# POWER SUMS OF POLYNOMIALS OVER FINITE FIELDS AND APPLICATIONS: A SURVEY

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Abstract. In this brief expository survey, we explain some results and conjectures on various aspects of the study of the sums of integral powers of monic polynomials of a given degree over a finite field. The aspects include non-vanishing criteria, formulas and bounds for degree and valuation at finite primes, explicit formulas of various kind for the sums themselves, factorizations of such sums, generating functions for them, relations between them, special type of interpolations of the sums by algebraic functions, and the resulting connections between the motives constructed from them and the zeta and multizeta special values. We mention several applications to the function field arithmetic.

## 1. Introduction

In this expository survey, we explain some results, conjectures and applications of various aspects of the study of the sums of integral powers of monic polynomials of a given degree over a finite field.

The aspects include non-vanishing criteria, formulas and bounds for the degree and the valuation at finite primes for these sums, explicit formulas of various kind for the sums themselves, factorizations of such sums, generating functions for them, relations between them, special type of interpolations (Anderson's 'solitons' [T04, T01]) of the sums by algebraic functions, and the resulting connections between the motives constructed from them and the zeta and multizeta special values.

The combinatorics of cancellations can be very complicated, especially for non-prime finite fields. Thus there are many open questions and we mention several observations, guesses and conjectures.

After this, we mention and refer to several applications to several areas of function field arithmetic [G96, R02, T04] (there are, of course, many other applications which we do not cover). These include evaluations of and relations between the special values of zeta and multizeta, understanding of the zero distribution of Riemann hypothesis type for the Goss zeta function, both at infinite and finite primes, study of variation in the $p$-ranks of Jacobians of cyclotomic coverings.

We do not give proofs (sometimes just mentioning the key idea) and give only the convenient references, where the details can be found. In describing the applications, we sometimes do not give the definitions of all the objects, referring to the cited papers for them and hoping that the description would still help the readers with general background on number theory.

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## 2. Power sums

**2.1. Notation.**

$$
\begin{aligned}
\mathbb{Z} &= \{\text{integers}\}\\
\mathbb{Z}_{+} &= \{\text{positive integers}\}\\
\mathbb{Z}_{\geq 0} &= \{\text{nonnegative integers}\}\\
q &= \text{a power of a prime } p,\ q = p^f\\
A &= \mathbb{F}_q[t]\\
A+ &= \{\text{monics in } A\}\\
A_d+ &= \{\text{monics in } A \text{ of degree } d\}\\
A_{<d}+ &= \{\text{monics in } A \text{ of degree less than } d\}\\
K &= \mathbb{F}_q(t)\\
K_\infty &= \mathbb{F}_q((1/t)) = \text{completion of } K \text{ at } \infty\\
C_\infty &= \text{the completion of an algebraic closure of } K_\infty\\
[n] &= t^{q^n} - t\\
d_n &= \textstyle\prod_{i=0}^{n-1}(t^{q^n} - t^{q^i})\\
\ell_n &= \textstyle\prod_{i=1}^{n}(t - t^{q^i})\\
\ell(k) &= \text{sum of the digits of the base } q \text{ expansion of } |k|\\
\deg &= \text{function assigning to } a \in A \text{ its degree in } t,\ \deg(0) = -\infty
\end{aligned}
$$

Let $v$ be a finite prime of $A$ and let $\mathrm{val}_v$ and $\mathrm{val}_\infty$ denote the usual normalized valuations at places $v$ and $\infty$ respectively.

**2.2. Definition of basic power sums.** Let $k, k_i \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$. We define basic, cumulative, v-adic and iterative power sums:

$$
\begin{aligned}
S_d(k) &:= \sum_{a \in A_d+} \frac{1}{a^k} \in K,\\
S_{<d}(k) &:= \sum_{a \in A_{<d}+} \frac{1}{a^k} \in K,\\
S_{d,v}(k) &:- \sum_{\substack{a \in A_{d+}\\ (v,a)=1}} \frac{1}{a^k} \in K,\\
S_d(k_1, \cdots, k_r) &:= S_d(k_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(k_2) \cdots S_{d_r}(k_r) \in K.
\end{aligned}
$$

We also consider degrees and valuations

$$s_d(k) := \mathrm{val}_\infty(S_d(k)) = -\deg S_d(k), \quad v_d(k) := \mathrm{val}_v(S_{d,v}(k)).$$

Note that since we are in characteristic $p$, and because of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ invariance, all these power sums get raised to $p^n$-th power, and thus valuations get multiplied by $p^n$, when we multiply all relevant $k$'s and $k_i$'s by $p^n$.

Since $\sum \theta^{-k}$, where the sum is over $\theta \in \mathbb{F}_q^*$, is $-1$ or 0 according as $k$ is 'even' (i.e., divisible by $q-1$) or not, the variants of the first three sums obtained by dropping 'monic' condition differ just in sign, in the 'even' $k$ case, and vanish in the second.

We will first focus on the basic power sums $S_d(k)$.

## 3. Vanishing criteria

**Theorem 1.** *(1) If $k > 0$, then $S_d(k)$ does not vanish.*

*(2) If $k \leq 0$, then $S_d(k) = 0$, if and only if $d > \ell(kp^i)/(q-1)$, for some $0 \leq i < f$, where $q = p^f$.*

*(3) If $k \leq 0$, then $S_d(k) \neq 0$, if and only if $-k = m_0 + m_1 + \cdots + m_d$, with $m_0 \geq 0$, and for $1 \leq j \leq d$, $q-1|m_j, m_j > 0$, and the sum is such that there is no carry over of digits base $p$.*

This theorem is essentially due to Carlitz, Lee and Sheats, with another cohomological proof of part (2) by Boeckle and Pink. See [C35, C39, Le43, S98], for original papers as well as remarks [T04, Thm. 5.1.2, Sec. 5.6, 5.8], [T09b, A. 5] on the complicated history of claims with incomplete proofs.

The vanishing results hold in very general setting of many variables, or even vector and affine spaces over finite fields [T04, Thm. 5.1.2], but the non-vanishing is more subtle.

## 4. Degrees of power sums

While the degree of the term $1/a^k$, being just $-dk$, has extremely simple behavior, the degree we are interested in behaves quite erratically, because of complicated cancellation combinatorics in finite fields, especially non-prime finite fields.

Let $k$ be positive for this paragraph. While $-dk$ is clearly strictly decreasing function of $d$ or $k$, the degree of $S_d(k)$ is not a strictly decreasing as a function of $k$. The statement that it is a strictly decreasing function of $d$ is a subtle claim which implies non-vanishing (Sec. 16) of all the multizeta values. As $d$ jumps by one, $-dk$ changes by $k$. But again the claim that the jumps in the degree of $S_d(k)$ strictly increase with $d$ easily implies the Riemann hypothesis analog for the Goss zeta function (Sec. 15), as we explain below.

We now give some results from [T09b]. As we will see, the situation with $q$ non-prime is much more complicated. Note $S_0(k) = 1$ and $s_0(k) = 0$, for $k \in \mathbb{Z}$. First we record direct or recursive formulas, and a kind of interesting duality (part (5)) between values for positive and negative $k$'s.

**Theorem 2.** *(1) Let $q$ be a prime. Let $k \in \mathbb{Z}$ and $d > 0$. Then we have following recursive formula in $d$.*

$$s_d(k) = s_{d-1}(s_1(k)) + s_1(k) = \sum_{i=1}^{d} s_1^{(i)}(k),$$

*where both sides are either finite or both infinite.*

*(2) For general $q$, and $k > 0$, we have $s_1(k+1) = k + q$, if $q$ divides $k$. Let $q = p$ be a prime. If $q$ does not divide $k$, then $s_1(k+1) = s_1(k) + q^{ord_q(s_1(k))}$.*

*(3) We have, when $q = 2$, $k > 0$,*

$$s_d(k+1) = 2^{d+1} - 2^d + \sum (min(e_i - i + 1, d) - 2)2^{e_i} + 2^{max(e_i+1, d+i)},$$

*where the notation now is $k = \sum_{i=1}^{\ell(k)} 2^{e_i}$, with $0 \leq e_1 < e_2 < \cdots$.*

*(4) More generally, let $q = p$ be prime, $k > 0$. Write $k = \sum_{i=1}^{m} q^{e_i}$, with $e_i$ monotonically increasing with $i$ and no more than $q-1$ consecutive values being the same. Put*

$$e_i' = e_i - \lfloor \frac{i-1}{q-1} \rfloor.$$

*Then*

$$s_d(k+1) = \frac{q^{d+1}-q}{q-1} + \sum (min(e_i', d) + \frac{q}{q-1}(q^{max(d-e_i',0)}-1))q^{e_i}.$$

*(5) Let $k>0$, $q$ be a prime power, then we have*

$$|s_d(-k)| + |s_d(q^n-k)| = dq^n, \text{ for } q^n > k,$$

*if $s_d(-k)$ is finite. In particular, under these conditions we have $s_d(q^{n+m}-k) - s_d(q^n-k) = d(q^{n+m}-q^n)$.*

The part (1) reduces the complexities of $s_d$ to $s_1$, which is partly described in (2), but for more on its complexities, see [T09b, Appendix]

The part (1), for $k>0$, would also follow from the following guess [T09b, Pa. 538]:

When $q$ is a prime, 'min $s_{d-1}(s_1(k)+j)+s_1(k)+j$, where the minimum is taken over $j \geq 0$, with $(q-1)|j$ and $\binom{s_1(k)+j-1}{k-1}$ nonzero modulo $p$, is unique and occurs at $j=0$'

Parts (3) and (4) were first found by Greg Anderson by computer experimentation. The part (5) describes interesting relation between degrees for the positive versus negative powers.

Next we have results on degree inequalities and divisibilities.

**Theorem 3.** *(1) If $q=2$ and $k>0$, we have*

$$dk + 2(2^d-1) - d \leq s_d(k) \leq 2(2^d-1)k,$$

*with each inequality being an equality for infinitely many $k$'s.*

*(2) We have $s_d(k) \equiv dk \mod (q-1)$, for $d \geq 0$ and $k>0$, and further $q$ divides $s_d(k)$ if $q$ is a prime.*

*(3) We have $s_1^{(i)}(k)$ is divisible by $q^i$ and $s_d(k)-s_{d-1}(k)$ divisible by $q^d$, when $q$ is a prime.*

*(4) If $k<0$, then $s_d(k) \equiv dk \mod (q-1)$. Assume $k<0$, $q$ is prime and $s_d(k)$ is finite, then (i) $q$ divides $s_d(k)$, (ii) $|s_{d+1}(k)| \geq |s_d(k)|$, and (iii)*

$$(q-1)\sum_{i=1}^{d-1}(d-i)q^{d-i} \leq |s_d(k)| \leq |dk| - q(q^d-1)/(q-1) - d.$$

For more general results, implications on conjectures on degree bounds etc. see [T09b].

The numerical data suggests [T09b, H3] the guess that $s_d(k) < s_d(k+1)$ when $k$ is not divisible by $p$.

## 5. Generating functions

We start with a general set-up [LTp3] and then specialize. Let $F$ be a field containing $\mathbb{F}_q$, and let $f(z) = \sum f_i z^{q^i} \in F[[z]]$ be a non-constant power series. We consider

$$h(z) = zf'(z)/f(z), \qquad a(z) = zf'(z)/(1-f(z))$$

and write $h(z) = \sum h_i z^i \in F[[z]]$, $a(z) = \sum a_i z^i$.

In the special case where $f(z)$ is a polynomial, writing $u = 1/z$, we consider power series expansions in $u$ of $h$ and $a$, normalising as follows. We write $-h(z) = \sum H_i u^i$, $-a(z) = \sum A_i u^i$.

We have

$$h_i = H_i = 0 \quad \text{if} \quad i \not\equiv 0 \bmod (q-1).$$

Also, if $f$ is of degree $q^d$, then

$$H_i = A_i = 0 \;\; \text{if} \quad i < q^d - 1.$$

The connection with our power sums comes via the choice of $f$ to be the Carlitz's analog of binomial coefficient:

$$f(z) := \binom{z}{q^d}_c := \sum_{i=0}^{d} \frac{z^{q^i}}{d_i \ell_{d-i}^{q^i}} = \frac{1}{D_d} \prod_{a \in A, \deg a < d} (z-a),$$

giving

$$h_k = S_{<d}(k), \quad H_k = S_{<d}(-k), \quad a_k = S_d(k), \quad A_k = S_d(-k),$$

where the first two equalities hold for positive $k$ 'even' and the last two for any $k \geq 1$.

## 6. Multiplicative relations

In full generality of the set-up above, we have the following multiplicative relations [LTp3] between the coefficients of these logarithmic derivatives:

**Theorem 4.** *(1) For $1 \leq s \leq q$, and $0 \leq k_j \leq k$, with $1 \leq j \leq s$, we have*

$$\prod_{j=1}^{s} h_{q^k - q^{k_j}} = h_{\sum (q^k - q^{k_j})}.$$

*(2) For $1 \leq s \leq q$ and $k_i \geq 1$, with $1 \leq i \leq s$, we have*

$$\prod_{i=1}^{s} H_{q^{k_i}-1} = H_{q^{k_1}+\cdots+q^{k_s}-s}.$$

*(3) For $1 \leq s < q$ and $0 \leq k_i < k$, with $1 \leq i \leq s$, we have*

$$\prod_{i=1}^{s} a_{q^k - q^{k_i}} = f_0^{(s-1)q^k} a_{q^k - \sum q^{k_i}}.$$

*(4) For $1 \leq s < q$ and $k_i \geq 0$, with $1 \leq i \leq s$, we have*

$$\prod_{j=1}^{s} A_{q^{k_j}-1} = f_0^{s-1} A_{q^{k_1}+\cdots+q^{k_s}-1}.$$

We will give applications of these relations in power sums case later, but we note that these power sums relations and proofs of all the parts follow from specialization/generalization arguments via the following very general theorem and from direct proofs of some of the parts.

**Theorem 5.** *Let $F$ be a field containing $\mathbb{F}_q$. (1) Let $b_1, \ldots, b_d \in F$ be $\mathbb{F}_q$-linearly independent and let $B_{ij} \in F$, for $i = 1$ to $d$ and $j = 1$ to $s \leq q$. Then*

$$\prod_{j=1}^{s} \sum_{(\theta_1,\ldots,\theta_d) \in \mathbb{F}_q^d - \{0\}} \frac{\sum_i \theta_i B_{ij}}{\sum_i \theta_i b_i} = (-1)^{s-1} \sum_{(\theta_1,\ldots,\theta_d) \in \mathbb{F}_q^d - \{0\}} \frac{\prod_j (\sum_i \theta_i B_{ij})}{(\sum_i \theta_i b_i)^s}.$$

*(2) Let $M_j, \mu, b_1, \dots, b_d, B_{ij} \in F$ ($i = 1$ to $d$ and $j = 1$ to $s < q$). Then (assuming no zeros in the denominators, or in other words, make common denominators and look at the polynomial identity for numerators)*

$$\prod_{j=1}^{s} \sum_{(\theta_1,\dots,\theta_d)\in\mathbb{F}_q^d} \frac{\sum_i (M_j+\theta_i B_{ij})}{\sum_i(\mu+\theta_i b_i)} = (\sum_{(\theta_1,\dots,\theta_d)\in\mathbb{F}_q^d} \frac{1}{\sum_i(\mu+\theta_i b_i)})^{s-1} \sum_{(\theta_1,\dots,\theta_d)\in\mathbb{F}_q^d} \frac{\prod_j(\sum_i(M_j+\theta_i B_{ij}))}{\sum_i(\mu+\theta_i b_i)}.$$

It would be very interesting to get a direct proof of this theorem which uses only finite field cancellation combinatorics.

## 7. Explicit formulas and factorizations

It is interesting that the values at $k = \pm(q^i - 1)$ coming up in the last section are the values where the power sums have clean explicit formulas from which their (numerators and denominators, if any) factorizations can be easily read off.

**Theorem 6.** [C35, C39, Le43, LTp3] *We have,*

$$S_{<d}(q^i-1) = \frac{\ell_{d+i-1}}{\ell_i \ell_{d-1}^{q^i}}, \qquad S_d(q^i-1) = \frac{\ell_{d+i-1}}{\ell_{i-1}\ell_d^{q^i}},$$

*and for $i \geq d$,*

$$S_{<d}(-(q^i-1)) = \frac{d_{i-1}^q}{\ell_{d-1} d_{i-d}^{q^d}}, \qquad S_d(-(q^i-1)) = \frac{d_i}{\ell_d d_{i-d}^{q^d}}.$$

**Theorem 7.** [Ge88] *Let $q^d - 1 \leq k < q^{d+1}-1$ , and $k = \sum k_j\, q^j$ be the base $q$ expansion (i.e., $0 \leq k_j < q$). Then*

$$S_d(-k) = (-1)^{\gamma} \cdot M \cdot \prod_{j \leq d} (L_{d-j}^{q^j(k_j-(q-1))} d_j^{k_j}),$$

*where $\gamma := d + \sum_{j<d}(d-j+1)k_j$ and $M$ is the multinomial coefficient $\binom{k_d}{k_0',\dots,k_d'}$. $k_j' := q-1-k_j (j < d)$ and $k_d' := \ell(k) - d(q-1)$.*

To understand factorizations, we need to recall the well-known results of Carlitz and others [C35, G96, T04]), that $[n] := t^{q^n} - t$ is the product of all the monic irreducibles of degree dividing $n$, $D_m := \prod_{i=0}^{m-1}[m-i]^{q^i}$ is the product of all the monic polynomials of degree $m$, whereas $L_n := \prod_{i=1}^{n}[i]$ is the least common multiple of all the monic polynomials of degree $n$. We have $d_i := D_i$, $\ell_i := (-1)^i L_i$.

In particular, note that for the families of power sums occurring in the last two theorems, as well as for the families obtained from them by the multiplicative relations through the theorems of the last section, the valuation of the power sum at a particular prime depends only on the degree of the prime, since the same is true for $[i]$'s, $\ell_i$'s and $d_i$'s occurring in these formulas.

By the Carlitz generating function in Section 5, we have the following more general formula expressing power sums as convenient sums of products (rather than just products, and thus loosing immediate understanding of the factorization of power sums as in the families above):

Let $k \geq 0$ and $d \geq 0$. Then $S_d(k+1)$ is the coefficient of $x^k$ in $(1+\sum+\sum^2+\cdots+\sum^k)/\ell_d$, where $\sum = \sum_{i=0}^{d} x^{q^i}/d_i \ell_{d-i}^{q^i}$, i.e.,

$$\begin{aligned} S_d(k+1) &= \frac{1}{\ell_d} \sum_{\sum_{i=0}^d k_i q^i = k, k_i \geq 0} \binom{\sum k_i}{k_0, \cdots, k_d} \prod \left( \frac{1}{d_i \ell_{d-i}^{q^i}} \right)^{k_i} \\ &= \frac{1}{\ell_d^{k+1}} \sum_{\sum_{i=0}^d k_i q^i = k, k_i \geq 0} \binom{\sum k_i}{k_0, \cdots, k_d} \prod \left( \frac{(\ell_d/\ell_{d-i})^{q^i}}{d_i} \right)^{k_i} \\ &= \frac{1}{\ell_d^{k+1}} \sum_{k = k_d q^d + \cdots + k_0, k_i \geq 0} \binom{k_d + \cdots + k_0}{k_d, \cdots, k_0} \prod_{i=1}^d \left( \frac{(-1)^i ([d] \cdots [d-i+1])^{q^i}}{d_i} \right)^{k_i}. \end{aligned}$$

For the discussion of the implications of the guess [T09b, H1] that in this decomposition, there is a unique term of the maximum degree among the terms corresponding to various decomposition of $k$, see [T09b].

We record some simple special cases:

$$S_d(a) = 1/\ell_d^a, \quad 0 < a \leq q,$$

$$S_d(q+b) = \frac{1}{\ell_d^{q+b}} (1 - b \frac{[d]^q}{[1]}), \quad b < q,$$

$$S_d(aq+b) = \frac{1}{\ell_d^{aq+b}} (1 + \sum_{j=1}^a (-1)^j \binom{b+j-1}{j} \frac{[d]^{jq}}{[1]^j}), \quad 0 < a, b < q,$$

$$S_d(q^2+1) = \frac{1}{\ell_d^{q^2+1}} (1 + \sum_{j=1}^q (-1)^j \frac{[d]^{jq}}{[1]^j} + \frac{[d]^{q^2} [d-1]^{q^2}}{[2][1]^q}).$$

The first of these is the simplest, but the most important, as it makes connection with 'poly-log' coefficients in the Carlitz module setting, which motivated the author to the results of [AT90]. Several proofs have been given for this special case, in addition to the generating function proof of Carlitz above, see e.g., [HHM12] and [T10p, Sec. 6].

## 8. Leading 'zeta' term and its factorization

We saw that $k > 0$, $S_d(-k) = 0$ if and only if $d > \ell(k)/(q-1)$, for $q$ prime. More generally, for any $q$, this is true (by results of Sheats [S98], as pointed out by Böckle, see [T09b, A.5], [B12p]) if and only if $d > \mathcal{L}/(q-1)$, where $\mathcal{L}$ is the minimum of $\ell(p^i k)$'s, over $i$ (which can be restricted between 0 and $f-1$, when $q = p^f$). (See also [Per14, Thm. 3.1] for a multivariate result along these lines). Hence, for $k > 0$, $S_{\lfloor \mathcal{L}/(q-1) \rfloor}(-k)$ turns out to be the leading term of the Goss zeta $\zeta(-k, X) = \sum_{d=0}^\infty S_d(-k) X^d$, which is a polynomial in $X$, when $k > 0$.

**Theorem 8.** [T13] *Let $q$ be any prime power. Let $k > 0$ and $\ell(k) = (q-1)d + r$, with $0 \leq r < (q-1)$, so that $d = \lfloor \ell(k)/(q-1) \rfloor$. Write the base $q$-expansion $k = \sum_{i=1}^{d(q-1)+r} q^{k_i}$. Then*

$$S_d(-k) = (-1)^d \sum t^{\sum_{i=1}^d (i-1) \sum_{j=1}^{q-1} q^{k_{i,j}} + d \sum_{j=1}^r q^{k_{m,j}}},$$

*where the outer sum is over all the assignments to $i$'s of groups of $q-1$ of the powers $q^{k_j}$'s corresponding to indices in partitions of $d(q-1)+r$ indices into $d$*

*groups (denoted by $k_{i,j}$'s) of $q-1$ each and one group (denoted by $k_{m,j}$'s) of $r$ powers.*

*In particular, when $q=2$, we have a product formula*

$$S_d(-k) = \prod_{d \geq n > m} (t^{2^{k_n}} + t^{2^{k_m}}).$$

*More generally, for any $q$, but for the special family $k = (q-1)\sum_1^d q^{k_i} > 0$ (with $k_i$ distinct) we have the leading term*

$$S_d(-k) = (-1)^d \prod_{d \geq n > m} (t^{q^{k_n}} - t^{q^{k_m}})^{q-1}.$$

The product formula in the $q=2$ case was obtained earlier by Pink using a cohomological formula for the leading power sum. See [B12p, 7.1] and [B10p] for this, as well as the proof of the last part of the Theorem using the Vandermonde determinant formula combined with cohomological machinery.

When $q > 2$, we do not have a product formula involving only monomials in $[n]$'s, in the general case, for the leading term, even if $q$ is a prime. For example, when $q=3$, $k=13$, $S_1(-13) = -(t^3-t)(t^3-t+1)(t^3-t-1)$. On the other hand, for many families of $q, k$, we can prove the product expression (for the leading term $S_d(-k)$ as above)

$$c\prod (t^{q^j} - t^{q^i})^{r_{i,j}},$$

where $c \in \mathbb{F}_q$ expressed in terms of multinomial coefficient, product being over $i < j$ such that $k_i + k_j = q - 1 + r_{i,j}$, with $r_{i,j} > 0$, where $k = \sum k_j q^j$ is the base $q$ expansion of $k$.

Note that the first formula of the theorem above allows us to immediately write down the evaluation of the leading term as a polynomial, and also allows us to prove results on divisibility by the products as above. For several results and conjectures on the factorizations, see [BTp].

We say that $\wp$ is exceptional for (a positive integer) $k$, if $k < q^{\deg(\wp)} - 1$ and $\wp$ divides the leading coefficient $S_d(-k)$ of the polynomial $\zeta(-k, X)$ We say that a positive integer $k$ is exceptional, if there is $\wp$ exceptional for $k$.

If $\wp(t)$ is exceptional for $k$, then so are $\wp(t+\theta)$ and $\wp(\theta t)$ for any $\theta \in \mathbb{F}_q^*$. If $\wp$ is exceptional for $ps$, then it is for $k$, and if it is for $k$ and $pk < q^{\deg(\wp)} - 1$, then it is exceptional for $pk$.

**Theorem 9.** [BTp] *(i) The lowest $k$ for which the leading coefficient $S_d(-k)$ is divisible by an exceptional prime is $k = q^2 + q + (q-2)$ (with digit sum $q$), if $p \neq 2$. The degrees of the corresponding exceptional primes divide $p-1$, if $p \geq 5$.*

*(ii) If $p = 2$, $q > 2$, the smallest such $k$ is $q^3 + q^2 + q + (q-3)$, and the degrees of all the corresponding exceptional primes is 4.*

*(iii) Let $p > 2$. Put $Q := (q-1)/2$. When $k = q^2 + Qq + (q-2)$, the leading coefficient is $-Q[1]^Q([1]^{q-1} - 1)$. Hence when $q$ is prime, the exceptional primes in this case are exactly the Artin-Schreier primes $t^q - t - \theta$, $\theta \in \mathbb{F}_q^*$ occurring to multiplicity one.*

*(iv) Let $q$ be a prime. If one prime of form $t^q - t - \theta$, $\theta \in \mathbb{F}_q^*$ is exceptional for some $k$, then all are. Moreover, if one such prime has an exceptional $k$, then $k'$ formed from $k$ by reversing the order of its base $q$ digits is also exceptional.*

## 9. Special types of interpolations

We now describe a special kind of interpolation of the basic power sums by special two variable functions specialized on graphs of powers of Frobenius. We will explain various applications, but refer to the original papers [AT90, AT09, T01, T04] for the relevant theory of Anderson's $t$-motives, Drinfeld's Shtukas and related theory of integrable systems.

Given $f = \sum_{i=0}^{\infty} a_i T^i \in \mathbb{C}_\infty[[T]]$, put $f^{(n)} = \sum_i a_i^{q^n} T^i$ and extend this rule entrywise to matrices with entries in $\mathbb{C}_\infty[[T]]$. Choose $\tilde{t}$ satisfying $\tilde{t}^{q-1} = -t$ and consider

$$\Omega(T) = \tilde{t}^{-q} \prod_{i=1}^{\infty} (1 - T/t^{q^i}) \in \mathbb{C}_\infty[[T]].$$

Then $\tilde{\pi} := 1/\Omega(t) \in \mathbb{C}_\infty$ is a period of the Carlitz module. Given $n \in \mathbb{Z}_{\geq 0}$ we define the Carlitz gamma factorial $\Pi$ by $\Pi_n := \prod d_i^{n_i}$, where $n = \sum n_i q^i \quad (0 \leq n_i < q)$ is the base $q$ expansion of $n$. Define $H_s(T) \in A[T]$ by generating series identity

$$\sum_{s=0}^{\infty} \frac{H_s(T)}{\Pi_s|_{t=T}} x^s = (1 - \sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{(T^{q^i} - t^{q^j})}{(T^{q^i} - T^{q^{j-1}})} x^{q^i})^{-1}.$$

**Theorem 10.** [AT09, 3.7.4] *With notation as above, we have*

$$(H_{s-1}\Omega^s)^{(d)}(t) = \Pi_{s-1} S_d(s)/\tilde{\pi}^s$$

*for all $d \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_+$.*

We will see applications of these special type interpolations of power sums in Sections 14 and 16.

## 10. Products as sums

In [T10], it was shown, bypassing earlier explicit combinatorial conjectures, that the products of the power sums can be expressed as sums of several (iterated) power sums. In [L10, L11, L12], explicit formulae were established making the recipe given in [T10] explicit.

Huei Jeng Chen (unpublished) simplified the above recipe considerably, in the simplest case, to

**Theorem 11.**

$$S_d(a)S_d(b) = S_d(a+b) + \sum((-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1})S_d(a+b-j, j),$$

*where the sum is over $j$ which are multiples of $q-1$ and $0 < j < a+b$.*

For applications to the multizeta relations, see Section 16.

## 11. Relations between basic and iterated power sums

We record some examples of simple relations between the basic and iterated power sums. The applications to relations between the mutizeta values, topic which is still not even conjecturally understood, will be given in Section 16.

**Theorem 12.** [T09, LTp] *We have*

$$S_d(m, m(q-1)) = S_d(m)S_{<d}(m(q-1)) = \frac{[d]^m/[1]^m}{\ell_d^m \ell_{d-1}^{m(q-1)}} = \frac{(-1)^m}{[1]^m \ell_{d-1}^{mq}} = \frac{1}{\ell_1^m} S_{d-1}(mq).$$

$$S_d(q^2)(\frac{1}{\ell_1} + \frac{1}{\ell_2}) = S_d(q^2-1)\frac{1}{\ell_{d+1}} + S_{<d+1}(q^2-1)\frac{1}{\ell_{d+2}}.$$

$$S_d(q^i)(\frac{1}{\ell_{i-1}} + \frac{1}{\ell_{i-2}}) = S_d(q^i-1)\frac{1}{\ell_{d+i-1}} + S_{<d+1}(q^i-1)\frac{1}{\ell_{d+i}}.$$

$$S_d(1, q-1, q(q-1), \cdots, q^n(q-1)) = \frac{1}{\ell_{n+1}\ell_n^{q-1}\ell_{n-1}^{q(q-1)} \cdots \ell_1^{q^{n-1}(q-1)}} S_{d-(n+1)}(q^{n+1}).$$

## 12. $v$-adic situation

Let $v$ be a prime of $A$. The $v$-adic situation is much less explored, especially for primes $v$ of $A$ of degree more than one. But here are some partial results [T13]. Recall $v_d(k)$ from subsection 2.2.

(1) If $v$ is a prime of degree 1, then $v_d(1) = q^d - (d+1)$.

(2) If $q = 2$, $v$ is a prime of degree 1, then $v_d(3) = 2^{d+1} - 3d - 1$, for $d \geq 1$.

(3) If $q = 2$, $v$ is the prime of degree 2, namely $v = t^2 + t + 1$, then $v_d(1) = 2^{d-1} - \lfloor (d+1)/2 \rfloor$, for $d \geq 1$.

(4) If $q = 2$, $v = t^2 + t + 1$, then $v_d(3) = 2^d - 3\lfloor d/2 \rfloor + (-1)^{d-1}$, for $d > 1$. We have $v_1(3) = 1$, $v_0(3) = 0$.

Here is a guess made from a small numerical data:

If $q = 2$, $v = t^2 + t + 1$, $d > 1$,

$$v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor + (-1)^{d-1}, \text{ if } n \text{ is even,}$$

and

$$v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor - \lfloor (d+1)/2 \rfloor + \lfloor d/2 \rfloor, \text{ if } n \text{ is odd.}$$

We have some trivial lower and upper bounds: (i) If $k > 0$, and $d > m \deg(v)$, then $v_d(k) \geq m$.

(ii) For $k > 0$, $d \geq \deg(v)$, we have $v_d(k) \leq dk(q^d - q^{d-\deg(v)} - 1)/\deg(v)$.

We also have some periodicity coming from strong congruences as follows. In contrast to $(\mathbb{Z}/p^n)^*$ which is cyclic for odd prime $p$, the analog $(A/v^n)^*$ is far from cyclic in general, when $n > 1$. If $v$ has degree $D$, then $(A/v^n)^*$ has order $(q^D - 1)q^{D(n-1)}$, but it has exponent

$$e_n = (q^D - 1)p^{\lceil \log_p(n) \rceil}.$$

So $S_{d,v}(k) \equiv S_{d,v}(k + me_n) \mod v^n$. In particular,

$$\text{if } v_d(k) < n, \text{ then } v_d(k) = v_d(k + me_n), \; m \in \mathbb{Z}.$$

(Hence for any fixed $d$, $v_d(k)$ can be small even for large $k$. )

**Theorem 13.** *Let us write $\oplus$ for the sum with no carry overs in base $p$.*

(i) *Let $k$ be negative and $m = -k$. Then either $s_d(k) = -dm + \min(m_1 + 2m_2 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, $m_i \geq 0$, and for $i \geq 1$, $(q-1)$ divides $m_i > 0$, or $s_d(k)$ is infinite, if there is no such decomposition. When the decompositions exist, the minimum is uniquely given by the greedy algorithm. (By this we mean here and below that among all valid* *decompositions with $m_d \leq \cdots \leq m_1$, we choose ones with least $m_d$, among these the ones with least $m_{d-1}$ and so on).*

(ii) *Let $k$ be positive. Then $s_d(k) = dk + \min(m_1 + \cdots + dm_d)$, with $(k-1)+m = (k-1) \oplus m$ and $m = m_1 \oplus \cdots \oplus m_d$, with $m_i$ positive and divisible by $q-1$. The minimum is uniquely given by the greedy algorithm.*

(iii) *Let $k$ be negative and $m = -k$. Let $v$ be a prime of $A$ of degree one. Then either $v_d(k) = \min(m_1 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, where $(q-1)$ divides $m_i > 0$ for $0 < i < d$, and $q-1$ divides $m_0 \geq 0$; or $v_d(k)$ is infinite, if there is no such decomposition. When the decompositions exist, the minimum is uniquely given by the greedy algorithm.*

*(Thus, if $r$ is the least non-negative residue of $m$ (mod $q-1$), then the minimal $m_d$ is the least sum of $p$ powers chosen from the $p$-expansion of $m$ which is $r$ (mod $q-1$). If $q = p$ a prime, then it is just the sum of the least $r$ of the $p$ powers chosen from the expansion, because $p^n$ is 1 (mod $q-1$) then.)*

(iv) *Let $k$ be positive. Let $v$ be a prime of $A$ of degree one. Then $v_d(k) = \min(m_1 + \cdots + dm_d)$, with $(k-1)+m = (k-1) \oplus m$, $m = m_1 \oplus \cdots \oplus m_d$, and $(q-1)$ divides $m_j > 0$ for $j < d$ and $(q-1)$ divides $k + m_d$. The unique minimum is given by the greedy algorithm.*

**Theorem 14.** *Let $v$ be a prime of degree one.*

(i) *Let $k$ be positive. Then $v_{d+1}(k) = s_d(k) - dk$ if $q-1$ divides $k$.*
(ii) *Let $k$ is negative. Then $v_{d+1}(k) = s_d(k) - dk$ if $q-1$ divides $k$ and $s_d(k)$ is finite.*
(iii) *If $q-1$ divides $k$, then $v_d(k)$ is also divisible by $q-1$.*
(iv) *If $q$ is prime and $k$ is divisible by $q-1$, then $v_{d+1}(k) = v_d(v_2(k)+k) + dv_2(k)$.*

**Theorem 15.** *Let $q = p$ be a prime, $v$ a prime of $A$ of degree one and $-m = k < 0$. Write $m = \sum_{i=1}^{\ell} p^{e_i}$, with $e_i$ monotonically increasing and with not more than $p-1$ of the consecutive values being the same (i.e., consider the base $p$-digit expansion sequentially one digit at a time). Also, let $r$ be the least non-negative residue of $m$ (mod $q-1$). Then $v_d(k)$ is infinite if $\ell < (p-1)(d-1)+r$, and otherwise*

$$v_d(k) = d\sum_{s=1}^{r} p^{e_s} + \sum_{j=1}^{d-1} j \sum_{s=1}^{p-1} p^{e_{(d-1-j)(p-1)+r+s}}.$$

For many more interesting conjectural patterns we refer to [T13].

## 13. Sums for higher genus co-ordinate rings and more general sums

In this survey, we have focused on power sums for $A = \mathbb{F}_q[t]$.

We only give references now to some works [T92, A94, A96, T04, T09a, Ta12, B12p] which deal with generalizations to sums for co-ordinate rings of higher genus curves (from this genus zero situation), to multi-variable sums [P12, Per13, LTp3] or to sums with characters (preprints by Papanikolas) or deal with the corresponding zeta and L-values.

## 14. Applications: Zeta values

Consider for a positive integer $k$, the zeta value

$$\zeta(k) := \sum_{d=0}^{\infty} S_d(k) = \sum_{a\in A+} a^{-k} \in K_\infty$$

Carlitz studied these values as an analog of the Riemann zeta values, whereas Goss realized that when $k \leq 0$, while the last equality does not make sense, the first equality does make sense and is, in fact, in $A$, (in analogy or contrast with rationality of Riemann zeta values at negative integers), as the sum is finite, by the vanishing results in Section 3.

Goss also proved [G96, T04] (using induction and a recursion formula) that for $k < 0$, 'even', $\zeta(k) = 0$, in analogy with the statement for Riemann zeta values. This can also be seen [T04, Thm. 5.3.1] by vanishing results of Section 3 on power sums as follows. Since by the first vanishing result on the basic power sums, for $k < 0$, $\zeta(k) = S_{<d}(k)$ for sufficiently large $d$. But then it vanishes for 'even' $k$ by the general vanishing result [T04, Thm. 5.1.2] mentioned in Section 3.

Carlitz [C35] proved the analog $\zeta((q-1)m)/\tilde{\pi}^{(q-1)m} \in K$ of Euler's result that $\zeta_{\mathbb{Z}}(2m)/(2\pi i)^{2m} \in \mathbb{Q}$, where $\tilde{\pi}$, the Carlitz analog of $2\pi i$, is a fundamental period (see e.g., section 9 for a formula) of the Carlitz module [G96, T04].

In [AT90, 3.8.2], using the special interpolations of the power sums of Section 9, an algebraic point (torsion point if and only if $q-1$ divides $s$) on $s$-th tensor power $C^{\otimes s}$ of the Carlitz motive $C$ was constructed. Its logarithm connected to the Carlitz zeta value $\zeta(s)$. Equivalently, an extension over $A$ of this power by the trivial module which has $\Gamma_s\zeta(s)$ as its period was constructed.

Combining this with the transcendence theory of the logarithms, poly-logarithms and periods related to $t$-motives of Anderson [A86], developed by Wade, Jing Yu, Anderson, Brownawell, Papanikolas, Chang etc., it is proved that

**Theorem 16.** [CY07] *Only algebraic relations between $\zeta(s)$'s, for $s > 0$, come from Carlitz relation at 'even' $s$ mentioned above and $\zeta(sp) = \zeta(s)^p$.*

*In particular, all the zeta values at 'odd' $s$ not divisible by $p$ are algebraically independent from each other and from $\tilde{\pi}$.*

Consider the Goss $v$-adic zeta function.

$$\zeta_v(k) := \sum_{d=0}^{\infty} S_{d,v}(k) \in K_v.$$

The motivic result [AT90] mentioned above also contained $v$-adic counterparts, but the $v$-adic transcendence theory is not yet as developed, so we have following, comparatively weaker (but much stronger than its number field counterpart) application:

**Theorem 17.** [Y91] *If $n > 0$ is not divisible by $q-1$, then $\zeta_v(n)$ is transcendental over $K$.*

## 15. Applications: Zero distribution for the Goss zeta functions

More generally, to go from discrete values at $k$ to a continuous space of exponents, Goss [G96, T04] defined the two variable Goss zeta function as follows. Define exponent spaces $S_\infty := C_\infty^* \times \mathbb{Z}_p$ and $S_v := C_v^* \times \lim \mathbb{Z}/(q^{\deg v} - 1)p^j\mathbb{Z}$.

For $s = (x, y) \in S_\infty$, put

$$\zeta(s) = \sum_{d=0}^{\infty} x^d \sum_{a \in A_d+} (a/t^d)^y \in C_\infty.$$

Note that for $y$ an integer, the coefficient of the $d$-th term in this power series in $x$ is nothing but $S_d(-y)t^{-dy}$, hence $\zeta(k)$ as above is $\zeta(t^{-k}, -k)$.

For $s = (x, y) \in S_v$, put

$$\zeta_v(s) = \sum_{d=0}^{\infty} x^d \sum_{\substack{a \in A_d+ \\ (a,v)=1}} a^y \in C_v.$$

Note that for $y$ an integer, the coefficient of the $d$-th term in this power series is nothing but $S_{d,v}(-y)$, hence $\zeta_v(k)$ above is $\zeta_v(1, -k)$.

See [G96, T04] and references there for more details on these interpolations, and their analytic properties. The analog of the Riemann hypothesis ('zeros restricted to a real line inside the complex plane') due to Wan and Sheats in this context is

**Theorem 18.** [W96, DV96, S98, G96, T04] *For given $y \in \mathbb{Z}_p$, the zeros of $\zeta(x, y)$ are simple and lie in $K_\infty \subset C_\infty$.*

For a power series in $K_\infty[[x]]$ ,such as $\zeta$, in the non-archimedean setting, one gets such a result, just by knowing that the corresponding Newton polygons have increasing slopes at every integral step. This translates into degree inequalities on the basic power sums. Wan had proved this using estimates for the degrees. The author had noticed [T90, Pa. 10] that the proof of Carlitz claim (3) of Theorem 1 (which was incomplete) would also imply degree formula which would imply this. The proof was fixed for $q$ prime, in [DV96], and in general, only after much more effort by Sheats [S98] (with $q = 4$ done by Poonen a little earlier).

It should be noted that for a prime $q$, the slopes increase and thus the analog of the Riemann Hypothesis for $\mathbb{F}_p[t]$ is immediate from the recursion (1), without needing the 'initial values' $s_1(k)$, except for the trivial fact $s_1(k) > k$, for positive $k$.

Thus, it would be still very interesting to get some direct recursion for general prime power $q$ case, as this might simplify the quite complicated combinatorial proof by Sheats.

As for the $v$-adic case, Wan proved [W96, G00] 'simple real zeros' statement in the case $v$ is of degree one and when $y$ is multiple of $q - 1$. Using the similar results of Section 12 on $v$-adic valuations, Diaz-Vargas has recently generalized by dropping the condition on $y$, thus fully settling the degree one case. The situation in the case when $v$ is of degree more than one is not fully understood, but the author showed [T13] using the valuation results as in Section 12 that the statement in this case will need a modification.

See also [B12p] about some higher genus, class number one case results, using the higher genus power sums work mentioned in Section 13.

## 16. Applications: Multizeta values

We define [T04, T09, T10, AT09, T09a] the multizeta values of depth $r$ and weight $\sum s_i$ as

$$\zeta(s_1, \cdots, s_r) := \sum_{d=0}^{\infty} S_d(s_1, \cdots, s_r).$$

*Non-vanishing*: For integers $s, s_1 > 1$ and $s_i > 0$, the Riemann zeta $\sum n^{-s}$, where the sum is over positive integers $n$, and the Euler multizeta values $\sum n_1^{-s_1} \cdots n_r^{-s_r}$, where the sum is over positive integers $n_1 > \cdots > n_r$, are clearly non-zero, as they are sums of positive terms. In the function field case, the non-vanishing is not obvious. Indeed, the Carlitz-Goss zeta, at negative integers, are also given by sums, but vanish for negative 'even' integers. For positive $k$, $\zeta(k)$ does not vanish for the simple reason that $d = 0$ term, being 1, is of degree zero, whereas the terms for $d > 0$ are of negative degree and thus there is no chance of cancellation.

The non-vanishing of the multizeta for $s_i > 0$ is a little harder, but can be proved [T09b, Thm. 4] by using the degree recursions and inequalities in Section 4.

*Mixed motives and periods*: The Euler multizeta values occur in various parts of mathematics and physics such as Grothendieck-Ihara program to study the absolute Galois group over $\mathbb{Q}$ through the algebraic fundamental group of projective line minus three points, and related studies of iterated extensions of Tate motives, quantum groups, knot invariants, Feynman path integral renormalizations etc.

Using the special interpolations in Section 9, we can define [AT09] explicit iterated extensions of Carlitz-Anderson-Tate $t$-motives whose periods are these multizeta values.

The linear and algebraic relations of multizeta values thus have several structural implications in many areas of mathematics.

*Shuffle relations*: For the Euler multizeta, the $\mathbb{Q}$-linear span of the multizeta values is an algebra, because the product of multizeta values can also be expressed as sum of some multizeta values, for a simple reason called sum shuffle, which just shuffles inequalities chains in $n_i$'s using the trichotomy law that either $n_i > n_j$ or $n_i < n_j$ or $n_i = n_j$, so that, for example, $\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a + b)$. In the case of polynomials over finite fields, there is no natural order and such trichotomy arguments as well as these relations fail, but by the relations mentioned in Section 10, we do get same 'algebra' property, with more complicated relations such as

$$\zeta(a)\zeta(b) = \zeta(a,b)+\zeta(b,a)+\zeta(a+b)+\sum((-1)^{a-1}\binom{j-1}{a-1}+(-1)^{b-1}\binom{j-1}{b-1})\zeta(a+b-j,j),$$

where the sum is over $j$ which are multiples of $q-1$ and $0 < j < a+b$.

Note that when we have $a, b, a+b \leq q$, then since $S_d(i) = 1/\ell_d^i$, for $i = a, b, a+b$, we do get the classical sum shuffle relation.

We refer to [T09b, LTp2] for the use of power sum identities that we considered in classifying special kinds of shuffle relations such as the classical sum shuffle relation and some other special ones.

*Multizeta relations with $\mathbb{F}_q[t]$-coefficients:* While all the relations for the Euler multizeta are conjecturally understood, the situation is not even conjecturally understood for these multizeta values for polynomials over finite fields. But the relations in Sections 7, 11, factorizations (which allow cancellations when multiplied) in Section 7 and multiplicative relations in Section 6, allow us to prove fails of relations with coefficients in $\mathbb{F}_p[t]$ (rather than in the prime field $\mathbb{F}_p$ of characteristic $p$, analogous to the prime field $\mathbb{Q}$ of characteristic zero!). We refer to [T09, L10, LTp3, LTp] for such relations and give here only some examples which follow from relations mentioned in previous sections.

$$\zeta(1, q^2-1) = \zeta(q^2)(\frac{1}{\ell_1} + \frac{1}{\ell_2}),$$

$$\zeta(q^n - \sum_{i=1}^{s} q^{k_i}, (q-1)q^n) = \frac{(-1)^s}{\ell_1^{q^n}} \prod_{i=1}^{s} [n-k_i]^{q^{k_i}} \zeta(q^{n+1} - \sum_{i=1}^{s} q^{k_i}),$$

where $n > 0$, $1 \leq s < q$, $0 \leq k_i < n$.

For any $q$,

$$\zeta(1, q-1, (q-1)q, \ldots, (q-1)q^n) = \frac{(-1)^{n+1}}{[1]^{q^n}[2]^{q^{n-1}} \cdots [n+1]^{q^0}} \zeta(q^{n+1}).$$

In other direction, we refer to [Ch12, CPY] for very strong transcendence and algebraic independence results and criteria using motivic connection.

## 17. Bernoulli factorization

Bernoulli numbers, ubiquitous in mathematics (with their factorizations having relevance in number theory, as well as differential topology and homotopy!) arose in the study of finite (positive) power sums by Bernoulli, and reappeared in the zeta values at both positive and negative integers, related by functional equation, in the famous calculations of Euler. In the function field arithmetic, there are two analogs of Bernoulli numbers [T04, 5.3.9, 4.16], coming from the zeta values at positive and negative integers, there being no known functional equation. The values at negative integers are essentially finite power sums we considered, but because of general relations mentioned in both the infinite and the finite case in Sections 5. 6, we understand their factorizations well for some families. This has many applications in understanding of the class groups and modules in the setting of the cyclotomic extensions, by analogs of Kummer-Herbrand-Ribet theorems due to Goss, Sinnott, Okada, Taelman [G96, T04, Ta12]. These theorems relate the arithmetic of these abelian extensions to factorizations of Bernoulli number analogs, which are power sums. So we have applications of the factorization and multiplicative relation results we have mentioned.

In another directions, we refer to [T12] to see how this knowledge of factorizations helped the author to give families of counterexamples to a conjecture of Chowla about the usual Bernoulli numbers.

## 18. Applications: $p$-rank questions for cyclotomic Jacobians

Just as Kummer-Herbrand-Ribet theorems link the $p$-part of the class group (which is the group of $\mathbb{F}_q$-points of the Jacobian, in the simplest incarnation) of the cyclotomic function fields to the divisibility by $p$ of appropriate Bernoulli numbers, the $p$-ranks of these Jacobians vary (see [R02, 5.9, 11.20] and [S13, BTp]) according to the divisibility by $\wp$ of the leading Goss zeta term. Thus the results and conjectures of Section 8 on the factorization of these leading terms, in addition to their intrinsic interest, also have implications to understanding interesting variation of these $p$-ranks.

## 19. Miscellaneous applications

*Zeta measure* The surprising (in terms of the known analogies) calculation of the divided power series corresponding to the Zeta measure [T90] for $\mathbb{F}_q[t]$ boils down to certain specialized relations between power sums for negative powers.

*Wieferich-Wilson primes criteria* In [Tp], we considered analogs of Wieferich and Wilson primes, i.e., the primes for which the Fermat or Wilson congruences (which represent basic structure for finite field theory) hold moduli higher powers, in the context of theory of Carlitz module for $\mathbb{F}_q[t]$ and showed that prime $\wp$ of degree $d$ is Wieferich if and only if it is Wilson if and only if $\wp$ divides $\sum_{i=0}^{d-1} S_i(1)$. Apart from the theoretical interest, it also gives a convenient way to search for such primes efficiently, because of the simple formula for $S_i(1)$ mentioned in Section 7.

*Higher genus, $v$-adic and non-rational infinite place generalizations; log-algebraicity* We have focused mainly on $S_d(k)$'s for $\mathbb{F}_q[t]$, but there are many developments related to higher genus generalizations from the $\mathbb{F}_q(t)$ situation discussed here. There are log-algebraicity results generalizing some aspects of Section 14 [T92, A94, A96, T04, T09b], and such results for multivariable power sums in recent works of Papanikolas (unpublished), Pellarin and Perkins [P12, Per13]. The results of Sections 9 and 12 have also been used (work in progress) in studying the $v$-adic aspects of multizeta. The shuffle relations mentioned in Section 16 generalize [T10] to the higher genus situation when the infinite place is $\mathbb{F}_q$-rational. In [LTp2], it is shown that out of these shuffle relations only certain interesting families survive when the infinite place is not $\mathbb{F}_q$-rational.

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