Indian J. Pure Appl. Math., **45**(5): 787-801, October 2014

ZETA-LIKE MULTIZETA VALUES FOR $\mathbb{F}_q[t]$

José Alejandro Lara Rodríguez* and Dinesh S. Thakur**[1]

Dedicated to the genius of Srinivasa Ramanujan

**Facultad de Matemáticas, Universidad Autónoma de Yucatán,*
Periférico Norte Tab. 13615, *Mérida, Yucatán, México*
***Department of Mathematics, University of Rochester, Rochester,*
NY 14627 *USA*
e-mails: lrodri@uady.mx; dinesh.thakur@rochester.edu

(*Received* 17 *December* 2013; *after final revision* 20 *April* 2014;
accepted 27 *April* 2014)

Abstract. We prove and conjecture several relations between multizeta values for $\mathbb{F}_q[t]$, focusing on the zeta-like values, namely those whose ratio with the zeta value of the same weight is rational (or equivalently algebraic). In particular, we describe them conjecturally fully for $q = 2$, or more generally for any q for 'even' weight ('eulerian' tuples). We provide some data in support of the guesses.

Key words : Eulerian; periods; t-motives; transcendence.

1. Introduction

Relations between multizeta values defined by Euler have been investigated extensively for the last two decades, and the conjectural forms of these relations have many structural connections with several interesting areas (see [Ct2001, Z2012] and references there) of mathematics. In some sense, the relations have been at least conjecturally understood, though much remains to be proved and relating the general framework to specific instances is often hard.

We will look at the function field analog [T2004, AT2009, T2009, Tbanff, L2011, L2012], where the relations are still not conjecturally understood, though in contrast, there are also some very strong transcendence and linear/algebraic independence results [CY2007, Ch2012, CPY] proved.

[1]The authors supported in part by PROMEP grant F-PROMEP-36/Rev-03 SEP-23-006 and by NSA grant H98230-13-1-0244 respectively.

While for the Euler multizeta values, the relations come via comparing the two families of shuffle relations, in our function field setting, there is only one shuffle family [T2010]. While the rational number field is the prime field in characteristic zero giving coefficients for the relations, in the function field case the prime field is not the analogous rational function field, but just the finite field, which does not see all the relations. Some such relations were proved in [T2009, L2011].

While second author's student George Todd is doing extensive numerical study of general relations using an analog of the 'LLL method', in this paper we focus on the two term relations of special type, namely zeta-like multizeta, i.e., those whose ratio with the (Carlitz) zeta value of the same weight is rational (or equivalently algebraic). We provide several results, and conjectures, with full conjectural description for $q = 2$, or more generally for any q with 'even' weight ('eulerian' tuples).

We first fix the notation and give the basic definitions. Next we summarize the known and the new results on zeta-like values, and state the conjectures. Then we give the proof of the results. Finally we discuss the numerical data, calculated by the first author, giving some evidence for the conjectures made from it.

2. Notation and Basic definitions

$\mathbb{Z}$	$\{$integers$\}$,
$\mathbb{Z}_+$	$\{$positive integers$\}$,
q	a power of a prime p,
$\mathbb{F}_q$	a finite field of q elements,
A	the polynomial ring $\mathbb{F}_q[t]$, t a variable
A_+	monics in A,
K	the function field $\mathbb{F}_q(t)$,
K_∞	$\mathbb{F}_q((1/t))$ = the completion of K at ∞,
A_{d+}	$\{$elements of A_+ of degree $d\}$,
$[n]$	$t^{q^n} - t$,
ℓ_n	$\prod_{i=1}^n (t - t^{q^i}) = (-1)^n L_n = (-1)^n [n][n-1]\cdots[1]$,
'even'	multiple of $q-1$,

We first recall definitions of power sums, iterated power sums, zeta and multizeta values [T2004, T2009].

For $s \in \mathbb{Z}_+$ and $d \geq 0$, write

$$S_d(s) := \sum_{a \in A_{d+}} \frac{1}{a^s} \in K.$$

(This is $S_d(-s)$ in the notation of [T2004].)

Given integers $s_i \in \mathbb{Z}_+$ and $d \geq 0$ put

$$S_d(s_1, \ldots, s_r) = S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in K.$$

For $s_i \in \mathbb{Z}+$, we define multizeta values

$$\zeta(s_1, \ldots, s_r) := \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over all $a_i \in A_+$ of degree d_i such that $d_1 > \cdots > d_r \geq 0$. We say that this multizeta value (or rather the tuple $(s_1, \ldots, s_r)$) has depth r and weight $\sum s_i$. In depth one, we recover the Carlitz zeta.

We refer to [C1935, G1996, T2004] for background on this and general function field analogies. Carlitz proved analog of Euler's result that for 'even' s, $\zeta(s)$ is a non-zero rational multiple of $\tilde{\pi}^s$ (i.e., $\zeta(s)/\tilde{\pi}^s \in K$), where the Carlitz period $\tilde{\pi}$ is analog of $2\pi i$. In 1970's, in the context of the development of Drinfeld modules and their periods, David Goss independently proved such analogs of Euler's result in more general context and developed theory of the Goss zeta functions.

A multizeta value $\zeta(s_1, \ldots, s_r)$ of depth r (or the r-tuple $(s_1, \ldots, s_r)$) is *zeta-like* if the ratio

$$\zeta(s_1, \ldots, s_r)/\zeta(s_1 + \cdots + s_r)$$

is rational. (We always use depth $r > 1$ below, sometimes without mention, because in the $r = 1$ case everything is zeta-like by definition). A multizeta value of weight w is called *eulerian*, if it is a rational multiple of $\tilde{\pi}^w$. So eulerian is a special case of zeta-like for 'even' weight, by Carlitz result (mentioned above) which says that in depth one, all the zeta values of 'even' weight are eulerian.

A strong transcendence result [Ch2012] proved in the function field case shows that if the ratio in the definition of the zeta-like value is not in K, then it is not even algebraic over K, and in fact, the multizeta in the numerator and the corresponding zeta value in the denominator are then algebraically independent. Another strong transcendence result [CY2007] shows that the Carlitz zeta value of not 'even' weight and $\tilde{\pi}$ are algebraically independent.

Since $\zeta(ps_1, \ldots, ps_r) = \zeta(s_1, \ldots, s_r)^p$, in all the discussion we can restrict to tuples where not all s_i's are divisible by p. We call such tuples *primitive*.

3. Old and new results on Zeta-like values

For the Euler multizeta in the number field case, the classical sum shuffle relation specialized, namely $\zeta(k)^2 - \zeta(2k) = 2\zeta(k, k)$, immediately implies that $\zeta(2n, 2n)$ are eulerian. Combined with the usual transcendence conjectures, it also implies that $\zeta(2n+1, 2n+1)$ are not zeta-like. In the function field case, this classical sum shuffle relation does not hold in general, but it does hold [T2004, Thm. 5.10.6] if $2k \leq q$, so that when $p \neq 2$, $\zeta(kp^n, kp^n) = \zeta(k, k)^{p^n}$ is not zeta-like if $2k \leq q$, since $\zeta(2k)/\zeta(k)^2$ is then transcendental by [CY2007]. Another instance of different shuffle [L2012, Thm. 6.3] similarly shows that $\zeta(q^n-1, q^n)$ is not zeta-like, for $q > 2$. In [T2004, T2009, L2011], more examples of zeta-like and non-zeta-like values of 'even' and 'odd' weights were proved. Combining with general shuffle relations [T2010], some more such results can be proved. But we have now proved much stronger results, which we will recall below.

In [CPY], using the interpretation [AT2009] of multizeta values as periods of iterated extensions of tensor powers of Carlitz-Anderson t-motives, it was proved that if $\zeta(s_1, \dots, s_r)$ is zeta-like (eulerian in the first version), then $\zeta(s_2, \dots, s_r)$ is eulerian, so that all $\zeta(s_k, \cdots, s_r)$ are eulerian and s_i are 'even', for $i \geq 2$. (See [T2009, 5.3]).

Remark. This implies some, but not all, of the non-zeta-like special results mentioned above. Many can be proved by direct appeal to [CY2007] and shuffle and other results proved, a few (such as $\zeta(2,1)$ is not zeta-like for $q = 2$) were proved [T2004, Thm. 5.10.12] without using [CY2007].

While [CPY] was being proved for the eulerian case, we had conjectured this (and a few more implications) for zeta-like case, but only in depth 2 and were starting calculations in general depth, which give many interesting conjectural restrictions recalled below.

We now state some families of zeta-like (so eulerian, if the weight is 'even') multizeta values of depth two. Proofs will be given in the fifth section.

Theorem 3.1. *For any (prime power) q, we have*

$$\zeta(q^n - \sum_{i=1}^{s} q^{k_i}, (q-1)q^n) = \frac{(-1)^s}{\ell_1^{q^n}} \prod_{i=1}^{s} [n-k_i]^{q^{k_i}} \zeta(q^{n+1} - \sum_{i=1}^{s} q^{k_i}), \tag{1}$$

where $n > 0$, $1 \leq s < q$, $0 \leq k_i < n$.

Let $n \geq 0$, $0 \leq k_i \leq n+1$, $1 \leq s_1 \leq q$, $0 \leq s_2 \leq q - s_1$. Then for $a = s_1 q^n$ and $b = s_1(q^{n+1} - q^n) + \sum_{i=1}^{s_2}(q^{n+1} - q^{k_i})$, we have

$$\zeta(a, b) = \frac{1}{\ell_1^{s_1 q^n}} \zeta(a+b). \tag{2}$$

$$\zeta(q^2 - (q-1), (q-1)(q^2+1)) = \frac{1 - [2]^q}{\ell_1^{q^2-1}\ell_2} \zeta(q^3), \tag{3}$$

$$\zeta(2q-1, (q-1)(q^2+q-1)) = \frac{1 - [2]^q}{\ell_1^{q+1}\ell_2^{q-1}} \zeta(q^3). \tag{4}$$

$$\zeta(1, q^2-1) = \zeta(q^2)(1/\ell_1 + 1/\ell_2). \tag{5}$$

For $q > 2$, $n \geq 0$ and $-1 \leq j \leq n$,

$$\zeta((q-1)q^n-1,(q-1)q^{n+1}+q^n-q^{n-j}) = -\frac{[n+1]}{[1]^{(q-1)q^n}}\zeta(q^{n+2}-q^{n-j}-1). \tag{6}$$

Next we state a theorem (proved in section 5) giving zeta-like family of arbitrary depth.

Theorem 3.2. *For any q,*

$$\zeta(1,q-1,(q-1)q,\ldots,(q-1)q^n) = \frac{(-1)^{n+1}}{[1]^{q^n}[2]^{q^{n-1}}\cdots[n+1]^{q^0}}\zeta(q^{n+1}).$$

4. Observations, Guesses and Conjectures

Now we state some conjectures (based on the numerical data and on consistency with the theorems and the proof methods) with varying degrees of confidence and evidence!

Conjecture 4.1. Tuple restrictions *If $(s_1,\ldots,s_r)$ is zeta-like, then*

(1) $s_i \le s_{i+1}$, $i=1,\ldots,r-1$. Furthermore, $(q-1)s_i \le s_{i+1} \le (q^2-1)s_i$.
(2) $(s_2,\ldots,s_r)$ is eulerian and $(s_1,\ldots,s_{r-1})$ is zeta-like

Note that part 2 can be iterated by reducing the length of the tuple and thus reducing to the zeta case of Carlitz and thus implying that s_i are 'even' for $i \ge 2$ (already proved together with the first part of (2), in [CPY], as mentioned before).

Conjecture 4.2. Splicing of tuples

(1) Let $q=2$. If $(s_1,\ldots,s_k)$ and $(s_k,\ldots,s_r)$ are zeta like and the total weight $\sum_{i=1}^r s_i$ is a power of 2 or a power of 2 minus one, then $(s_1,\ldots,s_r)$ is zeta-like, except when the two tuples to be spliced are (1,1) and (1,1).
(2) Let q be arbitrary prime power. If $(s_1,\ldots,s_k)$ and $(s_k,\ldots,s_r)$ are eulerian and the total weight $\sum_{i=1}^r s_i$ is q^n-1 or $q(q-1)$, then $(s_1,\ldots,s_r)$ is eulerian.

Remarks. For general q, splicing conditions for zeta-like tuples seem to be much more restrictive and seem to depend (in the limited data we have) on the combinatorics of digit expansions.

Conjecture 4.3. Weight restrictions

(1) Eulerian multizeta value (in depth $r>1$) can occur only in weights $p^m(q^k-1)$, with primitive ones only in weights $q(q-1)$ or q^n-1 for $q>2$ and in weights 2^n-1 and 2^n, if $q=2$.
(2) When $q=p$, depth $r>1$, the weight of zeta-like but non-eulerian tuple is p^m times a number with no zero digit and at most one 1 digit in base p expansion.
(3) In depth r, the smallest weight of zeta-like value is q^{r-1}.

(4) *When* $q > 2$, *the smallest weight of eulerian value is* $q^r - 1$, $(q-1)q$, *and* $q-1$ *according to depth* $r > 2$, $r = 2$ *and* $r = 1$ *respectively.*

(5) *Weight* q^k *is not a zeta-like weight of a primitive tuple, if* $k > r > 3$, *and also when* $k > 3$ *if* $r = 3$, *or* 2, *where* r *is the depth.*

Remarks. (0) Let us recall the known results for the Euler multizeta. (We use the standard short-form $\{X\}_k$ standing for the tuple X repeated k times.) Euler proved that $\zeta(3,1) = \zeta(4)/4$ and $\zeta(2,1) = \zeta(3)$, which generalizes to zeta-like $\zeta(2, \{1\}_k) = \zeta(k+2)$ (special case of Hoffman-Zagier duality relation) and $\zeta(\{3,1\}_k) = \zeta(\{2\}_{2k})/(2k+1)$ (Broadhurst's result, conjectured by Zagier) which is known to be eulerian as $\zeta(\{2\}_k) = \pi^{2k}/(2k+1)!$ (see e.g., [Z2012]). In fact, $\zeta(\{2n\}_k)$ is also eulerian. (Proof: Let the induction hypothesis $P(k)$ be that $A_k := \zeta(\{2n\}_k)$ and $B_{k,m} := \sum \zeta(X(i))$ are eulerian for all n, m, where the sum runs over all tuples $X(i)$ of length k with $k-1$ entries $2n$ and one entry $2nm$. The sum shuffle gives $\zeta(2n)A_k = (k+1)A_{k+1} + B_{k,2}$ and $\zeta(2mn)A_k = B_{k+1,m} + B_{k,m+1}$ proving the result by induction. For a proof using generating functions, see [BBB1997]. The same reference conjectures that $\zeta(\{\{2\}_m, 1, \{2\}_m, 3\}_n, \{2\}_m)$ is eulerian. We thank J. Zhao for the reference). In our very limited numerical search (weight ≤ 50 for depth 2, and even lower for depths 3, 4), as well as limited search of the vast literature, we did not find any other zeta-like tuples. We do not know whether there are any more examples, or conjectures based on theoretical or numerical evidence.

Considering our conjectures in the function field case, note that the last eulerian family mentioned starts only in depth 5, and that we never looked numerically seriously beyond depth 3 or 4 in the function field case!

For Euler's multizeta, each even weight > 2 is eulerian, in the sense that it occurs as a weight of eulerian multizeta of some depth more than one, e.g., as $\zeta(\{2\}_k)$ is eulerian. In our case, for $q = 3$, even $\zeta(2,2)$ is not eulerian by [T2004, Thm. 5.10.12], and this conjecture predicts much stringent weight conditions. It is conjectured for the Euler multizeta that the eulerian case occurs only in even weights. In our case, we know by [Ch2012] that the eulerian case occurs only in 'even' weights. For the Euler's multizeta, $\zeta(2n, 2n)$ are eulerian of weight $4n$ and depth 2, though weight $4n+2$ does not seem to be eulerian weight in depth 2.

(1) The weight $p^m(q^k - 1)$, with $m > 0$ for eulerian value can occur with primitive tuples, e.g., $q = 2$ and $(1,1), (1,3), (3,5)$ or $q = 3$ and $(2,4)$.

(2) The parts 3 and 4 are known for depth 1, and the occurrence in predicted weights is either proved in our main theorem or also follows from the higher depth families conjectures below. So the 'smallest' is the real conjectural part. More data may allow to conjecture the depth dependence of possible m and k in the first part.

Conjecture 4.4. Depth 2, weight at most q^2 *All zeta-like primitive tuples of weight at most* q^2 *and depth* 2 *are exactly* $(i, j(q-1))$, $i = 1, \ldots, q$, $j = i, \ldots, \lfloor (q^2-i)/(q-1) \rfloor$ $(\lfloor (q^2-i)/(q-1) \rfloor$

equals $q+1$ *or* q*, depending if* $i=1$ *or* $i>1$*):*

$$\begin{array}{cccccc} (1,q-1) & (1,2(q-1)) & (1,3(q-1)) & (1,4(q-1)) & \dots & (1,(q+1)(q-1)) \\ & (2,2(q-1)) & (2,3(q-1)) & (2,4(q-1)) & \dots & (2,q(q-1)) \\ & & (3,3(q-1)) & (3,4(q-1)) & \dots & (3,q(q-1)) \\ & & & & & \vdots \\ & & & & & (q,q(q-1)) \end{array}$$

Note that our theorems imply that those tuples are zeta-like. The converse is the conjectural part.

Conjecture 4.5. Depth 2 *For* $q>2$*, in depth 2 primitive eulerian tuples are exactly* $(q-1,(q-1)^2)$*,* $(q^n-1,(q-1)q^n)$ *and* $(q^n(q-1),q^{n+2}-1-q^n(q-1))$*.*

For $q=2$*, the zeta-like (eulerian equivalently) primitive tuples of depth two are exactly* $(1,1),(1,3),(3,5)$ *and* $(2^n-1,2^n)$*,* $(2^n,2^{n+1}+2^n-1)$*.*

The first and the last two in the case $q=2$ are specialization of the $q>2$ conjecture for $q=2$. Again, Theorem 3.1 implies that all of these are eulerian, the converse is conjectural. (This seems to be true up to weight 128 from the numerical data). Note that 4.1, 4.2, 4.3 part (1) and 4.5 conjecturally completely describe all the eulerian tuples, by splicing from the depth two and by using the p-th power map.

Conjecture 4.6. Conjectural zeta-like families of arbitrary depth

(1) For any q*,* $n\geq 1$ *and* $r\geq 2$*, we have*

$$\zeta(q^n-1,(q-1)q^n,\dots,(q-1)q^{n+r-2})=\frac{[n+r-2][n+r-3]\cdots[n]}{[1]^{q^{n+r-2}}[2]^{q^{n+r-3}}\cdots[r-1]^{q^n}}\zeta(q^{n+r-1}-1).$$

(2) For any q*,* $n\geq 0$*,*

$$\zeta(1,q^2-1,(q-1)q^2,\dots,(q-1)q^{n+1})=\frac{[n+2]-1}{\ell_1[n+2]}\frac{1}{\ell_1^{(q-1)q^n}\ell_2^{(q-1)q^{n-1}}\cdots\ell_{n-1}^{(q-1)q^2}\ell_n^{q^2}}\zeta(q^{n+2}).$$

(3) For $q>2$*,* $n\geq 0$ *and* $r\geq 2$*,*

$$\zeta((q-1)q^n-1,(q-1)q^{n+1},\dots,(q-1)q^{n+r-1})$$

equals

$$\frac{(-1)^{r+1}[n+r-1][n+r-2]\cdots[n+1]}{[1]^{(q^{r-1}-1)q^n}[2]^{(q^{r-2}-1)q^n}\cdots[r-1]^{(q-1)q^n}}\zeta(q^{n+r}-q^n-1).$$

It seems quite likely that these families can be proved by a proof similar to that of Theorem 3.2 below, but this has not been carried out yet. Note also that in the depth 2 case, all of these are proved in Theorem 3.1. (Here the part (2) reduces for $n=0$ to part (5) of Theorem 3.1 by the usual conventions on empty products, sums, patterns and indexing).

5. Proofs

The following formulas, which are consequence of Theorems 1 and 3 in [LT], will be used in the proof of the main theorem.

(1) For $1 \le s < q$ and $0 \le k_i < k$ with $1 \le i \le s$, we have

$$S_d(q^k - \sum_{i=1}^{s} q^{k_i}) = \ell_d^{(s-1)q^k} S_d(q^k - q^{k_1}) \cdots S_d(q^k - q^{k_s}). \tag{7}$$

(2) For $1 \le s \le q$, and any $0 \le k_i \le k$, we have

$$S_{<d}(\sum_{i=1}^{s}(q^k - q^{k_i})) = \prod_{i=1}^{s} S_{<d}(q^k - q^{k_i}). \tag{8}$$

We also recall Carlitz' evaluations (see e.g., [T2009, 3.3.1, 3.3.2])

$$S_d(a) = 1/\ell_d^a, \quad (a \le q) \tag{9}$$

$$S_d(q^j - 1) = \ell_{d+j-1}/\ell_{j-1}\ell_d^{q^j} \tag{10}$$

$$S_{<d}(q^j - 1) = \ell_{d+j-1}/\ell_j \ell_{d-1}^{q^j}, \tag{11}$$

Proof of Theorem 3.1. Let $a = q^n - \sum_{i=1}^{s} q^{k_i}$ and $b = (q-1)q^n$. By definition, we have

$$\zeta(a, b) = \sum_{d=1}^{\infty} S_d(a, b) = \sum_{d=1}^{\infty} S_d(a) S_{<d}(b).$$

Using (7), (8), (10) and (11), by straight calculations we get

$$S_d(a)S_{<d}(b) = \frac{(-1)^s}{\ell_1^{q^n}} (\prod_{i=1}^{s} [n - k_i]^{q^{k_i}}) S_{d-1}(a + b).$$

By summing over d the claim (1) follows. The proofs of claims (2) and (6) are similar, once we note that for (2) we have

$$a + b = q^{n+2} - (q - s_1 - s_2)q^{n+1} - \sum_{i=1}^{s_2} q^{k_i},$$

and for (6), the requirement $q > 2$ guarantees that formula (7) can be applied for $S_d(a)$ and $S_{d-1}(a + b)$. In order to apply (8), note that $b = (q-1)(q^{n+1} - q^n) + q^{n+1} - q^{n-j}$.

Now, let $a = q^2 - (q - 1)$ and $b = (q - 1)(q^2 - q) + (q^2 - 1)$. Using formulas (7) and (8) again, a straight calculations yields

$$S_d(a, b) = \frac{1}{\ell_1^{q^2-1}\ell_2} \frac{t^q - t^{q^{d+2}}}{\ell_{d-1}^{q^3}}.$$

Recall that the inverse around origin of the Carlitz exponential $e_C(z)$ is the Carlitz logarithm $\log(z) = \sum z^{q^d}/\ell_d$ and it satisfies $t\log(z) = \log(tz)+\log(z^q)$. Therefore, $t\log(1) = \log(t)+\log(1)$ or equivalently $\log(t) = (t-1)\log(1)$. Since $\zeta(1) = \log(1)$ and $\zeta(1)^{q^3} = \zeta(q^3)$, by summing over d we get

$$\begin{aligned}\sum_{d=1}^{\infty} \frac{t^q - t^{q^{d+2}}}{\ell_{d-1}^{q^3}} &= t^q \sum_{d=0}^{\infty} \frac{1}{\ell_d^{q^3}} - \sum_{d=0}^{\infty} \frac{t^{q^{d+3}}}{\ell_d^{q^3}} \\ &= t^q \log(1)^{q^3} - \log(t)^{q^3} \\ &= t^q \zeta(q^3) - (t^{q^3} - 1)\zeta(q^3) \\ &= (-[2]^q + 1)\zeta(q^3),\end{aligned}$$

and claim (3) follows.

Now, for (4), let $a = 2q-1$ and $b = q^2 - q + (q-1)(q^2-1)$. We have

$$S_d(a,b) = \frac{1}{\ell_1^{q+1}\ell_2^{q-1}} \frac{(-[d+1]^q)}{\ell_{d-1}^{q^3}}.$$

By summing over $d \geq 1$, we obtain

$$\sum_{d=1}^{\infty} \frac{t^q - t^{q^{d+2}}}{\ell_{d-1}^{q^3}} = (-[2]^q + 1)\zeta(q^3),$$

and the result follows.

Finally, (5) is proved in [T2009, Thm. 5] □

Proof of Theorem 3.2. We claim that

$$S_d(1, q-1, q(q-1), \cdots, q^n(q-1)) = \frac{1}{\ell_{n+1}\ell_n^{q-1}\ell_{n-1}^{q(q-1)} \cdots \ell_1^{q^{n-1}(q-1)}} S_{d-(n+1)}(q^{n+1}).$$

Summing the claimed equality over d proves the Theorem.

For $n = 0$ and all d, this is proved in [T2009, 3.4.6]. We prove it by induction by assuming it for n replaced by $n-1$, and considering it for n as claimed.

For $d < n+1$ both sides are zero, and for $d = n+1$, it follows using (9) for $a = 1, q-1$ together with the obvious $S_d(q^n j) = S_d(j)^{q^n}$. We write $s_n(d) := \sum_{j=0}^{d-1} S_j(q-1, q(q-1), \cdots, q^n(q-1))$ and $f_n(d) := \ell_d/(\ell_{n+1}\ell_n^{q-1} \cdots \ell_1^{q^{n-1}(q-1)}\ell_{d-(n+1)}^{q^{n+1}})$. It is enough to show that $s_n(d) = f_n(d)$ for all $d > n+1$. Now $s_n(d+1) - s_n(d)$ is

$$S_d(q-1, \cdots, q^n(q-1)) = S_d(q-1) \sum_{j=0}^{d-1} S_j(q(q-1), \cdots, q^n(q-1)) = S_d(q-1)s_{n-1}(d)^q.$$

Now $s_{n-1}(d) = f_{n-1}(d)$ by induction, and a simple manipulation shows that $f_n(d+1) - f_n(d) = S_d(q-1)f_{n-1}(d)^q$ thus completing the proof of the claim and the theorem by induction. □

Remarks. It might be worthwhile to point out a very special low weight case of (2) of Theorem 3.1 that $(n, m(q-1))$ is zeta-like, if $1 \leq n \leq q$ and $n \leq m \leq q$.

6. Data

Theory of continued fractions for function fields was first developed by Emil Artin in his thesis. (See [T2004, Chap. 9] for a survey). We use them to find the zeta-like values as follows. We calculate the multizeta divided by zeta of the same weight numerically (i.e., approximation where we use first few degrees rather than all), and calculate its continued fraction. If the ratio of actual values is rational, the continued fraction thus calculated will be the same as the continued fraction of this rational for the first few partial quotients and then there will be very large partial quotient indicating small error in approximation. We detect this and then we double check by increasing the precision that we do get the stabilized part, followed by increasing partial quotient (corresponding to reducing error), followed by non-stabilized part.

We provide two examples at the suggestion of the referee.

Example 6.1. Let $q = 3$. We approximate $\zeta(1,2,6)/\zeta(9)$ by

$$f_d = \sum_{j=0}^{d} S_d(1,2,6) / \sum_{j=0}^{d} S_d(9)$$

for $d = 2, \ldots, 5$. The degrees of the numerator and denominator of the rational function f_d are in the last two columns of Table 6.1.1. Once we detect very large partial quotients we double check by increasing the precision. Thus we guess $\zeta(1,2,6)$ is zeta-like.

Table 6.1.1. Degrees of partial quotients of approximations to $\zeta(1,2,6)/\zeta(9)$

d	Degree of the partial quotients (degree of zero $= -1$)	D. num	D. den
2	$[-1, 18, 9, 3, 3, 60, 3, 3, 9]$	90	108
3	$[-1, 18, 90, 153, 9, 3, 3, 6, 3, 3, 9, 6, 33, 3, 3, 9]$	333	351
4	$[-1, 18, 333, 3, 3, 384, 3, 3, 9, 3, 3, 6, 3, 3, 9, 3, 3, 6, 3, 3]$	927	945
5	$[-1, 18, 1062, 1125, 9, 3, 3, 6, 3, 3, 9, 6, 33, 3, 3, 9, 6, 6, 6, 9]$	3249	3267

Example 6.2. As before, let $q = 3$. We guess $\zeta(1,1,1)/\zeta(3)$ is not zeta like as there are no large partial quotients as we can see from Table 6.2.1.

The calculation was done (in stages, with guesses verified with more data) over several months by programing in SAGE and using laptops and mainframes. In lower depths, and small

TABLE 6.2.1. Degrees of partial quotients of approximations to $\zeta(1,1,1)/\zeta(1)$

d	Degree of the partial quotients (degree of zero = -1)	D. num	D. den
2	[-1, 18, 3, 12, 3, 6, 3, 3, 6, 3, 3]	42	60
3	[-1, 18, 3, 3, 3, 3, 3, 3, 6, 3, 3, 9, 12, 6, 6, 9, 3, 3, 9, 3]	177	195
4	[-1, 18, 3, 3, 3, 3, 3, 3, 6, 3, 3, 9, 15, 6, 3, 3, 6, 3, 3, 3]	579	597
5	[-1, 18, 3, 3, 3, 3, 3, 3, 6, 3, 3, 9, 15, 6, 3, 3, 6, 3, 3, 3]	1797	1815

weights, small q's the calculation was exhaustive (i.e., going through all tuples looking for zeta-like values), and sometimes guesses of higher depth, weight, q's were checked separately to some extent. For $q = 2$, depth 2 and 3 and weight up to 128 and 32, respectively, and for $q = 3$, depth 2 and 3 and weight up to 81, calculation was exhaustive. For $q = 4, 5$, depths 2 and 3, we went through all weights up to q^3, but assuming that s_i is 'even' (called restrictive search) for $i \geq 2$, and $s_i \leq s_{i+1}$. However, we checked to some extent that the tuples not satisfying the increasing condition are not zeta-like. Also, often we decreased the precision, otherwise the calculation would have taken much more time.

We only list *primitive* tuples. The tuples marked with * are covered by the theorems.

6.1. **Data for** $q = 2$**.** Zeta like tuples of depth 2 and weight at most 128.

(1, 1)*	(1, 2)*	(1, 3)*	(2, 5)*	(3, 4)*	(3, 5)*	(4, 11)*
(7, 8)*	(8, 23)*	(15, 16)*	(31, 32)*	(16, 47)*	(32, 95)*	(63, 64)*

$q = 2$. Zeta like tuples of depth 3, weight at most $q^5 = 32$, and more.

(1, 1, 2)*	(1, 2, 4)	(1, 2, 5)	(1, 3, 4)	(3, 4, 8)	(7, 8, 16)
(15, 16, 32)	(31, 32, 64)				

$q = 2$. Some zeta like tuples of depth 4.

(1, 1, 2, 4)*	(1, 2, 4, 8)	(1, 3, 4, 8)	(3, 4, 8, 16)	(7, 8, 16, 32)
(15, 16, 32, 64)	(31, 32, 64, 128)			

$q = 2$. Some zeta-like tuples of depth 5.

(1, 1, 2, 4, 8)*	(1, 2, 4, 8, 16)	(1, 3, 4, 8, 16)	(3, 4, 8, 16, 32)
(7, 8, 16, 32, 64)	(15, 16, 32, 64, 128)	(31, 32, 64, 128, 256)	

$q = 2$. Some zeta-like tuples of depth 6.

(1, 1, 2, 4, 8, 16)*	(1, 2, 4, 8, 16, 32)	(1, 3, 4, 8, 16, 32)
(3, 4, 8, 16, 32, 64)	(7, 8, 16, 32, 64, 128)	(15, 16, 32, 64, 128, 256)
(31, 32, 64, 128, 256, 512)		

6.2. **Data for** $q = 3$**.** Zeta-like tuples of depth 2 and weight up to $q^4 = 81$:

(1, 2)*	(1, 4)*	(1, 6)*	(1, 8)*	(2, 4)*	(2, 6)*
(3, 14)*	(3, 20)*	(3, 22)*	(5, 12)*	(5, 18)*	(5, 20)*
(5, 22)*	(6, 20)*	(7, 18)*	(7, 20)*	(8, 18)*	(9, 44)*
(9, 62)*	(9, 68)*	(9, 70)*	(15, 62)	(17, 36)*	(17, 54)*
(17, 60)*	(17, 62)*	(18, 62)*	(23, 54)*	(25, 54)*	(26, 54)*

Zeta-like tuples of depth 3, weight $\leq q^4 = 81$ and more:

(1, 2, 6)*	(1, 6, 18)	(2, 6, 18)	(1, 6, 20)
(1, 8, 18)	(5, 18, 54)	(7, 18, 54)	(8, 18, 54)
(17, 54, 162)	(23, 54, 162)		

Some zeta-like tuples of depth 4.

(1, 2, 6, 18)*	(1, 6, 18, 54)	(2, 6, 18, 54)	(1, 8, 18, 54)
(5,18, 54, 162)	(7, 18, 54, 162)	(8, 18, 54, 162)	(17, 54, 162, 486)

6.3. **Data for** $q = 4$**.** Zeta-like tuples (restricted) of depth 2, weight $\leq q^3 = 64$:

(1, 3)*	(1, 6)*	(1, 9)*	(1, 12)*	(1, 15)*	(2, 9)*
(2, 21)	(2, 27)	(3, 9)*	(3, 12)*	(4, 27)*	(4, 39)*
(4, 51)*	(4, 57)*	(5, 18)	(5, 24)	(5, 27)	(7, 24)
(7, 36)	(7, 39)	(7, 48)*	(7, 51)	(7, 54)	(7, 57)*
(8, 39)*	(8, 51)*	(10, 51)	(11, 36)*	(11, 48)*	(11, 51)*
(12, 51)*	(13, 48)*	(13, 51)*	(15, 48)*		

Zeta-like tuples (restricted search) of depth 3 up to weight $q^3 = 64$:

(1, 3, 12)*	(1, 6, 24)	(1, 12, 48)	(3, 12, 48)
(1, 12, 51)	(1, 15, 48)		

6.4. **Data for** $q = 5$**.** Zeta-like tuples (restricted) of depth 2, weights ≤ 125:

(1, 4)*	(1, 8)*	(1, 12)*	(1, 16)*	(1, 20)*	(1, 24)*
(2, 8)*	(2, 12)*	(2, 16)*	(2, 20)*	(3, 12)*	(3,16)*
(3, 20)*	(4, 16)*	(4, 20)*	(5, 44)*	(5, 64)*	(5, 68)*
(5, 84)*	(5, 88)*	(5, 92)*	(5, 104)*	(5, 108)*	(5, 112)*
(5, 116)*	(9, 40)	(9, 60)	(9, 64)	(9, 80)	(9, 84)
(9, 88)	(9, 100)*	(9, 104)	(9, 108)	(9, 112)	(9, 116)*
(10, 64)*	(10, 84)*	(10, 88)*	(10, 104)*	(10, 108)*	(10, 112)*
(13, 60)	(13, 80)	(13, 84)	(13, 100)*	(13, 104)	(13, 108)
(13, 112)	(14, 60)	(14, 80)	(14, 84)	(14, 100)*	(14, 104)
(14, 108)	(15, 84)*	(15, 104)*	(15, 108)*	(17, 80)	(17, 100)*
(17, 104)	(17, 108)	(18, 80)	(18, 100)*	(18, 104)	(19, 80)*
(19, 100)*	(19, 104)*	(20, 104)*	(21, 100)*	(21, 104)*	(22, 100)*
(23, 100)*	(24, 100)*				

$q = 5$. Some zeta-like tuples.

(1, 4, 20)*	(1, 20, 104)	(1, 24, 100)	(2, 20, 100)
(4, 20, 100)	(3, 20, 100)	(3, 20, 100, 500)	(19, 100, 500)
(19, 100, 500, 2500)			

Summary of depth and weights classified by eulerian and zeta-like.

q	depth	Eulerian weights	Zeta-like weights
2	2	2, 3, 4, 7, 8, 15, 31, 63	
2	3	4, 7, 8, 15, 31, 63, 127	
2	4	8, 15, 16, 31, 63, 127, 255	
2	5	16, 31, 32, 63, 127, 255, 511	
2	6	32, 63, 64, 127, 255, 511, 1023	
3	2	6, 8, 26, 80	3, 5, 7, 9, 17, 23, 25, 27, 53, 71, 77, 79
3	3	26, 80	9, 25, 27, 77, 79, 233, 239
3	4	80, 242	27, 79, 81, 239, 241, 719
4	2	12, 15, 63	4, 7, 10, 11, 13, 16, 23, 29, 31, 32, 43, 46, 47, 55, 58, 59, 61, 62, 64
4	3	63	16, 31, 61, 64
5	2	20, 24, 124	5, 9, 10, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 49, 69, 73, 74, 89, 93, 94, 97, 98, 99, 109, 113, 114, 117, 118, 119, 121, 122, 123, 125
5	3	124	25, 122, 123, 125, 619
5	4		623, 3119

Acknowledgments. We are grateful to the referee for careful reading and for suggestions to improve clarity. The first author thanks the Centro de Cómputo del Cuerpo Académico Modelado y Simulación Computacional de Sistemas Físicos (UADY-CA-101) from the Universidad Autónoma de Yucatán as well as the University of Arizona for allowing the use of their computer servers for our computations.[2]

References

[AT2009] G.W. Anderson and D. Thakur. *Multizeta values for $\mathbb{F}_q[t]$, their period interpretation and relations between them.* International Mathematics Research Notices, 2009(11):2038–2055, May 2009. 1, 3

[BBB1997] J.M. Borwein, D.M. Bradley and D.J. Broadhurst. *Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k*, Electron. J. Combinatorics **4**(2) (1997) R5, 1-21. 4

[C1935] L. Carlitz. *On certain functions connected with polynomials in a Galois field.* Duke Math. J., 1(2):137–168, 1935. 2

[Ct2001] P. Cartier. *Fonctions Polylogarithmes, nombres polyzeta, et groupes pro-unipotents.* Sem. Bournaki no. 881, Mars (2001). 1

[Ch2012] C.-Y. Chang. *Linear independence of monomials of multizeta values in positive characteristic. To appear in Compositio Mathematica*, `http://arxiv.org/abs/1207.2326`. 1, 2, 4

[CPY] C.-Y. Chang, M. Papanikolas and J. Yu. *A criterion of Eulerian multizeta values in positive characteristic*, Preprint 2013. 1, 3, 4

[CY2007] C.-Y. Chang and J. Yu. *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. 216 (2007), 321-345. 1, 2, 3.

[G1996] D. Goss, *Basic structures of Function Field Arithmetic*, Springer Verlag, NY 1996. 2

[L2011] J. A. Lara Rodríguez. *Relations between multizeta values in characteristic p.* J. Number Theory, 131(4):2081–2099, 2011. 1, 3

[L2012] J. A. Lara Rodríguez. *Special relations between function field multizeta values and parity results.* Journal of the Ramanujan Mathematical Society, 27(3):275–293, 2012. 1, 3

[LT] J. A. Lara Rodríguez and D. Thakur. *Multiplicative relations between coefficients of logarithmic derivatives of $\mathbb{F}_q$-linear functions and applications. 'To appear in the special volume of the 'Journal of Algebra and its applications' in memory of Professor Shreeram Abhyankar'.* `http://arxiv.org/abs/1402.2178`.5

[T2004] D. Thakur, *Function Field Arithmetic*, World Sci., NJ, 2004. 1, 2, 3, 4, 6

[Tbanff] D. Thakur. *Multizeta in function field arithmetic.* To appear in the proceedings of Banff workshop to be published by EMS. 1

[2]Comments after Conjecture 4.5 easily imply that the only Eulerian tuples for $q > 2$ and of depth more than 2 are as in Conjecture 4.6 (1). We checked this out only after being informed by Jing Yu that a recent numerical computation (with a different algorithm based on [CPY]) by his student Yi-Hsuan observed only these tuples in depth more than 2. We thank them for this observation.

[T2009] D. Thakur, *Relations between multizeta values for* $\mathbb{F}_q[t]$, International Mathematics Research Notices, 2009(12):2318–2346. 1, 2, 3, 5, 5

[T2010] D. Thakur. *Shuffle Relations for Function Field Multizeta Values.* Int. Math. Res. Not. IMRN, 2010(11):1973–1980, 2010. 1, 3

[Z2012] D. Zagier. *Evaluation of the multizeta values* $\zeta(2, \cdots, 2, 3, 2, \cdots, 2)$. Annals of Math 175 (2012), 977-1000. 1, 4