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## Shtukas and Jacobi sums<sup>★</sup>

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**Summary.** We show how the analogues of Jacobi sums, in the context of function fields, introduced and studied in [T1, T2, T3] can be obtained from shtukas introduced and studied in [D2, D3, M]. We apply this to obtain some results on the prime factorization of analogues of Gauss sums and to prove an analogue of the Gross-Koblitz formula for general function field, generalizing the results in [T2]. For this purpose, we also introduce and interpolate a new analogue of gamma function.

### Introduction

If we try to imitate the classical cyclotomic theory over  $\mathbf{Q}$  in the case of a function field over a finite field, by adjoining roots of unity, we end up only with constant field extensions and do not get all abelian extensions in contrast to the classical case. Carlitz [C3], Drinfeld [D1] and Hayes [H1, H2] developed another type of cyclotomic theory over function fields by adjoining torsion of appropriate Drinfeld modules. (See introduction to [T3] for a quick summary). Mixing these two cyclotomic theories, analogues of Gauss sums, taking values in function fields, were defined and studied in [T1, T2, T3].

Analogues of well-known theorems such as the Stickelberger theorem, giving the prime factorization of the Gauss sums and the Gross-Koblitz theorem [G-K], expressing the Gauss sums in terms of special values of  $p$ -adic gamma function, were proved in the simplest case of  $\mathbf{F}_q[T]$  in [T1, T2]. In [T3] the prime factorizations, in the general case, were shown to be interesting, but strange in view of the established analogies. In particular, it was noted that the analogue of the Gross-Koblitz formula would not generalize with the gamma function under consideration there. (See 4.3).

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<sup>★</sup> Dedicated to the memory of Kumar Gandharva

The Gauss sums (we will sometimes drop the words ‘analogues’ from now on) can be viewed as character sums of special multiplicative characters and analogues of additive characters. But in contrast to the classical case, the Jacobi sums are not character sums of multiplicative characters, but they are still quotients of appropriate monomials in the Gauss sums. The crucial quantities turn out to be the basic Jacobi sums,  $g_{j-1}^q/g_j$ , in the notation explained below.

Inspired by Krichever’s theory of algebro-geometric solutions of K-dV equations, Drinfeld introduced [D2] a geometric concept of a shtuka (the Russian word literally means ‘a piece of something’) and established a natural bijection between a special case of shtukas and Drinfeld modules. A nice exposition can be found in [M]. The special case of rank one which is relevant here will be described in 0.3, following essentially the exposition in some notes of Greg Anderson.

In Sect. 1, we will show that the Jacobi sums made up from  $\wp$  torsion of a Drinfeld module can be interpreted as specializations at geometric points above  $\wp$  of a meromorphic function, obtained from the shtuka corresponding to the Drinfeld module, on curve cross its Hilbert cover. Hence the strange factorizations of the Gauss sums get related to the divisor of this function. In Sect. 2, we give some examples of this and apply it to show that for  $F_q(T)$  and any infinite place the prime factorizations of Gauss sums are similar to those in the  $F_q[T]$  case, which in turn are in analogy with the classical case. This generalizes the results of [T3], by removing the restriction on the degree of the infinite place there.

In Sect. 3, a new gamma function (agreeing with the one studied in [C2, G1, T4] for  $F_q[T]$ , but not in general) is introduced and interpolated at finite places of  $K$ . In Sect. 4, we show how the proof [T1, T2] of the analogue of the Gross-Koblitz theorem immediately generalizes by the results of Sect. 1. In Sect. 5, we discuss the interpolation of the gamma function at  $\infty$ . For a more complete story on gamma functions, the reader may refer to the expository paper [T6].

## 0 Background

### 0.0. Notation

$F_q$ : a finite field of characteristic  $p$  containing  $q$  elements

$X$ : a smooth, complete, geometrically irreducible curve over  $F_q$

$g$ : genus of  $X$

$K$ : the function field of  $X$

$\infty$ : a closed point of  $X$  (should be no confusion with usual usage)

$\delta$ : the degree of the place  $\infty$

$A$ : the ring of elements of  $K$  with no poles outside  $\infty$

$h$ : the class number of  $K$

$h_A$ : the class number of  $A$  ( $= h\delta$ )

Let  $M$  be an algebraically closed field containing  $F_q$  and of infinite transcendence degree. Let  $\bar{X}$  denote the fiber product of  $X$  with  $\text{Spec}(M)$  over  $F_q$ . We identify closed points of  $\bar{X}$  with  $M$ -valued points of  $X$  in the obvious way. For  $\xi \in X(M)$ , let  $\xi^{(i)}$  denote the point obtained by raising the coordinates of  $\xi$  to the  $q^i$ -th power. We extend the notation to the divisors on  $\bar{X}$  in the obvious fashion. For a meromorphic function  $f$  on  $\bar{X}$ , let  $f|_{\xi}$  denote the value (possibly infinite) of  $f$  at  $\xi$  and let  $f^{(i)}$  denote the pull-back of  $f$  under the map  $\text{id}_X \times \text{Spec}(F^i): \bar{X} \rightarrow \bar{X}$ ,

where  $F := x \rightarrow x^q: M \rightarrow M$ . For a meromorphic differential  $\omega = fdg$ , let  $\omega^{(i)} := f^{(i)}d(g^{(i)})$ . Fix a  $\bar{\mathbb{F}}_q$ -point  $\bar{\infty}$  above  $\infty$ .

Fix a local parameter  $t^{-1} \in \mathcal{O}_{\bar{x}, \bar{\infty}}$  at  $\bar{\infty}$ . For a nonzero  $x \in \mathcal{O}_{\bar{x}, \bar{\infty}}$ , define  $\deg(x) \in \mathbb{Z}$  and  $\text{sgn}(x) \in M$  to be  $\delta$  times the exponent in the highest power of  $t$  and the coefficient of the highest power respectively, in the expansion of  $x$  as Laurent series in  $t^{-1}$ , with coefficients in  $M$ . Note that  $\text{sgn}(x) \in \mathbb{F}_q^*$  for nonzero  $x \in K$ .

0.1. Drinfeld modules (See [D1, H1] for more details).

0.1.1 Let  $\xi: A \rightarrow M$  be an embedding of  $A$  in  $M$ . Let  $M\{F\}$  denote the noncommutative ring generated by the elements of  $M$  and by a symbol  $F$ , with the commutation relation  $Fm = m^qF$ , for all  $m \in M$ . By a Drinfeld  $A$ -module  $\rho$  relative to  $\xi$  (in fact ‘normalized with respect to  $\text{sgn}$ , of rank one and generic characteristic’, but we will drop these words), we will mean an injective homomorphism  $\rho: A \rightarrow M\{F\}$  ( $a \in A \rightarrow \rho_a \in M\{F\}$ ) such that,  $\rho_0 = 0$  and for all  $a \in A - \{0\}$ ,

$$\rho_a = \sum_{i=0}^{\deg(a)} \rho_{a,i} F^i, \rho_{a,i} \in M \quad \rho_{a,0} = \xi(a), \quad \rho_{a,\deg(a)} = \text{sgn}(a).$$

0.1.2 Two Drinfeld  $A$ -modules  $\rho, \tilde{\rho}$  are considered isomorphic if there is a nonzero  $m \in M$  such that  $m\rho_a = \tilde{\rho}_a m$  for  $a \in A$ .

0.1.3 Let  $H_1$  be the subfield of  $M$  generated by  $\rho_{a,i}$ ’s, for all  $i$ ’s and all  $a \in A$ . All  $\rho_{a,j}$ ’s are integral over  $A$ . The class field theoretic description of  $H_1$  can be found in [H2]. It is an abelian extension of  $K$  (here and sometimes in what follows, we identify  $K$  with its copy in  $M$  relative to  $\xi$ ), abelian and totally ramified over  $\infty$  of degree  $(q^\delta - 1)/(q - 1)$  over  $H$ , where  $H$  is the maximal abelian unramified extension of  $K$  in  $M$ , split at  $\infty$ . In particular,  $H_1 = H$ , when  $\delta = 1$ . The field of constants in  $H_1$  is  $\mathbb{F}_q$ . There are  $h$  nonisomorphic Drinfeld  $A$ -modules over  $H_1$ .

0.1.4 The exponential  $e(z) = e_\rho(z) \in M[[z]]$  of  $\rho$  is a power series characterized by  $e(az) = \rho_a(e(z))$ , for all  $a \in A$  and  $e(z) = z +$  higher order terms in  $z$ . Let the logarithm  $l(z) = l_\rho(z)$  be the formal inverse to the exponential. Using the functional equations for  $a \in \mathbb{F}_q$ , it is easy to see that the coefficients of  $z^k$  in  $e(z)$  or  $l(z)$  are zero, unless  $k$  is a power of  $q$ . Write  $e(z) = \sum e_i z^{q^i}$  and  $l(z) = \sum l_i z^{q^i}$ .

0.1.5. Example. (See [C1, T4]). Let  $A = \mathbb{F}_q[T]$ ,  $t = T$ . Then  $H_1 = H = K$  and the unique Drinfeld  $A$ -module  $C$  (called Carlitz module) is determined by  $C_T = T + F$ . We have  $e_C(z) = \sum z^{q^i}/D_i$  and  $l(z) = \sum z^{q^i}/L_i$ , where  $D_i := (T^{q^i} - T) \cdot (T^{q^i} - T^q) \cdot \dots \cdot (T^{q^i} - T^{q^{i-1}})$  and  $L_i := (T - T^{q^i})(T - T^{q^{i-1}}) \cdot \dots \cdot (T - T^q)$ .

0.2. Gauss and Jacobi sums (See [T2, T3] for more details).

0.2.1 For  $a \in A$ , define ‘ $a$ -torsion of  $\rho$ ’ as  $\Lambda_a := \{m \in M : \rho_a(m) = 0\}$ . For an ideal  $I$  of  $A$ , define ‘ $I$ -torsion of  $\rho$ ’ as  $\Lambda_I := \{m \in M : \rho_i(m) = 0, \text{ for all } i \in I\}$ . It is an  $A$ -module under  $\rho$ . As mentioned in the introduction, by adjoining  $\Lambda_I$  ( $I$  nonzero) to  $K$ , we get another type of cyclotomic extensions of  $K$ .

0.2.2 Let  $\wp$  be a prime of  $A$  of degree  $d$ . Choose an  $A$ -module isomorphism  $\psi: A/\wp \rightarrow A_\wp$  (an analogue of additive character) and let  $\chi_j$  ( $j \pmod d$ ) be  $\mathbb{F}_q$ -homomorphisms  $A/\wp \rightarrow M$ , indexed so that  $\chi_j^q = \chi_{j+1}$  (special multiplicative

characters which are  $q^j$ -powers of ‘teichmuller character’). We can identify  $\chi_j(z)$  with  $a|_{\theta^\psi}$ , for some geometric point  $\theta$  above  $\wp$ , if  $a \in A$  is such that  $a \bmod \wp$  is  $z$ . Then we define Gauss sums

$$g_j := g(\chi_j) := - \sum_{z \in (A/\wp)^*} \chi_j(z^{-1})\psi(z).$$

0.2.3 The  $g_i$  are nonzero. We define Jacobi sums  $J_j$  by  $J_j := g_{j-1}^q/g_j$ . Then  $J_j$  is independent of the choice of  $\psi$ . We put  $J := J_0$ .

0.2.4 For the Carlitz module given in 0.1.5, it was shown in [T1, T2] that  $J_j = -(T - \chi_j(T))$  and Stickelberger factorization of  $g_j$  was easily obtained from this. Note for example that  $g_j^{q^d-1} = J_j J_{j-1}^q \cdots J_{j-d+1}^{q^d-1}$ .

0.3. *Shtukas and associated functions* (See [D2, D3, M] for details).

**0.3.1. The Drinfeld vanishing lemma.** Let  $\xi, \eta \in X(M)$ , a divisor  $V$  of  $\bar{X}$  of degree  $g$  and a meromorphic function  $f$  on  $\bar{X}$  be given such that

$$V^{(1)} - V + (\xi) - (\eta) = (f)$$

If  $\xi \neq \eta^{(s-1)}$  ( $|s| < g$ ), then

$$H^i(\bar{X}, \mathcal{O}_{\bar{X}}(V - (\eta^{(-1)}))) = 0 \quad (i = 0, 1).$$

*Proof.* There is nothing to prove if  $g = 0$ , so assume  $g > 0$ . Define divisors  $V_i$  for  $i \in \mathbf{Z}$  by the rules

$$V_0 := V - (\eta^{(-1)}), \quad V_{i+1} := (\eta^{(i-1)}) + V_i$$

and set  $\mathcal{L}_i := \mathcal{O}_{\bar{X}}(V_i)$ . By the Riemann-Roch theorem, it will be enough to prove that  $h^0(\mathcal{L}_0) = 0$ . Set

$$S := \{1 - g \leq s \leq g : h^0(\mathcal{L}_s) = h^0(\mathcal{L}_{s-1}) + 1\}.$$

Then the cardinality of  $S$  is  $g$ , because  $h^0(\mathcal{L}_{-g}) = 0$  and  $h^0(\mathcal{L}_g) = g$ , by the Riemann-Roch theorem. Now let  $s \in S$  be such that  $s < g$ . Then there exists a global section  $e$  of  $\mathcal{L}_s$  not vanishing modulo  $\mathcal{L}_{s-1}$  and consequently,  $fe^{(1)}$  is a global section of  $\mathcal{L}_{s+1}$  not vanishing modulo  $\mathcal{L}_s$ , since  $\xi \neq \eta^{(s-1)} = \text{supp}(\mathcal{L}_{s+1}/\mathcal{L}_s)$  by the hypothesis. Therefore  $s + 1 \in S$ . So  $S = \{1, 2, \dots, g\}$ , hence the lemma is proved.

0.3.2 *Remark.* In the original version of the lemma,  $\eta$  is  $\infty^{(-1)}$  and  $\xi \neq \infty^{(s)}$  for all  $s$  and hence the condition such as  $\xi \neq \eta^{(s)}$  for  $|s| < g$  is not made explicit. This refined version is due to Greg Anderson, who has made use in his work of the flexibility thus obtained. In what follows though, we only need the original version.

**0.3.3. Corollary.** Let  $\xi, \eta \in X(M)$ . Let  $V$  be an effective divisor of degree  $g$  on  $\bar{X}$  such that  $V^{(1)} - V + (\xi) - (\eta)$  is principal. Then  $\eta^{(-1)}, \xi$  (resp.  $\mu$ , where  $\mu$  is a  $\mathbb{F}_q$ -rational point of  $X$ ) can not belong to the support of  $V$ , provided that for  $|s| < g$ ,  $\xi \neq \eta^{(s-1)}$  (resp.  $\xi \neq \eta^{(s)}$ ). If  $W$  is an effective divisor of degree  $d < g$  and  $\xi \neq \eta^{(g-d+s-1)}$  for  $|s| < g$ , then  $W^{(1)} - W + (\xi) - (\eta)$  can not be principal.

*Proof.* The fact that  $\eta^{(-1)}$  is not in the support of  $V$ , follows immediately from the lemma. If  $\xi$  (resp.  $\mu$ ) belongs to the support of  $V$ , we apply this fact to  $W = V - (\xi) + (\eta)$ ,  $\xi^{(1)}, \eta^{(1)}$  (resp.  $W = V - (\mu) + (\eta)$ ,  $\xi, \eta^{(1)}$ ) in place of  $V, \xi$  and  $\eta$  to get a contradiction. To prove the last claim, apply the lemma with

$V := W + (\eta) + (\eta^{(1)}) + \dots + (\eta^{(g-d-2)}) + (\xi^{(-1)})$  if  $d < g - 1$  (resp.  $V := W + \eta$  otherwise), and with  $\xi^{(-1)}$  and  $\eta^{(g-d-1)}$  (resp.  $\xi$  and  $\eta^{(1)}$ ) in place of  $\xi$  and  $\eta$  in the lemma.

**0.3.4** Fix a transcendental point  $\xi \in X(M)$ . Then evaluation at  $\xi$  induces an embedding of  $A$  into  $M$ . By solving the corresponding equation on the Jacobian of  $X$ , we see that for some divisor  $V$ ,  $V^{(1)} - V + (\xi) - (\infty)$  is principal. A Drinfeld divisor  $V$  relative to  $\xi$  is defined to be an effective divisor of degree  $g$  such that  $V^{(1)} - V + (\xi) - (\infty)$  is principal. From 0.3.1 and Riemann-Roch, it follows that Drinfeld divisor is the unique effective divisor in its divisor class. (In particular, there are  $h$  such divisors). Hence there exists a unique function  $f = f(V)$  with  $\text{sgn}(f) = 1$  and such that  $(f) = V^{(1)} - V + (\xi) - (\infty)$ . By abuse of terminology, we call  $f$  shtuka. (In fact, in our context shtuka is a line bundle  $\mathcal{L}$  on  $\bar{X}$  with  $\mathcal{L}^{(1)}$  being isomorphic to  $\mathcal{L}(-\xi + \infty)$  and in our case, with  $\mathcal{L} = \mathcal{O}_{\bar{X}}(V)$ ,  $f$  realizes this isomorphism.)

**0.3.5 Drinfeld bijection.** The set of Drinfeld divisors  $V$  (relative to  $\xi$ ) is in natural bijection with the set of Drinfeld  $A$ -modules  $\rho$  (relative to  $\xi$ ) as follows. (See [M] for details of the proof). Let  $f = f(V)$  as in 0.3.4. Then

$$1, f^{(0)}, f^{(0)}f^{(1)}, f^{(0)}f^{(1)}f^{(2)}, \dots$$

is an  $M$ -basis of the space of sections of  $\mathcal{O}_{\bar{X}}(V)$  over  $\bar{X} - \infty$ . Define  $\rho_{a,j} \in M$  by the rule

$$a := \sum_j \rho_{a,j} f^{(0)} \dots f^{(j-1)}. \tag{**}$$

Then the  $\rho$  corresponding to  $V$  is given by  $\rho_a := \sum \rho_{a,j} F^j$ . (We remark here that if we take  $\xi$  to be a  $\bar{\mathbb{F}}_q$ -point of  $\bar{X}$  not above  $\infty$ , instead of a transcendental point, there is still a Drinfeld bijection with so called Drinfeld modules of finite characteristics. But we will not consider this case in this paper.)

**0.3.6. Proposition.** *We have*

$$e(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{(f^{(0)} \dots f^{(n-1)})|_{\xi^{(n)}}}.$$

*Proof.* Note that by 0.3.3 the coefficients on the right hand side are never infinite. To see that the right hand side satisfies the correct functional equations for  $e(z)$ , divide both sides of (\*\*) by  $f^{(0)} \dots f^{(n-1)}$  and evaluate at  $\xi^{(n)}$ .

**0.3.7** By Riemann-Roch theorem, there exists a unique global section  $\omega = \omega(V)$  of  $\Omega_{\bar{X}/M}(-V + 2(\infty))$  with the leading term in the Laurent expansion in  $t$  at  $\infty$  being  $dt$  times an element of  $\text{sgn}$  one.

**0.3.8. Proposition (Anderson)** *We have*

$$l(z) = \sum_{n=0}^{\infty} \left( \text{Res}_{\xi} \frac{\omega^{(n+1)}}{f^{(0)} \dots f^{(n)}} \right) z^{q^n}.$$

*Proof.* The necessary relations to show that  $l(z)$  is the formal inverse of  $e(z)$  come from the fact that the sums of the residues of differentials figuring in the expression vanish.

**1 Interrelations**

1.0 From now on we will drop the words ‘relative to  $\xi$ ’.

**1.1. Lemma.** *A  $\bar{F}_q$ -point  $\theta$  of  $X$  over a prime  $\wp$  of  $A$  can not belong to the support of  $V$ .*

*Proof.* If  $\theta$  belongs to the support of  $V$ , the specialization of the defining equation of  $f$  to  $\xi = \theta$  contradicts 0.3.3.

**1.2. Theorem.** *Let  $V$  and  $f$  be the Drinfeld divisor and shtuka respectively, corresponding to a Drinfeld module  $\rho$  via the Drinfeld bijection. Then with the Jacobi sums  $J_j$  defined using  $\wp$ -torsion of  $\rho$  and normalized as in 0.2.2 and 0.2.3, we have*

$$f|_{\theta^{(j)}} = J_j.$$

*Proof.* Let  $h_0$  be a  $q^d - 1$ -th root of  $(ff^{(1)} \cdots f^{(d-1)})|_{\theta}$  and define  $h_i$  inductively by  $h_i := h_{i-1}^q / f|_{\theta^{(i)}}$ . Then  $h_i$  depends only on  $i$  modulo  $d$ . Specializing (\*\*) at  $\theta^{(i)}$  gives

$$a|_{\theta^{(i)}} h_i = \sum \rho_{a,j} h_{i-j}^{q^j}$$

Hence with  $\phi(z) := \sum \chi_i(z) h_i$ , we see that  $\phi(az) = \sum \rho_{a,j} \phi(z)^{q^j}$  and that  $\phi(z)$  is a  $\wp$ -torsion point. In other words,  $\phi$  is an analogue of additive character, as in 0.2.2 and so  $\phi(z) = \psi(uz)$  for some  $u \in (A/\wp)^*$ . From 0.2.2, it then follows that  $h_j = \chi_j(u) g_j$ . Hence

$$f|_{\theta^{(j)}} = \frac{h_{j-1}^q}{h_j} = \frac{g_{j-1}^q}{g_j} = J_j$$

as claimed.

1.3 The relation  $\xi \leftrightarrow V$  defines a correspondence, say  $W$  on  $X \times Y_1$ , where  $Y_1$  is the ‘Hilbert cover’ corresponding to  $H_1$ . Let  $U$  be the restriction of the transpose of  $W$  at  $\theta \times Y_1$ . Then the theorem and integrality of  $\rho_{a,j}$  mentioned in 0.1.3 show that  $(J) = qU^{(-1)} - U + (\theta) - I$ , where  $I$  is supported on  $\infty$ . This determines the divisor of  $J$  completely, if  $h = \delta = 1$ . In general,  $I$  is calculated in [T7].

1.4 Where  $\delta = 1$ , using 0.3.1 together with the moduli description of theta divisor in terms of cohomology jumps of line bundles, Greg Anderson (to appear) has given a nice interpretation of  $W$ , in terms of the theta divisor as follows: Let  $\alpha: X \rightarrow J$  be the embedding of  $X$  in the jacobian  $J$  of  $X$ , via  $x \rightarrow x - \infty$ . Define the Hilbert cover  $\pi: Y \rightarrow X$  as a pullback  $\beta: Y \rightarrow J$  of  $\alpha$  under the Lang’s isogeny  $F - 1: J \rightarrow J$ . Then  $W$  is the pullback of the theta divisor under  $-\alpha \circ p_1 - \beta \circ p_2: X \times Y \rightarrow J$ .

**2 Examples and application**

2.1 We first look at the case of Carlitz module 0.1.5. Since the genus is zero in this case, the Drinfeld divisor is empty and it is easy to see that  $f = T - T|_{\xi}$  and that  $\omega = dT$ . Note that the formulae provided by 0.3.6 and 0.3.8 are consistent with those in 0.1.5. Also comparison with 0.2.4 verifies the results of section 1 in this case.

2.2 Now let  $K$  be a rational function field with  $\infty$  being any place of  $K$ . Let  $\rho$  be a Drinfeld  $A$ -module. We can write  $H = F_{q^d}(T)$ , with  $\text{sgn}(T) = 1$ . Then it is easy to

see that  $f = T - T|_{\xi}$ . Combining this with the results of Sect. 1, we see that the prime factorization of Gauss sums in this case is quite analogous to the case 0.1.5 (treated in [T2]), except that place  $\infty$  corresponding to  $\rho$  replaces  $\infty$ . This generalizes the results in [T3] of the case of  $\infty$  of degree not more than two.

2.3 Apart from  $\mathbf{F}_q[T]$ 's, there are exactly four  $A$ 's with  $h_A = 1$ . (See [T3] and references there). The corresponding Drinfeld modules are computed in [H1]. In [T3, p. 248] we computed  $J$  and its divisor, but with different notation and in [T5] we computed  $f$ . In both cases, the computation consists of eliminations on the relations occurring in the proof of the Theorem 1.2 with  $a = x, y$ . Now we summarize the results of those computations, these are easy to verify directly and provide verification of results of Sect. 1 in these cases. (It may be worthwhile to point out here that in practice usually, as in the case of these examples, to produce  $f$  explicitly so that  $f|_{\xi^{(i)}} = e_{i-1}^q/e_i$  and  $f|_{\theta^{(i)}} = g_{i-1}^q/g_i$ , we do not need considerations of 0.3 or Sect. 1 at all and it can be achieved as described above.) We put  $\bar{x} := x|_{\xi}$  and  $\bar{y} := y|_{\xi}$ . While giving the divisor of  $J$ , by abuse of notation, instead of the quantities corresponding to  $\theta$ , we use those corresponding  $\xi$ .

(a)  $A = \mathbf{F}_2[x, y]/y^2 + y = x^3 + x + 1$ : We have

$$f = \frac{\bar{x}(x + \bar{x}) + y + \bar{y}}{x + \bar{x} + 1}.$$

If  $\xi + 1$  is the point where  $x$  is  $\bar{x} + 1$  and  $y$  is  $\bar{x} + \bar{y} + 1$ , then  $V = (\xi + 1)$ . Note that this point corresponds to the automorphism  $\sigma$  of example 1 of p. 248 of [T3]. By 1.3, it follows (and can be verified from [T3]) that

$$(J) = 2(\xi + 1)^{(-1)} - (\xi + 1) + (\xi) - 2(\infty)$$

where  $\xi + 1$  is the point where  $x$  is  $\bar{x} + 1$  and  $y$  is  $\bar{x} + \bar{y}$ .

(b)  $A = \mathbf{F}_4[x, y]/y^2 + y = x^3 + \zeta_3$ : We have

$$f = \frac{\bar{x}^2(x + \bar{x}) + y + \bar{y}}{x + \bar{x}}.$$

If  $\xi$  denotes the point where  $x$  is  $\bar{x}$  and  $y$  is  $\bar{y} + 1$ , then  $V = \xi$  and

$$(J) = 4(\xi)^{(-1)} - (\xi) + (\xi) - 4(\infty)$$

(c)  $A = \mathbf{F}_3[x, y]/y^2 = x^3 - x - 1$ : We have

$$f = \frac{-\bar{y}(x - \bar{x}) + y - \bar{y}}{x - \bar{x} - 1}.$$

If  $\xi + 1$  (resp.  $\xi - 1$ ) denotes the point where  $x$  is  $\bar{x} + 1$  (resp.  $\bar{x} - 1$ ) and  $y$  is  $\bar{y}$ , then  $V = (\xi + 1)$  and

$$(J) = 3(\xi - 1)^{(-1)} - (\xi - 1) + (\xi) - 3(\infty).$$

(d)  $A = \mathbf{F}_2[x, y]/y^2 + y = x^5 + x^3 + 1$ . We have

$$f = \frac{(\bar{x} + x)(\bar{x}^4 + \bar{x}^3 + (1 + x)\bar{x}^2) + \bar{y} + y}{\bar{x}^3 + x\bar{x}^2 + (1 + x)\bar{x} + x^2 + x}.$$

Let  $\xi + 1, \underline{\xi}, \underline{\xi} + 1$  be the points where  $x$  is  $\bar{x} + 1, \bar{x}, \bar{x} + 1$  resp. and  $y$  is  $\bar{y} + \bar{x}^2, \bar{y} + 1, \bar{y} + \bar{x}^2 + 1$  resp. Then  $V = (\underline{\xi}) + (\underline{\xi} + 1)^{(1)}$  and

$$(J) = 2((\underline{\xi}) + 2(\underline{\xi} + 1)^{(-1)})^{(-1)} - ((\underline{\xi}) + 2(\underline{\xi} + 1)^{(-1)}) + (\underline{\xi}) - 4(\infty).$$

2.4 To understand the factorization of Gauss and Jacobi sums, we need  $V$ , which is obtained by solving the defining equation of  $f$  on the jacobian of  $X$ . In other words,  $V$  is  $-(F - 1)^{-1}(\underline{\xi})$ . (The point  $\infty$ , when  $\delta = 1$ , is used in the embedding of the curve in the jacobian.) Examples (a), (b) and (c) are of genus one. Hence the Jacobian is the elliptic curve, so in fact  $V = -(F - 1)^{-1}(\underline{\xi})$  can be verified directly with the results given above by computing it on the elliptic curve. (Note that the theta divisor is just a point in this case, in the description 1.4).

### 3 A gamma function

3.0 For this section, let  $M$  be the completion of an algebraic closure of  $K_\infty$ .

3.1 Let  $\rho$  be a Drinfeld  $A$ -module. Then it is known [D1] that  $e(z)$  is everywhere convergent power series on  $M$  and  $l(z)$  converges in a small disc around origin and represents the inverse function to  $e(z)$ . Moreover, the kernel of  $e(z)$  is  $\tilde{\pi}\mathcal{A}$  for some ('period')  $\tilde{\pi} \in M$  and  $\mathcal{A}$  an ideal of  $A$  and hence we have a product formula:  $e(z) = z \prod (1 + z/\tilde{\pi}a)$ , where  $a$  runs through all nonzero elements of  $\mathcal{A}$ . Conversely, given an ideal  $\mathcal{A}$  of  $A$ , there is  $\tilde{\pi} \in M$  such that  $e(z)$  defined by the product formula as above is the exponential of a Drinfeld  $A$ -module.

We have seen in 0.1.4, that  $e(z)$  and  $l(z)$  are  $\mathbb{F}_q$ -linear and hence the coefficients of  $z^k$  in their power series expansion vanish if  $k$  is not a power of  $q$ . We have the following theorem (see also 5.8) dealing with the case when  $k$  is a power of  $q$ .

**3.2. Theorem.** *The coefficients  $e_n$  of  $z^{q^n}$  in  $e(z)$  are never zero, in each of the following situations. (a)  $X$  has a closed point of degree one (for example, when  $\delta = 1$  or  $q$  is large compared to  $g$ , or when  $g < 2$ ). (b)  $g < 6$  and  $K$  is not  $\mathbb{F}_2(x, y)/y^4 + xy^3 + (x^2 + x)y^2 + (x^3 + 1)y + x^4 + x + 1 = 0$  (This  $K$  can be characterized [LMQ] as the unique class number one field of genus three and having no prime of degree two.)*

*Proof.* We will first establish two criteria for nonvanishing of  $e_n$ 's. For simplicity, first consider the situation where  $\delta = 1$ . By the easy part of Riemann-Roch, in the contributions to the coefficient of  $z^{q^n}$  coming from the product formula given in 3.1, there is a unique term of highest degree and hence the coefficient is nonzero. This argument also gives nonvanishing of various coefficients of  $e(z)$ , depending on the gaps in Riemann-Roch, when  $\delta > 1$ , but in general there are many terms of same highest degree cancelling the top degree, so the full result does not follow this way. But using Galois conjugation we can conclude nonvanishing of  $e_n$  for  $\rho$  corresponding to one ideal class from the corresponding statement for different ideal class. This leads to the first criterion:

(C1): If there is a  $\mathbb{F}_q$ -divisor  $D$  with  $h^0(D) = n$ , then  $e_n \neq 0$ .

The case (a) is settled by (C1) since we can twist the divisor by appropriate multiple of the degree one prime to get the desired dimension.

On the other hand, note that  $e_n$  is zero if and only if  $\xi^{(n)}$  belongs to the support of  $V$ , so a priori there are at most  $g$  (so none if  $g = 0$ ) exceptions  $n$  to the following hypothesis:

$$e_n \neq 0 \quad n \in \mathbf{N}. \tag{\mathcal{H}}$$

The second criterion is

(C2):  $e_n \neq 0$ , if  $A$  has a prime of degree dividing  $n$ .

: Since otherwise the specialization of the defining equation of  $f$  to  $\xi$  at a  $\bar{\mathbf{F}}_q$ -point above the prime contradicts 0.3.3. This together with the  $\delta = 1$  case gives a different proof for the case (a).

In particular, if we let  $n_0$  to be the smallest (it is easy to calculate  $n_0$ ) positive integer such that  $A$  has primes of all degrees dividing  $n > n_0$ , then  $e_n \neq 0$  for  $n > n_0$ . Since there is always a  $\mathbf{F}_q$ -divisor of any prescribed degree,  $h^0 = n$  can be achieved, by Riemann-Roch, if  $n \geq g$ . Hence by (C1),  $e_n \neq 0$ , if  $n \geq g$ . We can also see this by Weil bounds and (C2), unless  $q = 2$  and  $g \leq 6$ , for then  $n_0 \leq g$ .

Let  $n$  be an exception to  $\mathcal{H}$ . By above,  $1 \leq n < g$ . Now  $A$  contains constants and hence  $h^0 = 1$  is possible, so  $n \neq 1$ . For  $D$  of degree  $2g - 2$ ,  $h^0(D) = g - 1 + h^0(\mathcal{X} - D)$ , where  $\mathcal{X}$  is the canonical divisor. If  $h > 1$ , we can choose  $\mathcal{X} - D$  nontrivial and get  $h^0 = g - 1$  showing  $n \neq g - 1$  in that case. But by [LMQ],  $g \geq 4$  implies  $h > 1$  and there are only two class number one, genus 3 fields. Among them the one not listed in (b) has a prime of degree two and (C2) applies. We have shown  $2 \leq n \leq g - 2$ , except possibly for  $K$  listed in (b). For  $D$  of degree  $2g - 3$ , we have  $h^0(D) = g - 2 + h^0(\mathcal{X} - D)$  and  $\mathcal{X} - D$  being of degree one, if its  $h^0 > 0$ , then we have a closed point of degree one and there are no exceptions. So in any case,  $2 \leq n \leq g - 3$ . Consider now  $g = 5$ . Only possibility to be ruled out is  $n = 2$ . For  $D$  of degree 2,  $h^0(D) = h^0(\mathcal{X} - D) - 2$ . If it is zero, we apply (C1); otherwise we apply (C2) to see that  $n \neq 2$ . This finishes the proof of the theorem.

3.3 We have proved that there are at most  $g - 4$  exceptions (except possibly for the  $K$  of the theorem) and it is easy to extend the last part of the proof to show that there are at most  $g - 5$ . The Weil bounds together with (C2) give much better asymptotic bound of the order  $2 \log_q 2g$ . We have more evidence for  $\mathcal{H}$  than that listed in the theorem and it is possible that  $\mathcal{H}$  is always satisfied. But as we can not settle it in full generality,

*We will assume  $\mathcal{H}$  below. See 5.7 for more remarks.*

3.4 We can then write  $e(z) = \sum z^{q^i}/d_i$ . By 0.1.3,  $d_i$  lie in  $H_1$ . For  $n \in \mathbf{N}$ , we define the factorial  $\Pi(n)$  of  $n$  as follows: Write  $n = \sum n_i q^i$ ,  $0 \leq n_i < q$  and put  $\Pi(n) := \prod d_i^{n_i}$ . One motivation for this definition, at least when  $n = q^i$ , is that classically the factorials are given by the reciprocals of the Taylor coefficients of the exponential. A motivation for the definition for arbitrary  $n$  and for the following  $\wp$ -adic interpolation is given in [T4]. (See also 3.8).

3.5 Let  $\wp$  be a prime of  $A$  of degree  $d$  and let  $\theta$  be an  $\bar{\mathbf{F}}_q$ -point of  $\bar{X}$  above  $\wp$ . If  $w \in K_\wp$  is as local parameter at  $\theta$ , we put

$$\tilde{d}_i := \tilde{d}_{i,w} := \frac{d_i}{d_{i-d} w^{l_i}}$$

where  $l_i$  is chosen so that  $\tilde{d}_i$  is a unit at  $\theta$ .

**3.6. Theorem.** *If  $u$  is any local parameter at  $\theta$ , there is a nonzero  $c \in \mathbf{F}_{q^a}$  such that with  $w = cu$ ,  $\tilde{d}_i$  tends to one  $\theta$ -adically as  $i$  tends to infinity.*

*Proof.* Let us write  $f_i := f|_{\xi^{i\theta}}$ . Then by 0.3.5, we have  $d_i/d_{i-1}^q = f_i$  and hence  $y_i := \tilde{d}_i/\tilde{d}_{i-1}^q = f_i/f_{i-d}w^{e_i}$  where  $e_i$  is such that  $y_i$  is a unit at  $\theta$ . Now as  $i$  tends to infinity,  $f_i$  tends  $f|_{\theta^{i\theta}} = f|_{\theta^{i-d\theta}} \neq 0$ . Hence  $y_i$  tends to one as  $i$  tends to infinity. Now if  $c$  is chosen, so that for some sufficiently large  $i_0$ ,  $\tilde{d}_{i_0}$  is a one-unit, then this shows that  $\tilde{d}_i$  is a one unit for  $i \geq i_0$ . But since  $\tilde{d}_i$  are one units and  $y_i$  tends to one,  $\tilde{d}_i$  tends to one (because  $q$  power map spreads the series expansions in characteristic  $p$ ) as claimed.

3.7 Let  $w$  be a local parameter at  $\theta$  such that  $\tilde{d}_i$  is a one unit for large  $i$ , then 3.6 implies that  $\tilde{d}_i$  tends to one. Define  $\wp$ -adic factorial  $\Pi_{\wp}(z) := \Pi_w(z)$  for  $z \in \mathbf{Z}_p$  as follows. Write  $z = \sum z_i q^i$ ,  $0 \leq z_i < q$  and put  $\Pi_{\wp}(z) := \prod \tilde{d}_i^{z_i}$ .

3.8 In the situation of 0.1.5 where  $A = \mathbf{F}_q[T]$ , Carlitz [C1] showed that  $d_i$  is the product of monic elements of  $A$  of degree  $i$ . If we choose  $w$  to be a monic prime  $\wp$  of  $A$ , then it is easily seen that  $\tilde{d}_i$  is then the product of monic elements of  $A$ , prime to  $\wp$ , of degree  $i$ . Goss [G1] showed that in this case  $-\tilde{d}_i \rightarrow 1$ . (We give another proof of this in 3.9). This can be reformulated as  $\tilde{d}_i \rightarrow 1$ , if  $w = -\wp$  and hence is compatible with the theorem. When  $h_A = 1$  we do not have the product interpretation for  $d_i$  or  $\tilde{d}_i$  but we can always take  $w = \wp$  a monic prime of  $A$ .

**3.9. Theorem.** *When  $h_A = 1$ , if we choose  $w$  to be a monic prime  $\wp$  of  $A$  of degree  $d$ , then  $-\tilde{d}_i \rightarrow 1$  as  $i$  tends to infinity*

*Proof.* Let us put  $f_i := f|_{\xi^{i\theta}}$ . Then by 0.3.6, we have

$$d_i/d_{i-d} = (f_i f_{i-1}^q \cdots f_{i-d+1}^{q^{d-1}})(f_{i-d} \cdots f_1^{q^{i-d-1}})^{q^d-1}.$$

Hence the  $\theta$ -adic sign of  $d_i/d_{i-d}$  is the same as that of the first bracket specialized at  $\theta$ . But by Theorem 1.2 it is then the  $\theta$ -adic sign of  $g_i^{q^d-1}$ . Now  $g_0^{q^d-1} \equiv -\wp \pmod{\wp^2}$  can be seen, when  $h_A = 1$  exactly as in the Lemma II and Theorem III of [T2]. Since  $(-1)^{q^d} = -1$ , this finishes the proof.

**4 Analogue of the Gross-Koblitz theorem**

4.0 For this section, let  $M$  be an algebraic closure of  $K_{\wp}$ ,  $\xi$  be the tautological point i.e. the  $M$ -valued point of  $X$  corresponding to  $K \subset K_{\wp}$  and let  $\theta$  be the Teichmuller representative in the residue disc of  $\xi$ . Note that even though we used the  $\infty$ -adic completion in the Sect. 3, the Taylor coefficients of  $e(z)$ , being in  $H_1$  by 0.1.3, can be thought of as elements of  $M$ .

**4.1. Theorem.** *Let  $0 \leq j < d$ . If  $\mu$  is the valuation of  $g_j$  at  $\xi$ , then we have  $g_j = \zeta w^{\mu}/\Pi_{\wp}(q^j/(1 - q^d))$ , where  $\zeta$  is a  $q^d - 1$ -th root of unity.*

*Proof.* (Compare with [T2]). For the purposes of this proof,  $\mu_i$  will be suitable integers. Let

$$M_j := \Pi_{\wp}(q^j/(1 - q^d)) = \lim_{m \rightarrow \infty} \tilde{d}_j \tilde{d}_{j+d} \tilde{d}_{j+2d} \cdots \tilde{d}_{j+md}.$$

Call the quantity after the limit sign  $T_m$ , so that  $M_j = \lim T_m$ . Then

$$T_m = \frac{d_j \cdots d_{j+md}}{d_j \cdots d_{j+md-d}} w^{\mu_1} = d_{j+md} w^{\mu_1}.$$

Since  $d_i = f_i d_{i-1}^q$ , we have

$$T_m = f_{j+md} f_{j+md-1}^q \cdots f_{j+1+(m-1)d}^{q^{d-1}} T_{m-1}^{q^d} w^{\mu_2}.$$

But as  $m$  tends to infinity, by Theorem 1.2,  $f_{l+md} \rightarrow f|_{\theta^{(d)}} = J_l$ . Hence

$$M_j^{1-q^d} = J_j J_{j-1}^q \cdots J_{j-d+1}^{q^{d-1}} w^{\mu_3} = g_j^{q^d-1} w^{\mu_3}$$

as claimed.

4.2. *Remark.* Unlike the classical [G-K] or the  $\mathbb{F}_q[T]$  case [T2], this theorem is weaker in that it does not tell us what the valuation  $\mu$  or the root of unity is. When  $h_A = 1$ , both can be specified using 3.9 and the results of [T3]. But unlike the two cases above, as the results of [T3] or Sect. 2 show,  $\mu$  can depend on  $\wp$ , not only through its degree. For example, by (c) of Sect. 2 or equivalently by [T3, Theorem I, (3)], in that situation, the valuation at  $\wp$  of  $(g_0 g_1 g_2)^2$  is 3 or 1 according as whether  $\wp$  is  $y$  or  $y + x$ . In general, the dependence of  $\mu$  on  $\wp$  seems complicated and deserves further investigation.

4.3 Let  $\delta = 1$  and let  $D_i$  be the product of monic elements of  $\mathcal{A}$  of degree  $i$ . The gamma function of [G1, T1, T4] and its interpolations are similar to ones we are considering, but with  $D_i$  replacing  $d_i$ . Greg Anderson (to appear) has constructed a meromorphic function  $\Delta$  on the Hilbert cover of  $X$  cross itself whose specializations at  $\bar{\mathbb{F}}_q$ -points are ‘Jacobi sums’ corresponding to Brumer-Stark units of [H2] and at  $\xi^{(i)}$  are  $D_{i+g}/D_{i+g-1}^q$ . This implies an analogue of the Gross-Koblitz, with Brumer-Stark units in place of Gauss sums of this paper, by exactly similar proof. As we have already noted in 3.8,  $f = \Delta$  for  $g = 0$  and  $\delta = 1$  and the proof already appears in [T1, T2]. When  $g = \delta = 1$ , Theorems IV, V, XI of [T5] give a formula for  $\Delta$  by another method.

*Example.* In the notation of 2.3, for the example (c) there, we have

$$\Delta = \frac{((x - \bar{x})^2 + x - \bar{x} + 1 - y\bar{y})(y^3 - \bar{y} - y^3(x^3 - \bar{x}))}{(x^3 - \bar{x} + 1)(y^3 \bar{y} - 1 - (x^3 - \bar{x})^2 - x^3 + \bar{x})}.$$

It is easy to verify directly from this formula that the divisor of  $\Delta|_{\theta}$  is  $(\theta^{(-1)} + 3(\theta^{(1)} - 3(\theta) - (\infty)))$ . A straightforward calculation with partial zeta functions shows that this divisor agrees with the divisor corresponding to ‘Jacobi sum’ associated to the Stickelberger elements.

### 5 Interpolation at $\infty$

5.0 For this section, let  $M$  be an algebraic closure of  $K_\infty$ ,  $\xi$  be the tautological point i.e. the  $M$ -valued point of  $X$  corresponding to  $K \subset K_\infty$  and let  $\bar{\infty}$  be the Teichmuller representative in the residue disc of  $\xi$ . For a nonzero  $b \in \mathcal{O}_{\bar{x}, \bar{\infty}}$ ,  $\bar{b} := b/t^{\text{val } b}$  be the unit part of  $b$  with respect to  $t$ .

**5.1. Theorem.** *There exists a nonzero  $c \in \mathbb{F}_q$  such that as  $i$  tends to infinity,  $c^{q^i} \bar{d}_i$  tends to one,  $\infty$ -adically*

*Proof.* We have  $f_i := d_i/d_{i-1}^q = f|_{\mathcal{E}^{(i)}}$ . By 0.3.3 and 0.3.4  $f$  has a pole at  $\infty$  of order say  $m > 0$  (in fact,  $m = 1$  by 0.3.3, if  $\delta = 1$  and by 5.6 in general). This together with the fact that  $\text{sgn}(f) = 1$  implies that for given  $N$ , for sufficiently large  $i$ ,  $d_i/d_{i-1}^q = t^{mq^i} +$  terms of degree less than  $mq^i - N$  in  $t$ . This in turn implies that  $f_i$  tends to one, as  $i$  tends to infinity. Now, if  $c$  is chosen so that for some sufficiently large  $i_0$ , the unit  $c^{q^{i_0}} \bar{d}_{i_0}$  is in fact a one-unit, then the fact that the  $q$ -power spreads the series expansion in characteristic  $p$ , together with  $f_i \rightarrow 1$  prove the claim.

5.2 We first interpolate the one-unit part as follows. For  $z \in \mathbb{Z}_p$ , we define  $\infty$ -adic factorial by

$$\bar{\Pi}_\infty(z) := \bar{\Pi}_t(z) := \prod (c^{q^i} \bar{d}_i)^{z_i}$$

where as usual  $z = \sum z_i q^i$ , with  $0 \leq z_i < q$ .

**5.3 Theorem.** *As  $i$  tends to infinity, the degree of  $d_i$  tends to zero,  $p$ -adically.*

*Proof.* This follows from considering the degrees of both sides in the expression for  $d_i/d_{i-1}^q$  in 5.1 and induction on  $i$ .

5.4. *Remark.* Using the Theorem 5.3, we can define  $\bar{\Pi}_\infty$  by ‘putting back the degree part’. For details we refer to [T4].

5.5 We now give different analytic proofs for Theorem 5.1 and 5.3, when  $\delta = 1$ , using the product formula as in the proof of 3.2.

When  $\delta = 1$ , by the product formula, for large  $i$ , the degree and the sign of  $d_i$  is the same as that of  $\tilde{\pi}^{q^i-1}$  times the product of all nonzero elements of  $\mathcal{A}$  (there are  $q^i - 1$  of these) of degree not more than  $i + w$  with  $w = g - 1 + \text{deg } \mathcal{A}$  by Riemann-Roch. By Theorem 3.10 of [T4], the  $\tilde{\pi}$  power contributes the sign  $(-1)^{(q^i-1)/(q-1)} = (-1)^i$ . And as the product of all nonzero elements of  $\mathbb{F}_q$  is  $-1$ , the contribution from the product of the  $q^i - 1$  elements is also  $(-1)^{(q^i-1)/(q-1)} = (-1)^i$ , making the total contribution 1. Hence the sign is 1. Now we give a sketch of another proof of the fact that the gap between the powers of two top degrees of  $t$  in the Laurent expansion of  $d_i$  tends to infinity, so that  $\bar{d}_i \rightarrow 1$ : Let us put  $D_i$  to be the product of all monic elements of  $\mathcal{A}$  of degree  $i$ , then the term producing the top degree is  $\tilde{\pi}^{-q^i+1}/(D_0 \cdots D_{i+w})^{q-1}$  up to a sign, hence by the Theorem 3.10 of [T4], the one-unit part of this tends to one. Hence it remains to show that the difference of degrees between those of  $1/(D_0 \cdots D_{i+w})^{q-1}$  and sum of the reciprocals of all other possible products of  $q^i - 1$  distinct elements of  $\mathcal{A}$  tends to infinity with  $i$ . We group the terms in the sum by taking one element of the largest degree (say  $i + w + r$ , with  $r > 0$ ) of fixed sign as variable and keeping the other  $q^i - 2$  elements fixed. Then the gap is clearly at least  $-\text{deg}(\sum 1/a) + i + w + r$  where this sum is over monic elements of degree  $i + w + r$ . Now this tends to infinity as claimed, by the formulae (15) and (17) of [T5] for  $\sum 1/a$  and Riemann-Roch. But in doing this, we have included some terms involving products with repetitions, this leads to consideration of  $\sum 1/a^2$  etc. which also can be shown to have sufficiently small degrees by [T5]. This shows that the Theorem 5.1 holds with  $c = 1$ , when  $\delta = 1$ .

Now again by Theorem 3.10 of [T4], when  $i$  tends to infinity, the degree of the product tends  $p$ -adically to the degree of  $\tilde{\pi}$  and hence the total degree tends to zero  $p$ -adically. This gives another proof of the Theorem 5.3, when  $\delta = 1$ .

5.6 For any  $\delta$ , if  $q^i$  is the number of elements of  $\mathcal{A}$  of degree less than or equal to  $m$  say, then, just as in 5.5, there being unique contribution of the top degree, we can show that  $e_i \neq 0$  and calculate the degree and sign of  $d_i$  in a similar fashion to 5.5. For example, let  $n$  be sufficiently large and  $i = n\delta - w$ . Then by the argument above  $\deg(d_i)$  is, up to addition of a contribution independent of  $n$ ,  $(q^\delta - 1) \sum_{j=1}^n j\delta q^{j\delta - w - \delta} + \deg(\tilde{\pi}^{q^i - 1})$ . Hence the degree of  $d_i/d_{i-\delta}^{q^\delta}$  is up to addition of a contribution independent of  $n$  (in fact from what we show, this contribution is zero, but we do not need this for the argument) equal to  $\delta q^i$ . This is compatible with the formula for  $d_i/d_{i-1}^{q^\delta}$  in the proof of Theorem 5.1, exactly when  $m = 1$ . In other words,  $f$  has a simple pole at  $\infty$ . Together with 0.3.3, this means that  $\infty$  is not in the support of  $V$ . This argument also shows how to find a formula for  $c$ , in terms of  $\text{sgn}(\tilde{\pi})$ .

5.7. *Remark.* In case,  $\mathcal{H}$  is not satisfied, we can define the gamma function and interpolations on smaller subsets of  $\mathbf{Z}_p$ , so as to avoid the  $j$ -th digit in  $q$ -base expansion, whenever  $e_j$  is zero. We know that there are at most  $g$  such  $j$ 's. With this (awkward) restriction, the situation is then similar, even the Gross-Koblitz formula holds, for large  $d$  and for  $j$  nonexceptional. Another (equally awkward) alternative is to define, say  $d_i = 1$ , if  $e_i = 0$ . Then Gamma function is everywhere defined, but Gross-Koblitz statement has to be corrected by a correction factor in  $H_1$ .

5.8. **Theorem.** *The coefficients  $l_n$  of  $z^{qn}$  are never zero.*

*Proof.* By Galois conjugation, it is sufficient to prove the theorem in the case when the lattice corresponding to  $\rho$  is  $\tilde{\pi}A$ , i.e.  $\rho$  corresponds to the principal ideal class. Write  $z/e(z) = \sum \zeta_j z^j$ . Applying Theorem 8.1 of [C1] to  $f = e$ , we get  $l_i = \zeta_{q^i - 1}$ . Taking the logarithmic derivative of the product formula for  $e$ , this is in turn equal to  $-\tilde{\pi}^{1-q^i} \sum 1/a^{q^i - 1}$ , where the sum is over nonzero elements of  $A$ . The sum does not vanish, because the top degree contribution to it is the sum over  $a \in \mathbf{F}_q^*$  and equals  $q - 1 = -1 \neq 0$ . This proves that  $l_i \neq 0$ .

5.9. *Remark.* If we let  $(\omega) = \bar{V} + V - 2(\infty)$ , with  $\bar{V}$  being an effective divisor of degree  $g$ , then by 0.3.7 and 0.3.8,  $l_n = 0$  if and only if  $\xi \in V + \bar{V}^{(n+1)}$ , which by 0.3.3 is equivalent to  $\xi \in \bar{V}^{(n+1)}$ . So there are at most  $g$  exceptional  $n$ 's (none, if  $g = 0$ ). The fact that  $\xi$  does not belong to the support of  $\bar{V}^{(1)}$  follows exactly as in 0.3.1 and 0.3.3, by replacing  $V$  by  $\bar{V}^{(1)}$  and  $q^i$ -th powers by  $q^i$ -th roots. Hence by copying the proof of (C2) of 3.2, we obtain another proof of the fact that  $l_n \neq 0$ , under the hypothesis that  $X$  has a finite closed point of degree one. In fact, we thus see a stronger result that  $\xi$  does not belong to  $\bar{V}^{(n)}$ , for  $n \in \mathbf{Z}$ . This result is not true in general without the hypothesis as can be seen from (b) of Sect. 2.

When  $\xi$  is a transcendental point, the fact that  $\xi$  does not belong to the supports of  $V$  and  $\bar{V}^{(1)}$  also follows from 0.3.6 and 0.3.8 together with the analytic theory, which gives the existence of the exponential and the logarithm.

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