius transformations, which do not change exponents, to class IA elements.

5.9. Folding lemma of Mendes-France, Shallit. — Following nice lemma [**MF73, Sh79**] have been rediscovered and used many times, including by the author in getting non-Riccati algebraic explicit continued fraction (Section 5.10), getting algebraic continued fractions with bounded sequence of partial quotients (Section 5.11), higher diophantine approximation exponents (Section 6), getting explicit continued fractions for analogs of e and Hurwitz numbers (Section 10.5), as well as by Lasjaunias in a nice different fashion in his Finite Fields and their applications paper of 2006.

Let $[a_0, a_1, \cdots, a_n] = p_n/q_n$, with the usual notation of continued fractions, then $[a_0, \cdots, a_n, y, -a_n, \cdots, -a_1] = p_n/q_n + (-1)^n/yq_n^2$.

This pattern is a signed block reversal /folding pattern following the new term y.

5.10. Explicit algebraic continued fractions. — Consider function fields of finite characteristic. In addition to interesting variety of explicit algebraic continued fractions in Class I mentioned above and surveyed in [**L00, L09**], [**L09**] also mentions one example which is a square of an element in Class I, but not of Class I.

In [**T03**], we gave explicit continued fractions of non-Riccati family, by taking appropriate linear combinations of Mahler type elements, and also showed that their exponents cover all rationals in a large range:

Let

$$\alpha = \sum_{i=1}^{k} f_i \sum_{j=0}^{\infty} t^{-m_i q^j + b_i},$$

where $m_i \geq 0$ and b_i are rational numbers so that the exponents are integers. Let $m_{i+1} > 2m_i$ for $1 \leq i < k$ and $qm_1 > 2m_k$. Then

$$E(\alpha) = \mathrm{MAX}(m_2/m_1, \cdots, m_k/m_{k-1}, qm_1/m_k), \tag{8}$$

and given any rational value μ between $q/2^{k-1}$ and $q^{1/k}$, α can be chosen with $E(\alpha) = \mu$ if further that $q > 2^k$.

The algebraic equation for each term (corresponding to a fixed i) is immediate, since it is just a multiple of Mahler's example. The flexibility in the choice of m_i's and b_i's can be used to produce many families of α's not satisfying the rational Riccati equation.

5.11. Algebraic elements with bounded partial quotients sequence. — Any α as above with $q = 2^k$, $m_i = 2^{i-1}$ and $b_i > b_{i+1}/2$, for i modulo k will produce an explicit continued fraction with bounded sequence of partial quotients in characteristic two. Most of these do not satisfy the rational Riccati equation (and so are of degree more than 3). If α satisfying our general conditions has bounded sequence of partial quotients, then [**T03**] the characteristic p is two.

In Class I, there are many examples known [**L00, L09**] with bounded partial quotients sequence. Mills-Robbins show that if q is more than 1 plus degree of $AD - BC$, then this sequence is unbounded, so it is natural that more examples have been found with $q = 2$, or in characteristic two.

Lasjaunias conjectures (private communication) that algebraic elements in odd characteristic having continued fractions with bounded partial quotient sequences are necessarily of Class I.

6. Distribution of higher degree exponents

Just like Roth's upper bound for $E(\alpha)$ is broken for algebraic α in the finite characteristic function field case, so is Schmidt's upper bound for $E_d(\alpha)$ broken in this case. In fact, we can have very bad rational approximations and simultaneously very good higher degree ones, as was shown in the following results from [**T11, T12**].

(1) For $p = 2$, and any integer $m > 1$, we can construct infinitely many algebraic elements α of degree at most 2^{m^2} having continued fractions with folding pattern symmetries, and bounded sequence of partial quotients, so that $E(\alpha) = 2$, but with $E_2(\alpha) \geq 2^m > 3$.

(2) Let p a prime, q a power of p and $\epsilon > 0$ be given. Then we can construct infinitely many algebraic α, with explicit equations and continued fractions, such that

$$q \leq \deg(\alpha) \leq q+1, \quad E(\alpha) < 2+\epsilon, \quad E_2(\alpha) > q-\epsilon,$$

with explicit sequence of quadratic approximations realizing the last bound.

(3) Let p a prime, q a power of p and $m, n > 1$ be given. Then we can construct infinitely many algebraic $\alpha_{m,n}$, with explicit equations and continued fractions, such that

$$\deg(\alpha_{m,n}) \leq q^m + 1, \quad \lim_{n\to\infty} E(\alpha_{m,n}) = 2, \quad \lim_{n\to\infty} E_{q+1}(\alpha_{m,n}) \geq q^{m-1} + \frac{q-1}{(q+1)q},$$

with explicit sequence of degree $q+1$-approximations realizing the last bound.

6.1. Questions. — (I) A natural question raised by these considerations is whether there are algebraic α's of each degree d, with $E(\alpha) = 2$ (or even with bounded partial quotients) and for which the Liouville bound for the lower degree approximations is attained, or whether some of these requirements need to be relaxed.

(II) For real algebraic numbers α of degree more than d, by combining (i) Minkowski implication $w_d(\alpha) \geq d$, with (ii) Schmidt's subspace theorem implication, for algebraic α, that $w_d(\alpha) \leq d$, we get (iii) $w_d(\alpha) = d$ which then by Wirsing's inequalities (Section 3.7) implies (iv) $w_d^*(\alpha) = d$. But in finite characteristic function field case, we have seen that analog of the upper bound (ii) fails, so conceivably the lower bound $w_d^*(\alpha) \geq d$ may also fail for algebraic α. Does it? Note that by (8) of Section 3.2, such α would need to be of degree more than $d+1$.

(III) Similarly, is it possible that for approximation by algebraic integers of degree at most d case, the lower bound d for the exponent (analog of $w^* + 1$) is broken for algebraic α in finite characteristic case, rather than (countably many) special transcendental α's in [**R03**] for real numbers?

6.2. Remarks. — Applying a straight-forward function field analog of Lemma 6.1 of [**B12?**] shows that $w_2 \geq q - \epsilon$. Hence, by the Liouville inequality, we can put $\deg(\alpha) = q + 1$, in (2) of this section. It also shows that if $w_d(\alpha) = w_d^*(\alpha)$ (this equality is true in number fields, for all d, for algebraic α of degree more than d, but its status is unclear in function fields over finite fields) for $d = 2$, then the Liouville bound is best possible (within 'ϵ') for these α's. I thank Yann Bugeaud for these remarks.

7. Deformation and Exponents hierarchies

In finite characteristic situation, Osgood's theorem (Section 4.3) shows that for the elements in the complement of differentially closed subset obtained by throwing away solutions of rational Riccati equations, the Liouville exponent bound can be improved to Thue bound. In [**KTV00**], this line of thought was pushed further by associating certain curves over function fields to given algebraic power series and showing that bounds on the rank of Kodaira-Spencer map of this curves imply bounds on the diophantine approximation exponents of the power series, with more 'generic' curves (in the deformation sense) giving lower exponents. Further, transporting Vojta's conjecture on height inequality to finite characteristic by modifying it by adding suitable deformation theoretic condition, it was shown that the the exponents of the numbers giving rise to 'general' curves approach Roth's bound. This matches also with quantitative behavior (Section 5.8) noticed in Class IA above.

In the higher exponent direction, unconditional hierarchy of exponent bounds for approximation by algebraic quantities of bounded degree was given. We refer to [**KTV00**] for the precise details.

8. Diophantine classification

8.1. Mahler-Koksma classification. — We have two standard measures of diophantine approximations of α by algebraics β of degree at most d, namely how small error can get in terms of height of β versus how small can $P(\alpha)$ be for a polynomial of degree at most d in terms of height of P. For a complex α, we define $w(d,h)$, $w^*(d,h)$ by $|P(\alpha)| = h^{-dw(d,h)}$ and $|\alpha - \beta| = h^{-dw^*(d,h)-1}$ where $P(x)$ is a polynomial (integral, non-zero) of degree at most d and height at most h for which $|P(\alpha)|$ takes the smallest positive value, and β is algebraic of degree at most d and height at most h such that $|\alpha - \beta|$ takes the smallest positive value.

Next we define, following Mahler, $w(d)$ as $\limsup w(d,h)$ and w as $\limsup w(d)$ and v as the least d for which $w(d)$ is infinite, with $v = \infty$ if $w(d) < \infty$ for all d.

Finally, we say that α is in class A, S, T, U respectively according as whether ($w = 0, v = \infty$), ($0 < w < \infty, = \infty$), ($w = \infty, v = \infty$) or ($w = \infty, 0 < v < \infty$). We can also use sub-classification S_w and U_v and it is customary to call S number of type $\sup w(d)$. Note that U_1 corresponds to Liouville numbers. Another way to characterize these classes in terms of w_k is that A-numbers correspond to bounded sequence of w_k, S-numbers correspond to unbounded sequence, but with $w_k < ck$, T-numbers correspond to 'not $w_k < ck$' for any c, but with w_k finite, and U-numbers are those having some w_k infinite.

(i) A numbers are precisely the algebraic numbers.

(2) Algebraically dependent numbers belong to the same class.

(3) Almost all (full measure) numbers are S-numbers. Almost all real and complex numbers are S-numbers of type 1 and 1/2 respectively.

(4) There exist T-numbers, U_d-numbers for each d and S-numbers of arbitrarily large type.

(5) e is S-number of type 1 and π is S or T-number.

Similarly, we define, following Koksma, $*$ counterparts $w^*(d), w^*, A^*, S^*$ etc. The classes turn out to be identical, with subclasses related, but more subtly because of inequalities (Section 3.7 between w and w^* quantities.

It is known that $w^*(d) = 1$ for almost all real α, whereas it is an open question whether $w^*(d) \geq 1$ for all real α.

For more details and references for all this, see books by Baker, Schneider, Schmidt.

8.2. Function field case of Mahler, Koksma classification. — Bundschuh [**Bun78**], Dubois considered the same definitions over function fields over any field and in particular over finite fields, and proved some analogous results. Existence of U_2- numbers is shown for all F and for F finite, existence of S-numbers and U_d numbers is proved. See [**B04**, Sec. 9.4-9.5] for references. The question of existence of T-numbers is open.

9. Computational classification for function fields over a finite field

Now we look at computational classification of 'naturally occurring' numbers which are computable. This has some useful algebraic properties and is based not just on general computational complexity, but on a particular model inspired by automata-algebraicity correspondence wwhich works over finite field base. We recall it now.

9.1. Automata and Algebraicity. — Christol [**Chr79, CKMR80, All87, AS03, T97a, T04, T12b**] discovered nice combinatorial descriptions of algebraic power series over finite fields in terms of finite automata, a very robust concept which has been studied extensively from various angles by computer scientists, logicians and formal linguists. We will see in the next section how various ways of thinking of automata have helped giving transcendence proofs by completely different methods when the usual methods do not apply.

For a positive integer q, q-automata data consists of a finite set S (thought of as the set of its 'states'), $s_0 \in S$ (thought of as the initial state), action of digits base q on S, i.e., a map $\{0, 1, \cdots, q-1\} \times S \to S$ (thought of as the transition map showing how input affects the state). Finally, there is an output map from S to another finite set T. We can consider q-automata as input-output device, which on input an integer i fed in by its digits base q one by one, starts changing its states, starting from the initial, according to the actions of digits of i base q. At the end, you read the output at the end state. (There are various equivalent variants of this model).

Christol's theorem says that a power series $\sum f_i t^i$, with $f_i \in \mathbb{F}_q$ is algebraic over $\mathbb{F}_q(t)$, if and only if there is a q-automata with $T = \mathbb{F}_q$ which on input i produces output f_i.

For each $f \in \mathbb{F}_q$, consider the set S_f of i's such that $f_i = f$. It can be considered as the language of exponents i which are thought of all grammatical sentences in words being the base q digits of i. Grammars of different strengths and production rules have been studied and classified. Another characterization of algebraicity is the language of exponents is 'regular'.

We refer to the references above and in particular, to the two surveys [**T97a, T12b**] for more.

9.1.1. Exponents bounds using automata. — Firicel [**F11?**] gets an upper bound on the exponent of algebraic α in terms of the data (such as number of states) of any corresponding automata.

9.1.2. Characteristic dependence. — Cobham proved that if p and ℓ are distinct primes, then a sequence which is not eventually periodic can not be both p-automatic and ℓ-automatic. By the correspondence mentioned above, it means that algebraic irrational $\sum t^{n_i}$ in characteristic p has to be transcendental in all other finite characteristic. There is no simple 'algebraic' proof known of this fact. Cobham conjectured and Adamczewski, Bugeaud, Luca [**ABL04**] proved $\ell = 0$ analogy, using Schmidt subspace theorem. Namely, under the hypothesis above, the same series considered over $\mathbb{Q}(t)$, real number $\sum 10^{-n_i}$, and ℓ-adic number $\sum \ell^{n_i}$ (here ℓ can be p) are all transcendental.

9.1.3. Complex functions. — In contrast to the simple degree two passage from $\mathbb{R}$ to its algebraic closure $\mathbb{C}$, the passage in characteristic zero function fields from laurent series field to Puisseux series field, which is its algebraic closure, is of infinite degree. In finite characteristic, Puisseux series (which could have been handled by just substitution $t^{1/n}$ for t to use automata still) do not give algebraic closure, but Kedlaya [**K01**] in fact modified automata in this case to describe the algebraic closure. As we do not know natural applications of generalized series to special values, we will restrict to 'real' case.

9.2. Computational classification with algebraic properties. — We now briefly explain computational classification [**BT98**] with good algebraic properties, giving applications to refined transcendence classification of some important Laurent series in the next section. The usual numbers/Laurent series coming up in number theory and geometry are computable (already a small countable subclass) and like automata, computability has various incarnations studied by various viewpoints, such as Turing machines, languages generated by unrestricted grammar, recursive function theory, Post systems, Church's lambda calculus etc. Computer scientists, logicians, linguists have also studied intermediate strength classes. For example, linguists have Chomsky hierarchy of regular, context-free, context-sensitive, generative languages depending on strengths of grammar rules. Computer scientists classify input-output devices depending on their workings, memory requirements etc. into q-automata (at low end with zero or bounded memory), push down automata, linear space automata, Turing machines (at high end with infinite memory) Many classes converged to the same notions. So we examined these robust classes from computational, series perspective as in the automata characterization above and found that many of these have good algebraic properties, such as forming a field, a field algebraically closed in Laurent series etc.; in addition to closure, logical properties, such as closure under union, concatenation, complementation etc., explored before. The algebraic properties allow you to move by algebraic operations the problem about one series to another series which might be more convenient to deal by these generalized automata tools. See [**BT98**] for details.

10. Nature of and relations between special values

In this section, we restrict to function fields over a finite field $\mathbb{F}_q$. We will first introduce analogs in function field arithmetic of well-known special functions. We will only describe definitions, results, and identify methods, but leave the details of proofs, motivations, analogies and properties satisfied by these functions to references [**G96, T04**] and surveys, references identified in each subsection.

10.1. Special functions of function field arithmetic. — Basic analogs are

$$K := \mathbb{F}_q(t) \leftrightarrow \mathbb{Q}, \quad A := \mathbb{F}_q[t] \leftrightarrow \mathbb{Z}, \quad K_\infty := \mathbb{F}_q((1/t)) \leftrightarrow \mathbb{R}, \quad C_\infty := \widehat{\overline{K_\infty}} \leftrightarrow \mathbb{C}.$$

We think of $A+$, defined as the subset of A of polynomials monic in t, as an analog of the set of positive integers. Comparing sizes of A^* and $\mathbb{Z}^*$, which are $q-1$ and 2 respectively, in our situation, we call multiples of $q-1$ 'even' and other integers in $\mathbb{Z}$ 'odd'. Fundamental quantities related to function fields are

$$[i] = t^{q^i} - t, \quad l_0 = d_0 = 1, \quad l_i = -[i]l_{i-1}, \quad d_i = [i]d_{i-1}^q.$$

Then Carlitz-Drinfeld exponential and logarithm are given respectively by

$$e(z) = \sum_{i=0}^{\infty} z^{q^i}/d_i, \quad l(z) = \sum_{i=0}^{\infty} z^{q^i}/l_i.$$

The exponential is periodic with the period lattice $\tilde{\pi}A$, with

$$\tilde{\pi} = (-t)^{q/(q-1)} \prod_{n=1}^{\infty} (1 - t^{1-q^n})^{-1} = (-[1])^{q/(q-1)} \prod (1 - (t^{q^j} - t)/(t^{q^{j+1}} - t)),$$

being analog of $2\pi i$.

The Carlitz zeta value $\zeta(s)$, for s a positive integer is given by

$$\zeta(s) = \sum_{n \in A+} \frac{1}{n^s} \in K_\infty.$$

Next we have geometric gamma function

$$\Gamma(x) := \frac{1}{x} \prod_{a \in A+} (1 + \frac{x}{a})^{-1} \in C_\infty \cup \{\infty\}, \ \ x \in C_\infty,$$

and arithmetic gamma function defined by $\overline{\Gamma}(z+1) = \overline{\Pi}(z)$ and

$$\overline{\Pi} : \mathbb{Z}_p \to \mathbb{F}_q((1/t)), \ \ \sum n_i q^i \to \prod (d_i/t^{\deg d_i})^{n_i}.$$

Next we have two hypergeometric analogs. But first for $a \in \mathbb{Z}$ and integers $n \geqslant 0$, we define $(a)_n$ by $d_{n+a-1}^{q^{-(a-1)}}$, or $l_{-a-n}^{-q^n}$ or 0 according to whether $a \geq 1$, or $n \leq -a, a \leq 0$, or $n > -a \geq 0$ respectively.

We also define for $a \in C_\infty$ and integers $n \geq 0$, $(a)_n := \prod (a - f)$, with f running over all polynomials of degree less than n.

For all integers $r, s \geqslant 0$ and for all $a_i, b_j \in \mathbb{Z}$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s)$ with $b_j > 0$, consider the first hypergeometric function

$${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{d_n (b_1)_n \cdots (b_s)_n}\ z^{q^n}.$$

We denote it by ${}_rF_s(z)$, when the parameters are well understood

The second hypergeometric function ${}_r\mathcal{F}_s$ is similarly defined, but with parameters $a \in \mathbf{C}_\infty$, and $(a)_n$ defined above for them.

As in the classical case, various specializations of the hypergeometric series lead to interesting functions such as analogs of Bessel, Legendre, Jacobi, binomial functions.

We will concentrate on these and ignore results about (i) analogs of multizeta values (due to the author), (ii) Drinfeld Modular forms (due to Yu, Chang), (iii) higher rank or dimension objects in Drinfeld-Anderson theory (due to Yu, Chang and Papanikolas), and (iv) higher genus generalizations from rational function fields (due to Yu) and (v) classical algebraic geometry objects.

10.2. Application of Wade's approximation method. — We refer to surveys in [**T04**, Chapter 10] and [**W90, Y92, Br98**] for details, references and only select a few of the known results here.

10.2.1. e, π and zeta values. — Carlitz's student Wade, by traditional approximation techniques, but using simplifications obtained by making things $\mathbb{F}_q$-linear (the Wade method), proved transcendence of $e = e(1)$ and $\tilde{\pi}$, proved analogs of Hermite-Lindemann and Gelfond-Schneider theorems and proved transcendence of some series made up from $[n]$'s, d_n's and l_n's. The author in his Ph. D. thesis noted that slight modifications prove transcendence of $\zeta(sp^n)$ for $s \leq q$, and pushed such results to irrationality results of more classes of values and for ratios $\zeta(s)/\tilde{\pi}^s$, when s is 'odd', i.e., not divisible by $q-1$, by providing fast approximations which would have implied transcendence if Roth's analog were true in finite characteristic!

More results on irrationality measures of zeta values were obtained by de Mathan and Cherif. Soon after Jing Yu's proof [**Y92**] of transcendence for all s of $\zeta(s)$, for all 'odd' s of $\zeta(s)/\tilde{\pi}^s$ and v-adic interpolations $\zeta_v(s)$ mentioned below, Hellagouarch and Dammame proved the first part of these results using Wade's method.

10.2.2. Hypergeometric values. — Finally, we state [**TWYZ11**] result on transcendence of hypergeometric values obtained via Yao's generalization of Wade's method.

Let $r, s \geqslant 0$ be integers such that $r < s+1$, and let

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \text{ and } 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be integers. (This make the function ${}_rF_s$ entire, but not a polynomial). Then for all $\gamma \in \mathbf{C}_\infty \setminus \{0\}$ algebraic over $\mathbb{F}_q(t)$ and such that $\mathbb{F}_q(t)(\gamma)$ has less than q places above the infinite place of $\mathbb{F}_q[t]$ (In particular, γ can be any nonzero rational or nonzero algebraic of degree less than q), then ${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \gamma)$ is transcendental over $\mathbb{F}_q(t)$.

10.3. Applications of period methods. — We refer for details and references to surveys in [**Y92, Br98, T04, Pe07, T12c**] and references there.

10.3.1. Foundations of general transcendence theory. — In a series of papers, Jing Yu developed transcendence theory generalizing Wade's basic results to contexts of general Drinfeld modules over $\mathbb{F}_q[t]$, as well as more general rings in higher genus, and their higher dimensional versions, namely t-motives of Greg Anderson. Among many results, we will only mention here his analog of Baker's theorem in linear forms in logarithm, and analogs of Hermite-Lindemann, Wustholz's subgroup theorems for t-motives, all in the usual as well as v-adic contexts. See [**Y91, Y92, Y97**], [**T04**, Chapter 10] and references there.

10.3.2. Zeta values. — We continue the story from subsection 10.2.1. Giving formulas for $\zeta(s)$ in terms of s-th multilogarithm analog, [**AT90**] expressed it as the (canonical co-ordinate of) logarithm of s-th tensor power of the Carlitz module evaluated at an algebraic point, which is torsion point, if and only if s is 'even'. This combined with Yu's Hermite-Lindemann result [**Y92**], implies Yu's transcendence results on the zeta values and ratios with period power mentioned in subsection 10.2.1.

10.3.3. Recent algebraic independence results. — If g stands for classical or geometric or arithmetic gamma function above, the monomial $\prod g(f_i)^{n_i}$ is known to be algebraic (or more generally, an algebraic multiple of power of the period $2\pi i$, or $\tilde{\pi}$ respectively) (where f_i are proper fractions relevant in each case), if certain easily checkable combinatorial condition called bracket relation, which can be uniformly expressed in all 3 cases, is satisfied. We refer to [**T04**, Sec. 4.12] for details on this.

We now state some recent strong independence results in chronological order and then quickly explain some ideas behind the proofs.

(1) [**ABP04**] A set of Γ-monomials (i.e., subgroup of C_∞^* generated by $\tilde{\pi}$ and Γ-values at proper fractions in K) is $\overline{K}$-linearly dependent exactly when some pair of Γ-monomials is, and pairwise $\overline{K}$-linear dependence is entirely decided by bracket relation on their ratio. In particular, for any $f \in A_+$ of positive degree, the extension of $\overline{K}$ generated by $\tilde{\pi}$ and $\Gamma(x)$ with x ranging through proper fractions with denominator (not necessarily reduced) f, is of transcendence degree $1 + (q-2)|(A/f)^*|/(q-1)$ over $\overline{K}$.

Next, generalizing Yu's analog of Baker's linear independence result, as well as Denis' weaker result [**D06**] on algebraic independence of logarithms obtained by Mahler method, we have

(2) [**P08**] If $\ell_1, \cdots, \ell_n \in C_\infty$ are linearly independent over K with $e(\ell_i)$ algebraic over K, then ℓ_i are algebraically independent over $\overline{K}$.

(1) [**CY**] Only algebraic relations between $\zeta(n)$'s come from the Carlitz-Euler evaluation at 'even' n, implying for such n that $\zeta(n)/\tilde{\pi}^n \in K$, and $\zeta(pn) = \zeta(n)^p$. In particular, for n 'odd', $\zeta(n)$ and $\tilde{\pi}$ are algebraically independent and the transcendence degree of the field $K(\tilde{\pi}, \zeta(1), \cdots, \zeta(n))$ is $n+1-\lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/(p(q-1)) \rfloor$.

(2) [**CPYa**] Only algebraic relations between $\zeta_\ell(n)$'s for all ℓ and n, where ζ_ℓ denotes Carlitz zeta over $\mathbb{F}_{q^\ell}[t]$, are those as above coming from n 'even' or divisible by p. The periods $\tilde{\pi}_\ell$ of the Carlitz modules for $\mathbb{F}_{q^\ell}[t]$ are all algebraically independent.

(3) [**CPYb**] Only algebraic relations between $\zeta(n)$'s and geometric $\Gamma(z)$'s at proper fractions are those between zeta above and bracket relations for gamma.

(4) [**CPTY**] Only algebraic relations between $\zeta(n)$'s and arithmetic gamma values at proper fractions are those for zeta mentioned above and those for gamma coming from the bracket relations, and thus the transcendence degree of the field

$$K(\tilde{\pi}, \zeta(1), \cdots, \zeta(s), (c/(1-q^\ell))!)_{1 \le c \le q^\ell - 2}$$

is $s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor + \ell$.

10.3.4. Underlying tools: ABP criterion. — The main transcendence tool used in all these results is the following strong 'ABP' criterion [**ABP04**]:

Let $C_\infty\{T\}$ be the ring of power series over C_∞ convergent in closed unit disc. Consider $\Phi = \Phi(T) \in \mathrm{Mat}_{r\times r}(\overline{K}[T])$ such that det Φ is a polynomial in T vanishing (if at all) only at $T = t$ and $\psi = \psi(T) \in \mathrm{Mat}_{r\times 1}(C_\infty\{T\})$ satisfying $\psi^{(-1)} = \Phi\psi$, where (-1)-twist means replacing the coefficients of entries of the matrix by their q-th roots.

If $\rho\psi(t) = 0$ for $\rho \in \mathrm{Mat}_{1\times r}(\overline{K})$, then there is $P = P(T) \in \mathrm{Mat}_{1\times r}(\overline{K}[T])$ such that $P(t) = \rho$ and $P\psi = 0$.

10.3.5. Underlying tools: Anderson's t-motives and special values as their periods. — The main reason why this is useful is that in Anderson's theory of t-motives the periods arise exactly by specialization as in the criterion above. Thus $\overline{K}$-linear relations between the periods are explained by $\overline{K}[T]$-level linear relations (which in the t-motives set-up are the motivic relations and thus 'algebraic relations between periods are motivic', as analog of Grothendieck's conjecture for motives [**P08**] of Anderson). In terms of special functions of our interest, this makes the vague hope that 'there are no accidental relations and the relations between special values come from the known functional equations' precise and proves it.

Anderson's t-motives are simple, concrete linear algebra objects (no cycle-theoretic difficulties of classical motives!) and have tensor products via which algebraic relations between periods i.e., linear relations between powers and monomials in them reduce to linear relations between periods (of some other motives). In this sense, the ABP criterion above is similar to Wüstholz type sub-t-module theorem proved by Jing Yu.

Soon afterwards, Beukers proved [**Be06**] similar criterion for dependence of values of E-functions, but it does not have such strong applications to relations between periods of classical motives, because of differences in period connections in this case. The Tannakian formalism (based on linear algebra motivation) for t-motives developed in [**P08**] then expresses the transcendence degrees of field extensions generated by periods (appearing in the results above) as dimensions of 'motivic Galois groups' which allows their calculations using concrete difference Galois group descriptions of [**P08**].

Thus the strong algebraic independence results then follow from expressing special gamma, zeta values as periods (e.g., [**AT90, ABP04**]) of appropriate t-motives and calculating their dimensions. We refer to the references above and survey [**T12c**] for more details. Pellarin has shown how to derive some of these results more directly from Anderson's theory combined directly with Mahler method adapted to this case.

10.4. Applications of Automata methods. — We refer to [**T97a, T12b**] for detailed survey of tools and applications, so we will be brief and just mention some highlights.

First applications [**A90, B92**] of automata to special values occurring in function field arithmetic were applications to transcendence of $\tilde{\pi}$, some zeta values (known earlier by period methods and other methods mentioned above).

10.4.1. Gamma values. — When period methods only could prove very weak transcendence results (parallel to what is known for the usual gamma) on values of arithmetic gamma at proper fractions, automata methods [**T96, A96, MFY97, T97a**] proved very complete results settling transcendence of all values at proper fractions, of all values at p-adic integers which are not positive integers, and of all monomials in values at fractions which were not known to be algebraic. While the first and third results mentioned have now been superceded by algebraic independence results mentioned in Section 10.3.3, the second result above or the result on values of v-adic interpolation on gamma proved in [**T97a**] are still provable only by automata methods.

10.4.2. Refined transcendence of π by language tools. — Note $\pi := t^{-q/(q-1)}\tilde{\pi}$ is a Laurent series. It (or rather its reciprocal) is not [**BT98**] context-free (which gives, in particular, a language theoretic proof of its transcendence), but is context-sensitive. The tools here are language theoretic closure properties, moving to convenient series by algebraic properties and getting contradiction by 'pumping lemma' for context-free languages. In rough terms, 'pumping lemma' says that given sufficiently long grammatical sentence, some part of it can be 'pumped' many times retaining the grammatical structure. To give an example in natural language, in a sentence, 'He is a friend of mine', 'a friend of' can be pumped many times, to get e.g., 'He is a friend of a friend of mine'. We use this in language of exponents of series we have and get a contradiction.

10.4.3. Refined transcendence of e, θ. — Using computational and language tools, we [**BT98**] show that Carlitz analog of e, (known to be transcendental so non-automatic by Wade) is context-sensitive and theta series or set of squares is context-sensitive (even in logarithmic space under GRH), but (for $q = 2$) not context-free.

10.4.4. Transcendence of modular forms and Tate period by density argument. — In [**AlT99**] transcendence of some q-expansions of Eisenstein series (and also some related series in 'wrong' weights so that they are not immediately accessible through algebraic geometry techniques) is shown by showing that the asymptotics of coefficients (modulo p) does not match those classified by Cobham for sequences produced by automata. There are applications to transcendence of the multiplicative period of Tate elliptic curve. See [**T97a, T12b**] and [**T04**, Cha. 11] for details and references.

10.4.5. Hypergeometric functions. — Sharif-Woodcock and Harase generalized automata criterion somewhat from the case of finite base to more general bases. This is used to prove the following result [**TWYZ11**] characterizing the parameters for which the hypergeometric function is algebraic.

Let $r, s \geqslant 0$ be integers such that $r = s + 1$, and let

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \text{ and } 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be integers. Then the following properties are equivalent:

1. $a_j \geqslant b_{j-1}$, for all integers $1 \leqslant j \leqslant r$;
2. $({}_rF_s(z))^{q^\ell} \in \mathbb{F}_q[t][[z]]$, with $\ell = \max(a_r, b_s)$;
3. ${}_rF_s(z)$ is an algebraic function.

For the second hypergeometric analog, we do not have such complete results, but only algebraicity and transcendence results [**TWYZ11**] for particular parameter values and some more results of the following flavour.

Any function ${}_{s+1}\mathcal{F}_s(a_i; b_j; z)$, with a_i being any proper fractions and b_j being fractions with denominators of degree one, is algebraic.

10.5. Continued fraction for e and Hurwitz numbers. — Euler showed that $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \cdots]$ and Hurwitz showed (without giving a 'formula') that the continued fraction of $(ae^{2/n}+b)/(ce^{2/n}+d)$ (the so called Hurwitz numbers), where a, b, c, d are integers with $ad - bc \neq 0$, and n is a positive integer, consists eventually of arithmetic progressions. For example, in the case of e, there are 3 arithmetic progressions, two with common difference 0 and one with common difference 2.

The exact patterns of continued fractions, of quite different nature, for analogs of e and Hurwitz numbers in $\mathbb{F}_q[t]$ case are given in [**T97**] and reference there. Here we just recall the simplest case, when $q = 2$. Consider $[n] := t^{q^n} - t$, the building blocks of F_{q^n}. Then Carlitz-Drinfeld analog of e for $\mathbb{F}_2[t]$ has continued fraction (with repeating pattern as in folding lemma)

$$e = [1, \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \cdots].$$

In fact, it can also be described by saying that for $n > 0$, $a_n = [i]$, where i is the position from the right of the first occurrence of digit one in the binary expansion of n. Hence the continued fraction of the fundamental quantity e for $\mathbb{F}_2[t]$ is given by partial quotients $[n]$ which are building blocks of $\mathbb{F}_{2^n}$ in a pattern simply explained in terms of binary digits of n in this way!

10.6. Exponents for e and π. — Euler's continued fraction shows that $E(e) = 2$, also Salikhov in 2008 showed that $E(\pi) < 7.61$, but we do not know about their higher exponents.

In $\mathbb{F}_q[t]$ case, we know that $E(e) = q$, and $E_2(e) \geq 3 + 1/(q-1)$ or $\geq 2 + 1/(q-1)$, depending on $p = 2$ or $p > 2$. For analog $\pi := \prod_{n=1}^{\infty}(1 - t^{1-q^n})^{-1}$, we know $E(\pi) \geq q - 1$, with equality when $q \geq 5$. For another analog $\pi_1 := \prod(1 - (t^{q^j} - t)/(t^{q^{j+1}} - t))$, we have $E(\pi_1) \geq (q-1)^2/q$, with equality when $q \geq 5$. For these, see [**T99, T11, T12**] and references there.

10.7. Questions on special values. — We end by mentioning some interesting open problems on the nature of special values.

(1) Are Carlitz-Drinfeld analogs of e and $2\pi i$ algebraically independent?

(2) We know that Artin-Hasse exponential series $exp(\sum_0^\infty x^{p^n}/p^n)$ has p-integral coefficients. Is its reduction modulo p transcendental over $\mathbb{F}_p(x)$?

(3) Higher genus A or even v-adic counterparts of most of the results above are lacking.

(4) It would be of great interest to know the nature of $\Gamma(0)$, mentioned in [**T04**, Remark 8.3.11], in particular, its relation, if any, with $\tilde{\pi}$ for general A.

(5) What are all multizeta relations? For the period connection, see [**AT09**].

(6) What is the status of these special values in function field analog of Mahler-Koksma classification? Analogs of e, $\tilde{\pi}$ log(1) and $\zeta(s)$ (or rather twisted values, for $p \neq 2$) for $s \leq q$ were studied in [**Bun78**], and it was shown that some of these numbers are not U-numbers. But the results obtained are not strong enough to determine precise classes, so that there is no algebraic independence result from these results yet, unlike the periods approach.

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