

Remembering John Tate*

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In this article¹, I reminisce about my fond interactions with John Torrence Tate (13 March, 1925–16 October, 2019), my teacher, mentor and PhD advisor. It is hard to trace the influence of Tate’s work on other works accurately, as the basic objects, ideas and theorems that he introduced have permeated (and have been generalized by others into widely used theories) throughout mathematics. I briefly describe his many fundamental contributions to number theory and arithmetic geometry, and end with some personal anecdotes.

Tate’s Thesis

As Andrew Wiles put it aptly, “Tate helped shape the great reformulation of arithmetic and geometry which has taken place since the 1950s.”

Already in his 1950 PhD thesis, Tate applied harmonic analysis to give a beautiful local-global understanding of Hecke L -series; paving the way for similar analysis for more general automorphic L -functions. These important special functions, generalizing the famous Riemann zeta function, are highly suitable for packaging and unpacking many arithmetic structures and mysteries.

Cohomology Perspective

Following that came his work, using topological and categorical tools, on the cohomology of number fields and function fields. This, among other things, puts class field theory in a more general and useful perspective.



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¹ Dedicated to the memory of John Tate.

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With students, Tate's 60th birthday conference.

For the uninitiated, class field theory is a well-developed theory, explaining special arithmetic aspects of algebraic number systems with commutative symmetry (Galois) groups. It is considered as a crowning achievement of algebraic number theory in the first half of the 20th century.

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As first year graduate students at Harvard University, we worked hard towards learning this beautiful edifice of class field theory. The Bible book then, was the lecture notes of the Artin-Tate seminar. We were proud when we thought that we had gained a reasonable mastery over this subject with a lot of effort. But, soon we were floored when we heard Tate mention, in a conference talk, that “Such and such a fact is completely trivial: its proof requires absolutely nothing, *just* class field theory.” We realized that the frontier had moved on, and a long march lay ahead of us!

Lubin-Tate

Together with Lubin, Tate then developed what came to be known as Lubin-Tate's ‘explicit’ class field theory in a ‘local’ setting. They explicitly generated these special extensions with abelian Galois groups through formal groups (based on infinite power series) rather than on algebraic groups (which are based on poly-



nomials) that were used earlier in global theory. These are now called Lubin-Tate groups. Tate always humbly insisted though that they be called Lubin groups, as Lubin had introduced them, and that his joint involvement was only restricted to this application. Lubin and Tate also studied the deformations of these formal groups.

Lubin-Tate towers played an important role in the approach (e.g., by Harris and Taylor) to the local Langlands correspondence for GL_n , as well as in the earlier approach by Drinfeld to the Langlands correspondence for function fields.



Tate with some of his mathematical descendants at Arizona winter school, 13 March 2007 (his birthday).

Carlitz Theory

Both Tate and Drinfeld were unaware of some earlier work by Carlitz on explicit cyclotomic covers of fields of rational functions. Although this applies to a ‘global’ context, it has some important relations to Tate’s work. While attending Tate’s lecture on Lubin-Tate theory, David Hayes, a student of Carlitz, recognized the similarity of some formulas with those given by Carlitz and brought this to Tate’s attention. Unfortunately, many nice fundamental works of Carlitz in the 1930’s and 40’s had been

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forgotten, in part, because of unimaginative titles such as ‘Some properties of polynomials’, ‘Class of polynomials’, ‘Set of polynomials’ which gave no information on the contents! Then, as I learned from David Hayes, Tate encouraged and helped David to work out the details of Carlitz’s theory in modern language.



Tate at IAS, Princeton 03 May 2014.

Tate Curves and Rigid Analytic Geometry

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Tate then worked on developing the arithmetic of elliptic curves, often in collaboration with Serre, Mazur and others. Several generations of students were introduced to this fascinating subject by reading Tate’s lucid 1961 Haverford lecture notes. Even today, interested students can do no better than read his beautiful book on rational points on elliptic curves (mostly based on these lecture notes), followed by his more advanced and influential 1974 survey article published in the *Inventiones Mathematicae*. He worked out the Néron model algorithm for elliptic curves, and introduced techniques for determining rational torsion points. Tate also contributed to the general formulation of the Birch and Swinnerton-Dyer conjecture, explained and proved a large part of the analogue for function fields, and worked out p -adic versions. Further, Tate worked on reductions, liftings, and deformations, of the more general higher dimensional versions of elliptic curves, namely abelian varieties.

On one occasion, I mentioned to John that while classical al-



gebraic geometry of global function fields seems to work uniformly for all such fields (especially in the context of their zeta functions) in the context that I was studying, it seemed that each function field had interesting, individual subtleties. Even the rational function fields over a finite field become much harder to study (for example, in the question of distribution of the zeros of their zeta functions) when the finite field does not have a prime number of elements. To this comment, John made an interesting remark (in the nature of a confession). He said that having worked out class field theory using cohomological tools and having seen some important cohomological aspects lurking behind elliptic curves arithmetic, he was (over)-confident that he would find similar success in studying all elliptic curves. But then, he realized, only gradually, the fascinating individual subtleties and intricacies.

In another direction, by making a simple convergence calculation (the insight behind undertaking such a calculation is, of course, crucial), Tate first discovered p -adic uniformization of some elliptic curves, now called Tate curves. Using Grothendieck's ideas, he developed a rigid analytic geometry giving firm foundation to his intuitive conviction that deeper structures underlay the identities proved by these calculations. Tate curves and rigid analytic geometry are now indispensable tools for arithmetic geometers.

I applied the theory of automata to prove the transcendence of periods of a 'Tate elliptic curve' in finite characteristic. I was very happy to be able to dedicate this paper to him on his 70th birthday, and was further elated when he said that he appreciated this 'mathematical gift'.

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Special Values of L -functions

In yet another direction, Tate found arithmetic significance, through his discovery of connections with reciprocity laws and special values of zeta functions, in the so-called K-theory, which was of topological origin. Further important developments by Deligne, Beilinson, Bloch, Kato, Lichtenbaum, Borel, Langlands, etc., on

connections of K -groups with extensions of motives have led to vast areas of mathematics studying the structures behind the arithmetic of the special values of zeta and L -functions.



From Barry Mazur's 80th birthday conference at Harvard, June 2018.

Much later, when I was a graduate student, Tate had become interested in Stark's new conjectures on different aspects of special values of L -functions, and the ideas described in his lectures were written up (by others) in a very influential book.

Stark Conjectures

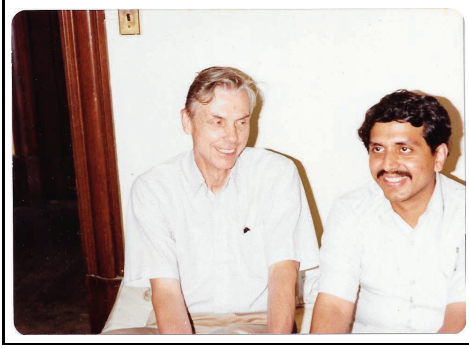
I worked on function field arithmetic, and though Tate did not work on similar aspects, he had a lot of influence on its developments via his earlier work on Tate conjectures, Birch-Swinnerton Dyer, Stark conjectures. I reviewed this influence in an article in [7], reprinted in [6]. Soon after this article was written, another breakthrough was made by Lenny Taelman, giving rise to clean introduction of "class modules and unit modules." This was directly influenced by Tate's formulation in the book mentioned above on Stark conjectures!

Tools for Resolving Mordell's Conjecture

Due to limitations of space, let me just list some of his other impressive works: his analysis and introduction of p -divisible



groups and p -adic Hodge theory has provided very important tools in arithmetic geometry.



At Harvard 1987.

At the end of the first year of my graduate studies at Harvard, Faltings's celebrated proof of the Mordell conjecture was published. It proved that over number fields, curve of genus at least 2 can have only finitely many rational points. Going through it, reveals to us how much Tate's contributions and viewpoints had permeated as key tools of number theory. An indication of the significance of Tate's work in providing tools for Faltings's marvelous proof of this concrete result of great generality on Diophantine equations is provided by the number of terms bearing his name that are used : Tate modules, classification of finite p -groups schemes, p -divisible groups, Tate conjectures, and related Tate isogeny theorem over finite fields.

So pervasive are his varied contributions, that in almost any number theory seminar, it is usually only a matter of time before his name comes up.

For more mathematical and personal details on his times, I refer the interested reader to the bibliography for his actual collected works, and to special issues published in his honor. See especially, his 'Abel interview', Shatz's recollections on mathematical atmosphere in the 1950's and 60's in [7], and a very nice detailed article on his works by Milne in [5]. Other sources are [2] and [4]. Finally, we must mention the celebrated Serre-Tate mathematical

correspondence [1], which is a real treasure for any number theorist. Many of these, together with some of his later lecture videos where his passion for math shines through, and interviews can be easily found on the internet too.

Personal Reminiscences

From the very start, he had told me that I can just call him John, in the American fashion, but being brought up in India in those days, I could not bring myself to abandon my natural inclination towards “Professor Tate” for a long time, and thus I usually just started talking to him without any specific address instead!

While Tate had a reputation as a no-nonsense, ‘brutally’ honest person, he was very friendly, generous and kind. He appreciated that people with very different sets of abilities can still be very good mathematicians: some problem solvers, some theory builders, some quick and sharp, some slow and vague, some who could easily absorb and comprehend involved mathematics like small talk and spot errors as a native speaker would instinctively spot grammatical errors (this is how he described Gabber and Deligne to me one day!) while others having to struggle to understand (he probably put himself in this class!). He believed that the love for the subject, curiosity, dedication, hard work and perseverance are that really mattered.



With John and Carol at their Texas home, April 2004.

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When I saw that the building next to the mathematics building at University of Minnesota was named John Tate building, I wrote to him and learned from him that it was named after his physicist father. John is, in fact, John T. Tate, Junior! As a matter of fact, John started as a physics graduate student, though he always loved math. As he explained, Bell's "Men of Mathematics" portrayal of mathematicians had given him an impression that math is reserved for great talent, while he thought he could do physics, since his father could! Soon, nature took its course, he transferred to mathematics and was very happy to learn with Emil Artin. After his doctorate, he spent three more years at Princeton, and ran some famous seminars with Artin.

In 1954, when Tate came to Harvard as a faculty, Indian mathematician Shreeram Abhyankar was a graduate student of Zariski there. Abhyankar told me that he was fortunate enough to learn some class field theory from John, who had then recently worked with Emil Artin on its reformulation. When I mentioned this to John, he said he remembers learning about higher rank valuations from Abhyankar!

Talking to John always gave us a better understanding because of his deeply penetrating, incisive comments always in search of beauty, clarity and the right way of thinking. Proofs did not satisfy him till we got the right ones. In fact, long after my PhD thesis defense, he simplified one of my proofs, finally getting the 'right' one!

Tate loved to calculate good examples. Often, he would quote Siegel complaining that these days there is a tendency to give only trivial examples such as a set which is empty, a group with one element and a field with two! Then he would go on to produce a beautiful, tantalizing and convincing example of just the right complexity. In his lectures, he always thought on his feet and often got carried away or even got stuck, while raising new questions, trying to improve on the spot. But rather than polished and smooth lectures, often it was his struggle that taught us a lot more! His ability to go straight to the heart of the matter, his appreciation for structural beauty, his care for the fine details of

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his examples, and his urge to try to see everything in the right perspective always showed through.



John and Carol at their Boston home, 31st December 2016.

Having more time on his hand when writing a paper (and indeed famous for taking his time!), his papers portrayed perfection, polish and a sense of architecture, on the other hand.

He had a professional attitude, taking pride in perfection and working hard towards it. When I started working as a graduate student and asked him for one of his old reprints, he gave it only after correcting a missed hypothesis in it and explaining to me that mistake.

If one of his students exhibited improved understanding of some published paper or even found a minor error in what he himself was saying, his face would light up with the joy of being corrected.

He had an uncanny sense of deep mathematical structures, probably further developed through his association with Emil Artin, Serre, Grothendieck and Bourbaki. He was a master of cohomological yoga and used categorical language as a tool, though he often used concrete calculations to supplant this.

On one occasion, I asked Tate about one particular theorem of Grothendieck, hoping to just get some basic insights. Initially, he did not even understand what the theorem said, then slowly he unwound the notation and specialized to the setting of Brauer



groups and class field theory, which was his home territory. As soon as he understood the crux of the special case, it was child's play for him to go back to the general setting and give a complete general proof. Those two hours, I really witnessed a miraculous tour de force giving a complete proof of an important hard theorem, starting from the scratch. Such was his intellectual strength.

The last course that I took with John discussed Raynaud's generalization of his work on group schemes, Fontaine's 'no non-trivial abelian schemes over \mathbb{Z} ' result (techniques used here are now fundamental tools in applications to Diophantine geometry), Fontaine-Winterberger fields of norms and Kazdan-Deligne tricks for moving from characteristic p to 0 in local setting (fundamental ideas recently beautifully generalized in Scholze's work). Looking back, I marvel at how much of a visionary John was in thinking towards the important work and issues of the future.

When I talked with him about some of my recent work on Wieferich-Wilson primes, which involved factorials in multiplicative and additive settings in function fields, he told me an amusing story from his childhood: When he first saw, in school, the factorial ' $n!$ ' which is the product of first n natural numbers, his immediate reaction was: why is nobody looking first at the sum of first n natural numbers? He proceeded introducing and studying ' $n?$ ' (his notation: 'question mark rather than exclamation mark!'). Of course, he soon realized why (recall the famous story of 'Gauss trick evaluation').

When Tate was the chair of the Harvard Mathematics Department, serving his turn by rotation, the department evaluating committee visited the mathematic department and then asked him directions to the applied math building. But as he told me, even after many years at Harvard, he had no idea of even where it was! It was just his presence of mind that saved him when he told them that his secretary will take them there. Of course, he loved mathematics in all its aspects, pure or applied, though he himself focused on a few pure math areas. Though in his early career he protested NATO or governmental funding, he had worked for Navy during the World War II crisis, and much later, worked for

the National Security Agency, helping applied security projects (some of his work on elliptic curves and cohomological dualities even found cryptological applications) after USA came under terrorist attacks in 2003.

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He was a thorough gentleman, and very kind to his students. When I visited UTexas a few times for seminars, he would come to fetch me at the airport in Austin, take me to the hotel, and invite me to their house. In Tucson, a few times he visited his granddaughter Ginger Clausen, who was a graduate student in philosophy then. In 2013, when I was in London, he and his wife Carol had come for such visit, and not only called my wife Jyoti, but insisted that they would take her out for dinner. She was touched and overwhelmed by their generosity and warmth.

I cannot resist telling a funny story from that visit, as it sheds light on mathematicians' passion for their field. In 1988–1989, at the Institute for Advanced Study, Princeton, Andre Weil, who had a soft corner for India, Indian philosophy, and Indian food, having spent a few years in India on his first job and having learnt the *Bhagawad Gita*, came to our house for dinner a few times. During one of these visits, he asked Jyoti whether she had heard of the Riemann hypothesis, and when she confessed that she had not, he spent 15 minutes patiently explaining it to her, and explaining how it was his unfulfilled life ambition to prove it. (After he left, Jyoti, in her innocence, said to me, 'why don't you prove the Riemann hypothesis? It will make him very happy'!). John was delighted to hear this story from Jyoti, and asked her what she remembered about the Riemann hypothesis. When she confessed that she does not remember anything beyond 'zeros of something being on some line of symmetry', he then proceeded giving her a long patient explanation of its statement and significance! I really wish I was there! After all, in addition to Weil's proof of Riemann hypothesis for curves, there is also Mattuck–Tate geometric proof of Castelnuova–Severi inequality leading to the Riemann hypothesis for curves (and trying to understand this proof led Grothendieck to his Hodge index theorem!).

On Easter day of 1964, Grothendieck gifted John a painted easter



egg (see the cover photograph in this volume) with some of their important theorems, such as Tate isogeny theorem and Grothendieck–Riemann–Roch theorem, carved on it in his own handwriting! We amused ourselves with the idea that when we refer to it, we would give the reference as ‘GRR theorem, see Grothendieck–Egg to Tate’.

John was an enthusiastic supporter and a frequent participant of our annual Arizona winter schools (now in its 23rd year) for number theory graduate students. He even wrote a (rare) preface giving the background history of the subject (as well as his blessings for our effort, I would like to think!) for the lecture notebook on p -adic geometry that we produced. The winter schools usually took place in March and often we celebrated his birthday (starting with his 75th birthday which was celebrated with toasts and anecdotes by Nick Katz and Jonathan Lubin) there when it fell in during its dates. This tradition continued, with unanimous enthusiasm and admiration from the gathered mathematics community, even when he stopped attending due to his advanced age.

He had retained his passion and enthusiasm for mathematics even in old age, when his physical faculties were diminishing. I was touched when he drove and came to my seminar at MIT, when he was over 90. We discussed some mathematics. He liked some new conjectures I had then made (*arXiv1512.02685*), tried to verify himself by calculating in a different way, and later sent me a few emails encouraging me and continuing discussions. Last I met him was at Barry Mazur’s 80th birthday conference in June 2018, when he made it a point to attend it at least for a day.

John was very proud of his mathematician grandson Dustin Clausen. When he realized Dustin’s strong interests and abilities in mathematics, he nurtured them by giving him challenging exercises. Dustin (among many other fine achievements) extended product formula to higher K -groups, and gave a new perspective to Artin reciprocity law (both areas where Tate had made crucial contributions) using modern topological insights. Tate’s article [3] (probably his last one, published after his collected works) mentions this work, and his last email to me in March 2019,

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proudly mentions that Dustin’s new work has been appreciated by Scholze.

I spent October 2019 at Max Planck Institute in mathematics at Bonn, and there I heard the sad news of John’s passing away through email from Jeremy Teitelbaum (we overlapped as fellow students of John). I derived at least some comfort learning that just the day before, when Carol read my email to him about Dustin’s excellent seminar at MPI on his work with Scholze, John had smiled.

Tate’s work has been recognized by many prestigious awards such as 1956 Cole Prize, and lifetime achievement recognitions such as 1995 Steele Prize, 2002/3 Wolf Prize, 2010 Abel Prize. With his usual modesty, he always warmly thanked the committees for “selecting my career from the many equally or more deserving ones for this award” and said, “a lifetime of mathematical activity is a reward in itself,” which summed up his passion and attitude towards mathematics².

²I thank J-P Serre and B Sury for their helpful comments and corrections on the draft.

Suggested Reading

- [1] *Correspondance Serre-Tate*, 2 volumes, Edited by P. Colmez, J. P. Serre, Soc. Math. France 2015.
- [2] *Collected Works of John Tate*, Edited by B. Mazur, J. P. Serre, Amer. Math. Soc., 2016.
- [3] John Tate, *Number theory in 20th century*, Part I, in [4]
- [4] Tate issue, *Bulletin of Amer. Math. Soc.* 54, No.4, October 2017.
- [5] *The Abel Prize 2008-2012*, Edited by Helge Holden, Ragni Piene, (John Tate Pa. pp.249–349), Springer, Heidelberg, 2014.
- [6] *Wolf prize in Mathematics*, Edited by Y. Sinai, E.Stein, Vol.4, (John Tate pa. pp.443–515, World Scientific, 2012.
- [7] *Special Issue: In Honor of John Tate*, *Pure and Applied Mathematics Quarterly*, Vol.5, No.4, 2009 (Part 1) and Vol.6, No.1, 2010 (Part 2).

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