ZETA-LIKE MULTIZETA VALUES FOR HIGHER GENUS CURVES

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Dedicated to Jean-Pierre Serre

ABSTRACT. We prove some (and conjecture more) relations between the multizeta values for positive genus function fields of class number one, focusing on the zeta-like values, namely those whose ratio with the zeta value of the same weight is rational (or conjecturally equivalently, algebraic). These are the first known relations between multizetas, which are not with prime field coefficients. We seem to have one universal family. We also find that, interestingly, the mechanism with which the relations work is quite different from the rational function field case, raising interesting questions about the expected motivic interpretation in higher genus.

1. INTRODUCTION

Recently studied connections of the multizeta values

$$\zeta(s_1, \cdots, s_r) := \zeta_{\mathbb{Z}}(s_1, \cdots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R},\ (s_i \in \mathbb{Z}, s_i \geq 1, s_1 > 1),$$

introduced by Euler, with the arithmetic fundamental groups have made them an important tool in the recent push towards non-abelian, homotopical directions in number theory. See e.g., [Z16] and references there to the huge body of work by several mathematicians.

For a survey of work on function field analogs of multizeta, with connections to Drinfeld modules and Anderson's t-motives (see [A86, G96, T04] for background), we refer to the survey [T17]. For the definitions of the multizeta values $\zeta(s_1, \cdots, s_r) = \zeta_A(s_1, \cdots, s_r)$, now defined for certain analogs A of $\mathbb{Z}$ and taking values in appropriate completions of the corresponding function fields, we refer to the Section 2 below.

Let us focus on very simple type of relations between the multizeta values. Following [LT14], we call a multizeta value zetalike, if its ratio with the zeta value of the same weight, i.e., $\sum s_j$ in the notation above, is rational. In the special case of even weight (for a function field over $\mathbb{F}_q$, this 'even' condition gets replaced by the analog 'q-even', i.e. multiple of $q-1$), we also call it Eulerian. (Often we restrict to multizeta of depth more than one, without mention, since only then the concept is really significant). In the case of rational number field (i.e., $\zeta = \zeta_{\mathbb{Z}}$), we know some eulerian families [LT14], but we speculated that only $\zeta(2m+1)$ and its 'dual' (see [Z16] for the explanation of this terminology) $\zeta(2, 1, \cdots, 1)$, where 1 is repeated $2m-1$ times, may be the only multizetas that are zetalike of odd weight.

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In contrast, we proved [LT14] (see also [CPY19, Ch17, T17]) some multizeta families to be Eulerian for the rational function field case, and conjectured that these are the only Eulerian multizetas, but could only prove and conjecture several zetalike families of weights which are not q-even, without getting full characterization, even conjecturally.

For the t-motivic interpretation of these notions of zetalike and eulerian, at least in the rational functions field case, we refer the interested reader to [CPY19].

In this paper, we investigate this question for higher genus function fields, where, up till now, the only relations known [T10] were the sum shuffle type relations (with prime field coefficients), and the obvious relations $\zeta(ps_1, \cdots, ps_r) = \zeta(s_1, \cdots, s_r)^p$ in characteristic p, which are also derivable from the more complicated sum shuffle type relations mentioned. In both number field and function field cases, the shuffle relations reduce the study of algebraic relations to linear relations.

The t-motivic period interpretation [AT09] in depth more than one is also only developed so far in genus zero. Now we find (and can prove some) much more interesting relations involving non-prime field coefficients. See the conjectures and theorems in Sections 3 and 5 below.

Several years ago, the second author had checked (numerically) that $\zeta(1, q-1)$ is not zetalike (for one class number one elliptic curve over $\mathbb{F}_q$, with $q = 2$), in contrast to what he had proved [T04, T09] in the rational function field case over $\mathbb{F}_q$. That this multizeta being zetalike is (conjecturally) the only non-trivial linear relation in weights at most q. So in contrast to the rational function fields, in higher genus, it seems that the relations start at higher weights. (It seems at weight $q^2 - 1$, see Section 3 for details.)

Now with more extensive use of computer aided numerical experiments, we have better understanding (see below) of what should happen and some 'positive identifications' of zetalike multizeta.

We then prove some of these conjectures by developing further, from the zeta case to multizeta, the 'polylog-algebraicity techniques' of [T92], where an appropriately constructed algebraic function on the curve (corresponding to the function field) cross itself (or its Hilbert cover cross itself, in general) specialized at the graph of the d-th power of the Frobenius endomorphism gives appropriate power sums of degree d (or at most d) times the d-th coefficient of appropriate polylog, for all d. (See Section 3, and Theorem 3.5 in particular, for several examples.) In [T92], these special algebraic functions (called F-functions) were used to give motivic algebraic incarnation of some zeta values (especially at 1, in class number one situation) generalizing partially the results of [AT90] to higher genus.

In 2009, these were used to verify [T20] that Taelman's beautiful analog of analytic class number formula [Ta10], which was then made only for the base $\mathbb{F}_q[t]$, works also for the higher genus cases of class number one. Aspects of log-algebraicity were developed much further in various directions e.g., [A94, A96, B13, F15, APT16, D16, GP18, M18] by Anderson, Anglès, Böckle, Debry, Fang, Green, Mornev, Ngo Dac, Papanikolas, Pellarin, Taelman, Tavares Ribeiro, and [T04][Sec. 8.9-8.10]. Since we do not make any use of these developments in this paper, we just refer the interested reader to the original papers, by only making a remark that to use them, we would need to extend these techniques to adapt to multizeta. We have not resolved this issue of the extent of log-algebraicity for multizetas fully in this paper, but have just developed it sufficient to prove our theorems and to illustrate the techniques.

Interestingly, the proofs as well as the mechanisms how these identities work out at infinite level, as limits from finite levels, are now quite different than in the genus zero case. It shows that we will need a better understanding of the underlying t-motivic mechanisms to understand the situation in general. In informal terms, the motives here are constructed through such algebraic functions on products of curves and the motivic identities are identities between such functions, and the iterated sum mechanism comes via the Frobenius-difference matrix equations [AT09][Sec. 2.5] that arise in Anderson's theory of t-motives. We expect (and know to a large extent by the works of Anderson, Brownawell, Papanikolas, Chang and Yu) that the multizeta values relations come from motivic identities, but in our higher genus, in contrast to what we know in genus zero, though it is certainly not ruled out, we have not been able to do this (as explained in Section 4), but have resorted to different mechanism of proofs, which raise some interesting questions and formulations. Roughly, in addition to the algebraic geometric techniques (which we will only identify by keywords such as Drinfeld modules, Shtukas, and Frobenius difference equations), in our proofs, we had to also use the limit processes. See Remark (II) in Section 4 for more on this.

We find (with only numerical evidence in low weights) exactly one zetalike/ eulerian family $\zeta(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+k})$ (where q is the cardinality of the field of constants $\mathbb{F}_q$) surviving from the rational function field case, for all (4 of them) class number one situations of higher genus. (Here we have restricted, without loss of generality, to 'primitive' tuples (s_i), i.e. those which are not multiples of p). We have not yet found any nontrivial zetalike example in weights which are not q-even, in the higher genus case.

We did not find any zetalike examples in higher class number function fields, and speculate that probably there are families of Hilbert class field coefficient linear combinations of multizeta values of different ideal classes (for the same tuple of s_i's) that are algebraic multiples of zeta of the same weight, but it might be rare or impossible for a single value to be zetalike in this case, unless you use variant definitions (see e.g., [G96, T04]) of the multizeta taking all ideal classes into account.

In the function field analog that we investigate (see [T17] for survey and references), the relations are still not conjecturally well-understood, though in contrast, there are also some very strong transcendence and linear/algebraic independence results (by Anderson, Brownawell, Chang, Papanikolas, Yu, et al) proved. Note that the various transcendence, independence results that have been proved for the zeta immediately carry over to the zetalike multizeta.

Here is the organization of the paper. We first fix the notation and give the basic definitions. Next, we state our conjectures on the zetalike families and give the proof of the special cases of conjectures in one example of class number one and positive genus.Then we give several remarks on possible generalization of the proof techniques, the contrasts with the genus zero case and the motivic implications. Then we discuss the relative situation, as well as the numerical methods, and the data, calculated by the first author, for some variants explored. Finally, we give all the computational details of the similar proofs of the cases for the remaining three class number one examples of positive genus.

2. Notation and definitions

Consider a function field K (of one variable over finite field $\mathbb{F}_q$), having a rational place i.e., a place of degree one. We choose any such place and label it ∞. Denote the corresponding ring of integers by A (consisting of elements of K having no pole outside ∞), the completion of K by K_∞, and the completion of its fixed algebraic closure by C_∞. Fix a uniformizer at infinity, so that we have corresponding sign (and degree) function. Let $A+$ (A_d+, respectively) denote the set of monic, i.e., of sign 1 (monic of degree d respectively) elements of A.

For $k, k_i, d \in \mathbb{Z}$, consider the power sums (sometimes denoted by $S_d(-k)$ in the references)

$$S_d(k) = \sum_{a \in A_d+} \frac{1}{a^k} \in K,$$

and extend inductively to the iterated power sums

$$\begin{aligned} S_d(k_1, \cdots, k_r) = &\qquad S_d(k_1) \qquad S_{<d}(k_2, \cdots, k_r) \\ = &\qquad S_d(k_1) \qquad \sum_{d > d_2 > \cdots > d_r} S_{d_2}(k_2) \cdots S_{d_r}(k_r), \end{aligned}$$

where $S_{<d} = \sum_{i=0}^{d-1} S_i$ as the notation suggests.

For positive integers s_i, we consider the multizeta values

$$\zeta(s_1, \cdots, s_r) := \sum_{d=0}^{\infty} S_d(s_1, \cdots, s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

of weight $\sum s_i$ and depth r (associated, a priori, to the tuple s_i rather than the value). (Here the second sum is over monic $a_i \in A$ of strictly decreasing degrees).

Call $\zeta(s_1, \cdots, s_r)$ *zetalike* (we only care, if $r > 1$) if $\zeta(s_1, \cdots, s_r)/\zeta(\sum s_i) \in K$.

In the case the weight $\sum s_i$ is q-even (i.e., a multiple of $q-1$), we also call the zetalike value *eulerian*, in recognition of the simple evaluation by Euler in the rational case, and analogous evaluations [C35, T04] by Carlitz and Goss in function fields.

Finally, for ρ a sign normalized rank one Drinfeld A-module (also called Hayes module), we denote the corresponding exponential and logarithm functions as $\exp_\rho(z) = \sum z^{q^i}/d_i$ and $\log_\rho(z) = \sum z^{q^i}/\ell_i$. While ℓ_i and d_i are polynomials in t in the $A = \mathbb{F}_q[t]$ case, in higher genus case, they are rational functions (non-integral in general) in the Hilbert class field. (see e.g., [T04], Chapter 2 for details).

3. Class number one situation: Conjectures and theorems

Apart from $A = \mathbb{F}_q[t]$'s (one for each prime power q), there are exactly four (see [T04] for references and corresponding Hayes modules) other A's of class number one:

- (i) $\mathbb{F}_2[x, y]$, with $y^2 + y = x^3 + x + 1$,
- (ii) $\mathbb{F}_3[x, y]$, with $y^2 = x^3 - x - 1$,
- (iii) $\mathbb{F}_4[x, y]$, with $y^2 + y = x^3 + w$, where $w^2 + w + 1 = 0$,
- (iv) $\mathbb{F}_2[x, y]$, with $y^2 + y = x^5 + x^3 + 1$.

Note that the first three are of genus 1 while the last one is of genus 2.

Conjecture 3.1. *For any class number one A with constant field $\mathbb{F}_q$, the multizeta values $\zeta(q^n-1,(q-1)q^n,\cdots,(q-1)q^{n+k})$ are zetalike (equivalently eulerian, in this case.)*

Remarks 3.2. (i) For the case of $A=\mathbb{F}_q[t]$'s, following more explicit form below was conjectured, proved in depth 2 in [LT14] and proved for any depth by Chen in [Ch17].

$$\zeta(q^n-1,(q-1)q^n,\ldots,(q-1)q^{n+k})=\frac{[n+k][n+k-1]\cdots[n]}{[1]^{q^{n+k}}[2]^{q^{n+k-1}}\cdots[k+1]^{q^n}}\zeta(q^{n+k+1}-1),$$

where $[n]=t^{q^n}-t$.

(ii) In genus zero case, there are more such families [LT14], but in higher genus, our limited exploration leads only to the family in the conjecture. (Of course, we restrict to 'primitive' tuples, i.e. not divisible by the characteristic).

Here are some more explicit conjectures, when $k=0$, in higher genus, class number one cases, listed above.

Conjecture 3.3. *Put $R_n=\zeta(q^n-1,q^n(q-1))/\zeta(q^{n+1}-1)$.*

For the case (i), we have

$$R_n=\frac{x^{2^{n+1}}+x^2}{y^{2^{n+1}}+y+x^{2^{n+1}+1}+x}.$$

For the case (ii), we have

$$R_n=\frac{(x^{3^n}-x)(y^{3^n}-y)^2+(x^{3^n}-x)(-x^{3^n}-x^3-x+1)}{x^{2\cdot 3^{n+1}}+x^{1+3^{n+1}}+x^{3^{n+1}}+y^{1+3^{n+1}}+x^2-x+1}.$$

For the case (iii), we have

$$R_n=\frac{(x^{4^n}+x)(y^{4^{n+1}}+y^4)+(x^{4^{n+1}}+x^4)\left(x^{4^n+2}+x^3+1\right)+(x^{4^n}+x)}{x^{2\cdot 4^{n+1}+2}+x^{4^{n+1}}y^{4^{n+1}}+x^{4^{n+1}}y+x^{4^{n+1}}+xy^{4^{n+1}}+xy}.$$

For the case (iv), we have

$$R_n=\frac{X^{22}+(1+x)(X^{20}+X^{18}+X^{16})+(1+x+x^2)(X^{12}+X^{10})+X^9+LR_n}{X^{24}+xX^{16}+(x+1)X^8+x^2+x},$$

where $X=x^{2^{n-1}}$ and $Y=y^{2^{n-1}}$ and $LR_n=xX^8+X^5+(Y+y)(X^2+X^4)+x^2+x$.

Note that in addition to the non-uniqueness of expressions coming through the relations between x,y, the fractions in the conjecture are not in the reduced form either, and in fact, there is a lot of cancellation (making it hard to guess from numerical data!). Compare, for example, the reduced forms in the special case of the theorems below. Thus, to make these guesses, we had to use theoretical ideas and expressions found in the proof of the first theorem below, then generalize and verify. See remark (V) of Section 4 for some indication. We have numerically verified the case (i) for $n\leq 11$, and (ii) for $n\leq 9$, (iii) for $n\leq 5$ and (iv) for $n\leq 12$.

We also have some more such explicit ratio conjectures, but not yet for large satisfactory families.

Our main theorems below prove the following cases of the conjecture in higher genus for $k=0$: the case (i), when $n=1,2$ (Theorems 3.4 and 5.1 respectively), the cases (ii, iii, iv) when $n=1$ (Theorems 8.1, 8.2, 8.3 respectively).

We start with case (i) and note here that for this example, the Taelman's class module [Ta10] is also trivial (as proved in [T20]), in addition to the class group. We will use the theory of [T92] (see also [T04][Sec. 4.15, 8.2]), but recalling everything (parts (i) to (v) of Theorem 3.5) that we need from there.

Theorem 3.4. *For $A = \mathbb{F}_2[x, y]/(y^2 + y + x^3 + x + 1)$, we have*

$$(x^2 + x + 1)\zeta(1, 2) = \zeta(3).$$

Proof. We first give some explicit '(poly-)log-algebraicity' formulae developed in [T92] for the relevant power sums, and in [T09] for the iterated versions, together with one extension needed for the multizeta case. These give expressions for polylog-coefficient ℓ_d^w times power sums $S_d(k_1, \cdots, k_i)$'s in terms of algebraic functions, on the curve (corresponding to K) cross itself, specialized at the graph of d-th power of Frobenius map.

We will now define several functions in $\mathbb{F}_2(x, y, X, Y)$, where x and X are independent transcendentals and $y^2 + y = x^3 + x + 1, Y^2 + Y = X^3 + X + 1$. For each such function, say f, put $f^{(k)}$ for the function resulting from f after substituting X^{2^k}, Y^{2^k} respectively for X, Y, and put $f(d) \in K$ for the function resulting from f after substituting x^{2^d}, y^{2^d} for X, Y.

More generally, we say [T92], in class number one case, that a function $h : \mathbb{Z}_{>c} \to C_\infty$ is F-function, if there is a rational function H on C cross itself such that H specialized to the graph of d-th power of Frobenius on C is $h(d + k)$, i.e., the value of h at $d + k$ (for sufficiently large d, fixed k). Since we use such usual function notation $h(k)$ below, only for the various power sums $S_d(k)$'s, there should be no confusion between the usual functional notation and the 'twist' notation introduced in the previous paragraph.

Put

$$B_x = X + x, \quad B_y = Y + y, \quad g = \frac{B_y + XB_x}{B_x^{(1)} + 1}, \quad F_1 = \frac{X + x^2}{B_y + xB_x + x^2 + x},$$

$$F_{<1} = g^2 + F_1^2 + F_1, \quad F_{12} = F_1 F_{<1}^2, \quad F_3 = F_1^2(g^2 + F_1^2),$$

$$g_m = \frac{Y^2 + y^4 + X^2(X^2 + x^4)}{X^4 + x^4 + 1}, \quad A_2 = F_1\left(\frac{g^4 g_m}{x^2 + x} + (g^2 + F_1^2 + F_1)^2(F_1^2 + F_1)\right),$$

$$F_{<3} = \frac{A_2}{F_1} + (g^2 + F_1^2 + F_1)^3,$$

$$C = B_y + xB_x + x^2 + x, C_m = Y + y^2 + x^2(X + x^2) + x^4 + x^2,$$

$$U = \frac{(X + x^2)(X^3 + X^2 x)}{x^2 + x} + \frac{X^2}{x} + Xx + 1, \quad J = \frac{U + C^2}{1 + ((g^{(1)})^3 C^2 F_{12})/C_m F_{12}^{(1)})},$$

$$F_{\leq 12} = \frac{JF_{12}}{C_m}.$$

Theorem 3.5. *For A as in the previous theorem, and for ℓ_d the (reciprocal) coefficients of the logarithm for the Hayes module for this A as defined at the end of Section 2, for $d \geq 2$ (check d=0, 1, 2) we have (i) $\ell_d S_d(1) = F_1(d)$, (ii) $\ell_d S_{<d}(1) = F_{<1}(d)$,*
(iii) $\ell_d^3 S_d(1,2) = F_{12}(d)$, (iv) $\ell_d^3 S_d(3) = F_3(d)$,
(v) $\ell_d^3 S_{<d}(3) = F_{<3}(d)$ and (vi) $\ell_d^3 S_{\leq d}(1,2) = F_{\leq 12}(d)$.

Proof. We use the generating functions $A_{d0}/(1 - \sum A_{di} t^{q^i}) = \sum S_d(k) t^{k-1}$ of [T92][(18)] for $S_d(k)$ and $(A_{d0}x)(\sum A_{di} x^{q^i})^{-1} = 1 + \sum S_{<d}(k) x^k$ of [T09][3.2] for $S_{<d}(k)$ given by one type of binomial coefficient [T92, T09], and the method of [T92] to calculate this in higher genus. (Here i runs from 0 to d and k from 1 to ∞, and $S_{<d}(k) = 0$ unless k is q-even.)

We first explain briefly, how (i)-(v) follow from theory developed in [T92], by unwinding and specializing the genus one formulas there to our specific A. Note the notations matches $B_x(i) = [i]_x, B_y(i) = [i]_y$. Once one uses the known coefficients of ρ (see Exa. C page 192 of [T92]) to get $x_1 = x^2+x, y_1 = y^2+y, y_2 = x(y^2+y)$, the recursions for ℓ_i, d_i (and this a_{ik} by (7), (14) of [T92]) from the functional equations of logarithm, exponential in terms of ρ, give formulas (we use $\ell_1 = 1, d_1 = 1$ in particular) for f_i, g_i, μ_i in (20), (27), (28), (23) of [T92] implying in particular that $g(i) = g_i = \ell_i/\ell_{i-1}$. This allows us to calculate A_{i0}, A_{i1}, A_{i2} of [T92][(14)] by comparing t, t^q, t^{q^2} coefficients in (21) (see also (14), (7)) of [T92], which is all we need from the generating function coefficients. (In fact, $A_2(d) = \ell_d^4 A_{d2}$ and $(g^2 F_1 + F_1^3 + F_1^2)(d) = \ell_d^2 A_{d1}$.) Then we need only to verify by straightforward manipulation that (we note here that $g_m(d) = g(d-1)^4$, $C_m(d) = C(d-1)^2$)

$$\ell_d S_d(1) = \ell_d A_{d0} = F_1(d),$$

$$\ell_d S_{<d}(1) = \frac{\ell_d^2 A_{d1}}{\ell_d A_{d0}},$$

$$\ell_d^3 S_d(3) = \ell_d A_{d0}(\ell_d^2 A_{d1} + \ell_d^2 A_{d0}^2) = F_3(d),$$

$$\ell_d^3 S_d(1,2) = (\ell_d S_d(1))(\ell_d S_{<d}(1))^2 = (\ell_d A_{d0})(\frac{\ell_d^2 A_{d1}}{\ell_d A_{d0}})^2 = F_{12}(d),$$

$$\ell_d^3 S_{<d}(3) = \frac{\ell_d^4 A_{d2}}{\ell_d A_{d0}} + (\frac{\ell_d^2 A_{d1}}{\ell_d A_{d0}})^3 = F_{<3}(d).$$

Finally, we verify (vi) by induction on d, using (ii) and the iterated definition: it is enough to check the initial value and the identity corresponding to $S_{\leq d+1} - S_{d+1} = S_{\leq d}$. Since $g^{(1)}(d) = \ell_{d+1}/\ell_d$, the identity is $F_{\leq 12}^{(1)} - F_{12}^{(1)} = (g^{(1)})^3 F_{\leq 12}$, which follows directly.

□

Now it is easy to finish the proof of the main theorem by just noticing that $(x^2+x+1)F_{\leq 12} - F_{<3}$ has negative degree in X, Y, so that as d tends to infinity, the 'error' $(x^2+x+1)S_{\leq d}(1,2) - S_{<d}(3)$ tends to zero, establishing the theorem. □

4. Some remarks

(I) **Explicit F-functions in standard forms:** To get more concrete perspective, we give some of these functions more explicitly. We split lower order part of numerators just for the display convenience.

$$F_3 = \frac{X^3 + x^2X^2 + Y + X + x^3 + x^2 + y + x + 1}{X^4 + x^2 + 1}.$$

$$F_{12} = \frac{X^4Y + xX^5 + X^3Y + (y+1)X^4 + x^2X^2Y + (x^3+y+x)X^3 + (x+1)XY + L_{12}}{X^6 + (x+1)X^4 + (x^2+1)X^2 + x^3 + x^2 + x + 1}$$

with $L_{12} = (x^2y + x^3 + x^2 + x)X^2 + x^3Y + (x^4 + xy + x^2 + y + x)X + x^3y + x^4 + x^2$,

$$F_{<3} = \frac{(x^2+x+1)X^6 + X^5 + (x^4+x^3+x^2+x)X^4 + X^2Y + (x+1)X^3 + L_{<3}}{(x^2+x)X^6 + (x^3+x)X^4 + (x^4+x^3+x^2+x)X^2 + x^5 + x}$$

with $L_{<3} = (x^4 + x^3 + x^2 + y)X^2 + xY + xX + x^6 + x^5 + x^3 + xy + x^2$

$$F_{\leq 12} = \frac{X^6 + (x^2+x+1)X^5 + (x^3+x^2)X^4 + (x^2+x+1)X^2Y + L_{\leq 12}}{(x^2+x)X^6 + (x^3+x)X^4 + (x^4+x^3+x^2+x)X^2 + x^5 + x}$$

with $L_{\leq 12} = (x^3+x^2+x+1)X^3 + (x^2y+x^3+xy+y)X^2 + x^3Y + x^3X + x^5 + x^3y + x^3$.

Note that the denominators are $(X^2+x+1)^2$, $(X^2+x+1)^3$, $(x^2+x)(X^2+x+1)^3$ (twice) respectively.

Comparing the dominating terms of the last two expressions, makes visible the last calculation of the proof above.

(II) **Comparison with the genus zero situation:** For the genus zero case $A = \mathbb{F}_q[t]$, we have [T09, T04] the F-function identity $S_d(q-1, q(q-1)) = S_{d-1}(q^2-1)/(t-t^q)^{q-1}$, which by summing over degrees up to d then gives corresponding F-function identity for $S_{\leq d}$, and then, by taking the limits, the identity at the multizeta level. The same is true in any depth by the formula for $S_n(d)$ in the proof on page 795 in [LT14]. On the other hand, in our case here, we have the identity only at the level of ζ, only leading terms matches at $S_{\leq d}$ level, and nothing at S_d level! In fact, for $q = 2$ case above (for example, by the theorem 2 and (I)), for $d > 2$, the degree of both $S_d(1,2)$ and $S_{d-1}(3)$ is -2^d. The degree of $S_d(1,2)+S_{d-1}(3)+(t^2+t)S_{d+1}(1,2)$ is -2^{d+1}, and the degree of $(t^2+t+1)S_{\leq d}(1,2)+S_{<d}(3) + (t^2+t)S_{d+1}(1,2)$ is -2^{d+2}.

Consider the genus zero zetalike Euler basic identity $\zeta(1, q-1) = \zeta(q)/(t-t^q)$, which is not eulerian, if $q > 2$. It corresponds to F-function identity at S_d level, we do not think that the resulting identity at $S_{\leq d}$ level obtained by summing is F-function identity.

We have checked (by computing the rank of the relevant matrices) that in the case (i), there is no non-trivial linear relation (leading to our theorem by summing) between the 8 quantities $S_k(3), S_k(1,2), S_k(2,1), S_k(1,1,1)$'s, with $k = d$ or $d+1$, of weight 3, (working for all d, or equivalently working for the corresponding F-functions), in contrast to the existence of such 'binary' relations [To18]. We have also checked that the same situation persists, even if we add $S_{d+2}(1,2), S_{d-1}(3), S_{d+2}(3)$, but have not tried adding more terms. Similarly, we have checked that there is no non-trivial linear relation (leading to our theorem by summing) between the 10 quantities $S_k(1,2), S_m(3)$, with $d \leq m \leq d+4, d <$

$k \leq d+5$, again in contrast to the genus zero situation [To18]. (See also [T17][Pa. 17-18] for relevant discussion of Todd's data.)

Since the F-functions involved in S_d-level identities were used to define [AT90, AT09] the corresponding motives in the genus zero case, we need to understand better the motivic mechanisms underlying these relations in higher genus.

(III) **Comparison of polylog-algebraicity for iterated sums and zetalike property:** As mentioned above, for a positive integer k and a positive q-even integer m, $\ell_d^k S_d(k)$ and $\ell_d^m S_{<d}(m)$ are F-functions [T92, A94, T09, GP18] (we say alternately that $S_d(k)$ and $S_{<d}(m)$ satisfy 'log-algebraicity' property).

In our situation, if weight $w = \sum s_i$ is q-even, and if $\ell_d^w S_{<d}(s_1, \cdots, s_r)$ is F-function, then as in the proof above, comparison of leading terms shows that $\zeta(s_1, \cdots, s_r)$ is zetalike (equivalently, eulerian, in this case).

For simplicity, let s_1, s_2 be q-even, $w = s_1 + s_2$. If $F_{\leq}$ is the F-function for $\ell_d^w S_{\leq d}(s_1, s_2)$, and if F is the F-function representing $\ell_d^w S_d(s_1, s_2)$, and g represents the F-function ℓ_d/ℓ_{d-1}, then $F_{\leq}(d) - F(d) = g(d)^w F_{\leq}(d-1)$, so to get such $F_{\leq}$ from (known) F and g, we need to solve the Frobenius-difference equation $F_{\leq} - g^w F_{\leq}^{(-1)} = F$. When exactly is it solvable? If this is understood, the method explained in the next section to solve it should then give (case-by-case) proofs for such multizeta relations through directly verifiable relations between such functions.

(IV) **Origin/explanation of some functions introduced in the proof:** For interested reader, we indicate how the formula (vi) was discovered (without knowledge of such algorithm). To guess what $F_{\leq 12}$ should be, comparison of the LHS of (vi) with $F_{12}(d)$ was made (for small d's) and factored to notice match of denominators, so their ratio was considered. (Note that without the factor $\ell(d)^3$, the relevant denominators do not match!). Again consideration of factors suggested that primes in denominators came from those of $F_{12}(d-1)$. Now the expressions show that cube of $C(d) = [d]_y + x[d]_x + [1]$ kills denominator of $F_{12}(d)$, here the square was enough so polynomials $E(d) = C(d-1)^2 \ell_d^3 S_{\leq d}(1,2)/F_{12}(d)$ were calculated for a few d's and it was noticed that except for the constant term which alternated between 0 and 1, the tail of $E(d)$ matched with $E(d-1)$, so the recursion between $E(d)$ and $E(d-1)$ was considered as F-polynomials are easy to guess explicitly (using geometric series). This led to function $U(d)$ satisfying $E(d) = U(d-1) + E(d-1)$. Next we consider equation coming from the relation $S_{\leq d} = S_d + S_{\leq d-1}$, which, after a simple straight manipulation, translates to $E(d) = C(d-1)^2 + E(d-1)g(d)^3 C(d-1)^2 F_{12}(d-1)/(C(d-2)^2 F_{12}(d))$. Solving these two equations led to F-function expression for E and thus for $F_{\leq 12}$. For more streamlined version developed later, see the next section.

(V) **Structure behind the explicit conjecture:** In the notation of the theorem, the first depth 2 explicit conjecture is $\zeta(q^n - 1, (q-1)q^n)/\zeta(q^{n+1} - 1) = [n]^2/C(n+1)$. We have similar but more involved descriptions for the rest. We remark that the denominators listed above in each case are (Frobenius twists of) denominators of F-function F_1 satisfying $\ell_d S_d(1) = F_1(d)$.

(VI) **Low F_i's:** If F_k denotes the F-function with $F_k(d) = \ell_d^k S_d(k)$, then for $F_{mp^n} = F_1^{mp^n}$, for $m \leq q$, by p-th powers and $\mathbb{F}_q$-linearity and power sums-symmetric sums argument [T14][Remark 6.1].

5. Another case

In order not to make the theorem and proof any more complicated by combining too many formulae at once, we decided to state the second case separately as the following theorem.

Theorem 5.1. *For $A = \mathbb{F}_2[x,y]/(y^2+y+x^3+x+1)$, we have*

$$(x^8+x^6+x^5+x^3+1)\zeta(3,4) = (x^4+x^2)\zeta(7).$$

Proof. We proceed in a similar way to the proof of the first case. In fact, the functions interpolating $\ell_d^3 S_d(3), \ell_d^3 S_{<d}(3)$, are already computed and since $S_d(2^n s) = S_d(s)^{2^n}$, the similar interpolating functions for $s=4$ are just fourth powers of the functions we calculated above for $s=1$.

This gives F_{34} such that $\ell_d^7 S_d(3,4) = F_{34}(d)$. Here it is explicitly: $F_{34} = N_{34}/D_{34}$, where $D_{34} = (X^2+x+1)^6$ and

$$\begin{aligned} N_{34} = {} & X^{11} + x^2 X^{10} + \left(x^3+x+1+y\right) X^8 + \left(x^4+1\right) X^7 + \left(x^6+x^3+x^2+x+y\right) X^6 \\ & + \left(x^4+x^2+1\right) X^5 + \left(x^7+x^6+x^5+x^4+x^2+yx^4\right) X^4 + \left(x^6+x^2+1\right) X^3 \\ & + \left(x^8+x^5+x^4+x^2+x+y(x^2+1)\right) X^2 + x^6 X + x^9+x^8+x^7+x^4+yx^6 \\ & + Y\left[X^8+X^6+x^4X^4+\left(x^2+1\right)X^2+x^6\right]. \end{aligned}$$

We claim that $F_{\leq 34} = N_{\leq 34}/D_{\leq 34}$ satisfies $\ell_d^7 S_{\leq d}(3,4) = F_{\leq 34}(d)$, where $D_{\leq 34} = (x^6+x^5+x^4+x^3+x^2+x+1)(X^2+x+1)^7$ and $N_{\leq 34}$ is

$$\begin{aligned} & \left(x^2+x\right) X^{14} + \left(x^2+x\right) X^{13} + \left(x^6+x^5+x^4+x^3+x^2+x+1\right) X^{12} \\ & + \left(x^6+x^5+x^4+x^2+1\right) X^{11} + \left(x^8+x^7+x^6+x^5+x^4+x^2+x+y(x^2+x)\right) X^{10} \\ & + \left(x^8+x^4+x^3+x^2\right) X^9 + \left(x^8+x^7+x^6+x^5+x^3+y(x^6+x^5+x^4+x+1)\right) X^8 \\ & + \left(x^9+x^7+x^6+x^4+x^3\right) X^7 \\ & + \left(x^9+x^7+x^5+x^2+x+1+y(x^8+x^6+x^5+x^3+x^2+x+1)\right) X^6 \\ & + \left(x^{10}+x^9+x^8+x^4+x^2+x+1\right) X^5 \\ & + \left(x^7+x^6+x^5+x^3+x+1++y(x^9+x^8+x^7+x^4+x^2+x+1)\right) X^4 \\ & + \left(x^{11}+x^{10}+x^9+x^7+x^5+x^4\right) X^3 \\ & + \left(x^{12}+x^{10}+x^9+x^8+x^6+x^5+x^4+y(x^{10}+x^7+x^5)\right) X^2 \\ & + \left(x^{11}+x^9+x^4\right) X + x^{10}+x^8+x^7+x^6+x^5+x^4+y(x^{11}+x^9+x^4) \\ & + Y\left[\left(x^2+x\right) X^{10} + \left(x^6+x^5+x^4+x+1\right) X^8 + \left(x^8+x^6+x^5+x^3+x^2+x+1\right) X^6\right] \\ & + Y\left[\left(x^9+x^8+x^7+x^5+x^4+x^2+x+1\right) X^4 + \left(x^{10}+x^7+x^5\right) X^2 + x^{11}+x^9+x^4\right]. \end{aligned}$$

This is proved by straightforward verification of the initial condition and the recursion identity $F_{\leq 34}^{(1)} - (g^{(1)})^7 F_{\leq 34} = F_{34}^{(1)}$.

Finally, similar methods as in the proof of the first case, gives $F_{<7}$ satisfying $F_{<7}(d) = \ell_d^7 S_{<d}(7)$ as follows

$F_{<7} = N_{<7}/D_{<7}$, where $D_{<7} = (x^8+x)(X^2+x+1)^7$ and $N_{<7}$ is

$$
\begin{aligned}
&\left(x^8+x^6+x^5+x^3+1\right)X^{14} + \left(x^8+x^6+x^5+x^3+1\right)X^{13}\\
&+\left(x^8+x^6+x^5+x^4+x^3+x^2+1\right)X^{12} + \left(x^9+x^7+x^6+x^2+x\right)X^{11}\\
&+\left(x^{12}+x^{10}+x^4+x^3y+x^2+y(x^8+x^6+x^5+x^3+1)\right)X^{10} + \left(x^{12}+x^8+x^7+x^4+x^3\right)X^9\\
&+\left(x^{12}+x^8+x^7+x^5+x^2+y(x^9+x^8+x^7+x^5+x^3+x^2+x+1)\right)X^8\\
&+\left(x^{13}+x^{10}+x^6+x^5+x^4+x^3+x\right)X^7\\
&+\left(x^{13}+x^{12}+x^9+x^7+x^5+x^3+y(x^{12}+x^9+x^5+x^4+x^2+x+1)\right)X^6\\
&+\left(x^{14}+x^{13}+x^{12}+x^{11}+x^9+x^6+x^4+x^3+x^2\right)X^5\\
&+\left(x^{14}+x^{13}+x^{12}+x^{11}+x^9+x^8+x^3y+x^3+y(x^{13}+x^{12}+x^{10}+x^9x^6+x^2+1)\right)X^4\\
&+\left(x^{15}+x^{12}+x^{11}+x^8+x^6+x^4+x^3+x^2+x+1\right)X^3\\
&+\left(x^{16}+x^{15}+x^{14}+x^{13}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+y(x^{14}+x^{11}+x^{10}+x^4+1)\right)X^2\\
&+\left(x^{15}+x^{14}+x^{12}+x^{10}+x^8+x^6+x^3+x^2+x\right)X\\
&+x^{15}++x^{14}+x^{12}+x^{10}+x^9+x^8+x^7+x^5+x^4+x^2\\
&+y(x^{15}+x^{14}+x^{12}+x^{10}+x^8+x^6+x^3+x^2+x)\\
&+Y\left[\left(x^8+x^6+x^5+x^3+1\right)X^{10} + \left(x^9+x^8+x^7+x^5+x^3+x^2+x+1\right)X^8\right]\\
&+Y\left[\left(x^{12}+x^9+x^5+x^4+x^2+x+1\right)X^6 + \left(x^{13}+x^{12}+x^{10}+x^9+x^6+x^3+x^2+1\right)X^4\right]\\
&+Y\left[\left(x^{14}+x^{11}+x^{10}+x^4+1\right)X^2 + x^{15}+x^{14}+x^{12}+x^{10}+x^8+x^6+x^3+x^2+x\right].
\end{aligned}
$$

The proof is thus complete, as before, by observing the ratio of the leading terms of $F_{<7}$ and $F_{\leq 34}$ is exactly (after simple cancellations) $(x^8+x^6+x^5+x^3+1)/(x^4+x^2)$. □

Remarks 5.2. (1) We solved by using SageMath, the Frobenius-difference equation

$$(X^2+x+1)^7[Z^{(1)} - (X^4+x+1)N_{34}^{(1)}] = (Y+y+X^4+X^3+X^2x+X+1)^7 Z,$$

where $Z = \sum_{k=0}^{14} a_k X^k + Y \sum_{m=0}^{12} b_m X^m$, by using the elliptic curve relation to get rid of higher powers of Y and then equating coefficients of X^n and YX^m, for $0 \leq n \leq 39,\ 0 \leq m \leq 38$ in the resulting linear system in 26 unknowns a_i, b_i. The unique solution obtained, in fact, proves the recursion relation. (We note here that $Z = N_{\leq 34}/(x^6+x^5+x^4+x^3+x^2+x+1)$.)

(2) We would have complete case-by-case algorithmic proof method for the whole family (at least in depth 2 and probably in general by induction on depth), if only we are assured of solvability of such equations resulting from our recursion. In the last section, we provide details of proofs of 3 more cases, done this way.

6. Dedekind type relative zeta situation

We also consider Dedekind type relative zeta and multizeta functions using norms from A to some corresponding $\mathbb{F}_q[x]$, say and explore corresponding zetalike multizetas.

More precisely, for a monic $a \in A$, we use the monic generator of $-k$-th power of the relative norm of a. For the class number one situation, this corresponds more closely to the Dedekind zeta. See [T04][Sec. 5.1].

First note that if the relative extension is Galois of degree p and A is class number one, then (argument of [T04][Pa 162] generalized to power sums) for an element of $A - \mathbb{F}_q[x]$, the p conjugates having the same norm, the total norm contribution is zero, where as for an element in the base, the norm is p-th power, so $\zeta_{A/\mathbb{F}_q[x]}(s_1, \cdots, s_r) = \zeta_{\mathbb{F}_q[x]}(ps_1, \cdots, ps_r)$. In particular, we get zetalike elements just from the genus zero case. This works for the three class number one examples A with $p = 2$, which are quadratic over $\mathbb{F}_q[x]$ of the form $y^2 + y = P(x)$, and the fourth class number one example with $q = 3$ of the form $y^2 = x^3 - x - 1$ considered as a cubic Galois extension (since $\mathbb{F}_3$ translations of a root are roots) over $\mathbb{F}_3[y]$.

Considered the class number one examples above of characteristic 2, as extensions of the relevant $\mathbb{F}_q[y]$'s, we did not find any zetalike examples, in numerical experimentation. Similarly, for $q = 3, y^2 = x^3 - x - 1$, and $Norm(f + yg) = f^2 - y^2g^2$, we have not yet found any zetalike examples.

We are in the beginning stages of exploration in the general relative situation and will report in the future paper about more refined conjectures on degrees, other F-functions and relations.

For now, we only make following simple remark that in higher genus, some power sums are zero, not only as the relevant sets are empty because of Weierstrass gaps (at the point at infinity), but also power sums can be zero, even if the relevant sets are not empty. For example, consider $\mathbb{F}_2[x, y]/y^2 + y = x^3 + x + 1$ over $\mathbb{F}_2[y]$. In this case, since the norm of x as well as of $x + 1$ is $y^2 + y + 1$, all the power sums for degree 2 also (for degree 1 they vanish for the reason above) vanish.

7. Numerical experiments

The numerical exploration to find zetalike values was done following the method of [LT14] using SageMath on laptop, using the continued fractions in $\mathbb{F}_q((1/x))$. Note that in cases (i, iii, iv), $S_d(k) \in \mathbb{F}_q(x)$ by invariance with respect to the Galois action $y \to y + 1$ of K over $\mathbb{F}_q(x)$. In the case (ii), if s_i's are even, we get the relevant Galois invariance. In these cases, the method of [LT14] using continued fractions for $\mathbb{F}_q((1/x))$ works immediately. In case (ii), in general, and in higher class number cases, (for low q, g), we used the norms to descend to this $\mathbb{F}_q((1/x))$ situation.

Apart from higher class number and Dedekind situation, we also looked for possible rational ratios of multizeta (of depth 2 or 3) of the same weight in class number one case, not explained by our conjecture on the zetalike family. We did not find any, in contrast to several examples in genus zero. For example, when $A = \mathbb{F}_2[t]$, we have rational ratios $\zeta(1, 3)/\zeta(2, 2)$, $\zeta(2, 3)/\zeta(3, 2)$, $\zeta(7, 4)/\zeta(4, 7)$, where only for the first example the numerators and denominators are zetalike (so the rationality of the first example is explained by this observation), but none of the numerators or denominators of the last 2 are zetalike. (We checked the class number one cases (i), (iv) for weights up to 32, and (iii) for weights up to 12, only for depth 2).

8. Details of the three other class number one cases proved

Theorem 8.1. *For $A = \mathbb{F}_3[x, y]/(y^2 - (x^3 - x - 1))$, we have*

$$(x^9 + x^6 + x^4 - x^3 + x^2 - 1)\zeta(2, 6) = (x^3 - x + 1)\zeta(8).$$

Proof. We proceed in a similar way to the proof of case i). We will now define several functions in $\mathbb{F}_3(x, y, X, Y)$, where x and X are independent transcendentals and $y^2 = x^3 - x - 1$ and $Y^2 = X^3 - X - 1$. For each function, say h, put $h^{(1)}$ for the function resulting from h after substituting X^3, Y^3 respectively for X, Y, and put $h(d) \in K$ for the function resulting from h after substituting x^{3^d} and y^{3^d} for X, Y. Put

$$F_1 = \frac{-(X - x^q)}{X^2 + (x+1)\,X + yY + x^2 - x + 1}, \qquad g^3 = \frac{-(Y-y)^3 + Y^q(X-x)^3}{Y(Y^3 - Y) - (X - x^3) - Y^3 Y(X^3 - X)},$$
$$F_{<2} = -yg^3 + F_1^3 - F_1^2, \qquad F_{26} = F_1^2 F_{<2}^3,$$
$$F_{<8} = -\frac{(x^3 - x + 1)g^9 (g^{(-1)})^9}{x^3 - x} + (yg^3 - F_1^3 + F_1^2)^4 + (F_1^3 - F_1^2)(y^3 g^9 - F_1^9 + F_1^6),$$
$$A_1 = F_1(yg^3 - F_1^3 + F_1^2), \quad A_2 = F_1\left(\frac{x^3 - x + 1}{x^3 - x} \cdot g^9 (g^{(-1)})^9 - (F_1^3 - F_1^2)(y^3 g^9 - F_1^9 + F_1^6)\right).$$

Notice that $A_1(d) = \ell_d^3 A_{d1}$ and $A_2(d) = \ell_d^9 A_{d2}$. Then, we have $\ell_d S_d(1) = \ell_d A_{d0} = F_1(d)$ and

$$\ell_d^2 S_{<d}(2) = \frac{-\ell_d^3 A_{d1}}{\ell_d A_{d0}} = \frac{A_1(d)}{F_1(d)} = (-yg^3 + F_1^3 - F_1^2)(d) = F_{<2}(d),$$
$$\ell_d^8 S_d(2,6) = (\ell_d^2 S_d(2))(\ell_d^2 S_{<d}(2))^3 = F_1(d)^2 F_{<2}(d)^3 = F_{26}(d).$$

Finally, we have

$$\begin{aligned}\ell_d^8 S_{<d}(8) &= \frac{(\ell_d^3 A_{d1})^4 - (\ell_d A_{d0})^3 \cdot \ell_d^9 A_{d2}}{(\ell_d A_{d0})^4} = \frac{A_1(d)^4 - F_1(d)^3 A_2(d)}{F_1(d)^4} \\ &= -\frac{(x^3 - x + 1)g^9 (g^{(-1)})^9}{x^3 - x} + (yg^3 - F_1^3 + F_1^2)^4 + (F_1^3 - F_1^2)(y^3 g^9 - F_1^9 + F_1^6), \\ &= F_{<8}(d).\end{aligned}$$

Explicitly $F_{26} = N_{26}/D_{26}$, where $D_{26} = (X^3 - x + 1)^8$ and N_{26} is

$$
\begin{aligned}
&\left(x^6 + x^4 + x^3 + x^2 - x + 1\right) X^{17} + \left(x^3 y - xy - y\right) X^{16} Y + \left(x^7 + x^6 + x^5 - x^4 - x^3\right) X^{16} + \\
&\left(x^6 y - x^2 y + xy - y\right) X^{15} Y + \left(x^8 - x^7 - x^4 - x^3 + x^2 - x + 1\right) X^{15} + \left(-x^3 y + xy\right) X^{14} Y + \\
&\left(x^7 + x^6 + x^5 - x^4 - x^3 + 1\right) X^{14} + (-xy - y) X^{13} Y + \left(-x^9 + x^8 - x^7 - x^4 + x^3 + x^2 - x\right) X^{13} + \\
&\left(x^7 y + x^6 y - x^5 y + x^4 y - x^3 y - x^2 y + xy - y\right) X^{12} Y + \\
&\left(-x^{12} - x^{10} - x^9 + x^7 + x^6 + x^5 + x^4 + x^2 + x + 1\right) X^{12} + \\
&\left(-x^9 y - x^6 y + x^3 y + x^2 y\right) X^{11} Y + \left(-x^9 + x^8 - x^7 - x^4 + x^3 + x^2\right) X^{11} + \\
&\left(-x^{10} y - x^9 y - x^7 y + x^6 y + x^5 y - x^4 y - x^3 y\right) X^{10} Y + \\
&\left(-x^{12} + x^{10} - x^9 + x^7 - x^6 + x^5 + x^4 - x^2 + 1\right) X^{10} + \\
&\left(-x^{11} y + x^{10} y + x^9 y + x^7 y - x^6 y - x^5 y + x^4 y - x^3 y\right) X^9 Y + \\
&\left(-x^{13} - x^{11} - x^{10} + x^9 + x^8 + x^7 + x^5 + x^3 - x^2 + x + 1\right) X^9 + \left(-x^6 y - x^4 y - x^2 y + y\right) X^8 Y + \\
&\left(-x^{12} + x^{10} + x^9 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 + x + 2\right) X^8 + \\
&\left(-x^9 y - x^7 y - x^6 y - x^5 y + x^4 y - xy - y\right) X^7 Y + \\
&\left(-x^{13} - x^{12} + x^{11} - x^{10} - x^8 - x^6 - x^4 + x^3 + x^2 + x + 1\right) X^7 + \\
&\left(-x^{12} y - x^{10} y - x^9 y - x^8 y - x^6 y + x^5 y - x^4 y + x^3 y - x^2 y + xy - y\right) X^6 Y + \\
&\left(-x^{14} + x^{13} - x^{12} - x^{10} - x^8 + x^3 + x + 2\right) X^6 + \left(x^9 y - x^7 y - x^6 y - x^5 y + x^2 y - xy + y\right) X^5 Y + \\
&\left(-x^{13} + x^{12} + x^{11} - x^8 + x^6 - x^4 - x^3 - x^2 - x\right) X^5 + \\
&\left(-x^9 y - x^8 y + x^7 y - x^6 y + x^5 y + x^4 y - x^3 y + xy + y\right) X^4 Y + \\
&\left(-x^{14} + x^{11} - x^{10} - x^9 - x^8 - x^7 + x^6 - x^4 - x^3 + x^2 - x + 1\right) X^4 + \\
&\left(-x^{13} y + x^{12} y - x^{10} y - x^9 y - x^8 y + x^7 y + x^6 y - x^5 y + x^4 y - x^3 y + x^2 y - xy + y\right) X^3 Y + \\
&\left(-x^{15} - x^{14} + x^{13} - x^{12} - x^{11} - x^{10} + x^8 + x^6 - x^4 + x^3 - x^2 + x\right) X^3 + \\
&\left(-x^8 y - x^7 y + x^5 y + x^4 y + x^3 y + x^2 y + xy + y\right) X^2 Y + \\
&\left(-x^{14} - x^{13} + x^{12} - x^{11} + x^{10} + x^9 - x^8 + x^7 - x^6 - x^3 + x^2 + 1\right) X^2 + \\
&\left(-x^{11} y - x^{10} y - x^9 y + x^8 y - x^7 y - x^6 y + x^5 y + x^4 y + x^3 y\right) XY + \\
&\left(-x^{15} + x^{14} + x^{11} - x^9 + x^7 - x^6 + x^5 + x^3\right) X + \\
&\left(-x^{14} y - x^{13} y - x^{12} y + x^{11} y + x^{10} y + x^9 y - x^8 y - x^6 y\right) Y - x^{16} - x^{14} + x^{13} - x^{12} \\
&+ x^{11} - x^{10} + x^9 + x^8 - x^7.
\end{aligned}
$$

In order to find $F_{\leq 26}$, such that $F_{\leq 26}(d) = \ell_d^8 S_{\leq d}(2,6)$, we use the recursion identity $F_{\leq 26}^{(1)} - (g^{(1)})^8 F_{\leq 26} = F_{26}$. Let $Z = (X^3 - x + 1)^8 F_{\leq 26}$. We solve by using SageMath, the equation

$$(X^3 - x + 1)^8 (Z^{(1)} - N_{26}^{(1)}) = (-X^6 Y + X^4 Y + (x-1) X^3 Y + (-x-1) XY + (-x-1) Y - y)^8 Z,$$

where $Z = \sum_{k=0}^{18} a_k X^k + Y \sum_{m=0}^{15} b_m X^m$. The unique solution obtained is $Z = N_{\leq 26}/(x^3 - x)$, so that $F_{\leq 26} = N_{\leq 26}/D_{\leq 26}$ where $D_{\leq 26} = (x^3 - x)(X^3 - x + 1)^8$

and $N_{\le 26}$ is

$$\begin{aligned}
&\left(x^3-x+1\right)X^{18}+\left(-x^6y-x^4y+x^3y-x^2y-xy\right)X^{15}Y+\left(-x^9-x^6-x^4+x^3-x^2+1\right)X^{15}+\\
&\left(x^6y+x^4y-x^3y+x^2y+xy\right)X^{13}Y+\left(-x^9y+x^7y-x^6y+x^5y+x^4y+x^3y+xy\right)X^{12}Y+\\
&\left(-x^{10}+x^9-x^7+x^6-x^5-x^4+x^2-x\right)X^{12}+\left(x^9y-x^7y-x^6y-x^5y+x^3y+x^2y\right)X^{10}Y+\\
&\left(-x^7y-x^6y-x^5y-x^3y+x^2y+y\right)X^9Y+\left(-x^{11}-x^{10}+x^9-x^8-x^7+x^6+x^4-x^3+x^2+1\right)X^9+\\
&\left(x^9y-x^3y-y\right)X^7Y+\left(-x^{10}y+x^7y+x^6y+x^5y+x^4y-x^3y-x^2y+y\right)X^6Y+\\
&\left(-x^{12}-x^{10}+x^7+x^5-x^3-x^2+x+1\right)X^6+\left(x^{10}y+x^9y-x^7y-x^6y-x^5y-x^4y+x^2y+y\right)X^4Y+\\
&\left(-x^{11}y-x^9y-x^8y-x^6y-x^5y-x^4y+x^2y+y\right)X^3Y+\\
&\left(-x^{13}-x^{12}-x^{11}-x^9+x^8-x^7+x^6-x^5-x^4+x^3-x^2-x+1\right)X^3+\\
&\left(x^{11}y+x^{10}y-x^9y+x^8y-x^7y\right)XY+\\
&\left(x^{11}y+x^{10}y-x^9y+x^8y-x^7y\right)Y+x^{15}-x^{14}-x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7.
\end{aligned}$$

$F_{<8}=N_{<8}/D_{<8}$ where $D_{<8}=(x^3-x)(X^3-x+1)^8$ and $N_{<8}$ is

$$\begin{aligned}
&\left(x^9+x^6+x^4-x^3+x^2+2\right)X^{18}+\left(-x^3y+xy\right)X^{15}Y-X^{15}+\left(x^3y-xy\right)X^{13}Y+\\
&\left(-x^{12}y+x^{10}y-x^4y+x^3y+x^2y-xy\right)X^{12}Y+\\
&\left(x^9+x^6+x^4-x^3+x^2-x\right)X^{12}+\left(x^{12}y-x^{10}y+x^4y-x^2y\right)X^{10}Y+\\
&\left(x^{12}y-x^{10}y+x^4y+x^3y-x^2y-xy-y\right)X^9Y+\left(-x^9-x^6-x^4+x^3+x^2-x+2\right)X^9+\\
&\left(-x^3y+xy+y\right)X^7Y+\left(x^{12}y-x^{10}y+x^4y-x^3y-x^2y-y\right)X^6Y+\left(x^9+x^6+x^4+x^3+x^2-x+2\right)X^6+\\
&\left(-x^{12}y+x^{10}y-x^4y+x^2y+xy-y\right)X^4Y+\left(-x^{12}y+x^{10}y-x^4y-y\right)X^3Y+\\
&\left(-x^4+x^3-x^2+x+2\right)X^3+\left(x^2y+xy\right)XY+\left(x^2y+xy\right)Y+x^{18}+x^{15}+x^{13}-x^{12}+x^{11}+x^6-x^5+x.
\end{aligned}$$

Notice that denominators of $F_{\le 26}$ and $F_{<8}$ match. The degree of $E=(x^9+x^6+x^4-x^3+x^2-1)F_{\le 26}-(x^3-x+1)F_{<8}$ is negative; more precisely, the degree of $E(d)$ is $-(-27+15\times 3^d)$; since the degree of ℓ_d is $-(3^{d+1}-3)/2$, the degree of the $E(d)/\ell_d^8$ is $-(-15+3^{d+1})$; therefore, $E(d)/\ell_d^8$ tends to zero as d tends to infinity. $\square$

Theorem 8.2. *For $A=\mathbb{F}_4[x,y]/(y^2+y+x^3+w)$, where $w^2+w+1=0$, we have*

$$\zeta(3,12)/\zeta(15)=(x^{12}+x^9+x^6+x^3+1)/(x^{24}+x^{18}+x^9+x^3+1)$$

Proof. We will now define several functions in $\mathbb{F}_3(x,y,X,Y)$, where x and X are independent transcendentals and $y^2+y=x^3+w$ and $Y^2=X^3+w$. For each function, say h, put $h^{(1)}$ for the function resulting from h after substituting X^4,Y^4 respectively for X,Y, and put $h(d)\in K$ for the function resulting from h after substituting x^{4^d} and y^{4^d} for X,Y. Put

$$F_1=\frac{X+x^q}{x^2X^2+XY+(y+1)X+xY+xy},\qquad g=\frac{Y+y+X^2(X-x)}{X^4+x}$$

$$F_{<3}=(x^4+x)g^4-F_1^4+F_1^3,\qquad F_{3,12}=F_1^3F_{<3}^4,\qquad A_1=F_1((x^4+x)g^4+F_1^4+F_1^3),$$

$$A_2=F_1\Big(\frac{(x^{12}+x^9+x^6+x^3+1)g^{16}(g^{(-1)})^{16}}{x^4+x}+(F_1^4+F_1^3)((x^4+x)g^4+F_1^4+F_1^3)^4\Big),$$

and

$$\begin{aligned}F_{<15} = ((x^4+x)g^4 + F_1^4 + F_1^3)^5 + \\ \frac{(x^{12}+x^9+x^6+x^3+1)g^{16}(g^{(-1)})^{16}}{x^4+x}(F_1^4+F_1^3)((x^4+x)g^4+F_1^4+F_1^3)^4.\end{aligned}$$

Note that $A_1(d) = \ell_d^4 A_{d1}$ and $A_2(d) = \ell_d^{16} A_{d2}$.

We have

$$\begin{aligned}\ell_d S_d(1) &= F_1(d)\\ \ell_d^3 S_{<d}(3) &= \frac{\ell_d^4 A_{d1}}{\ell_d A_{d0}} = \frac{F_1(d)((x^4+x)g^4 - F_1^4 + F_1^3)(d)}{F_1(d)} = F_{<3}(d),\\ \ell_d^{15} S_d(3,12) &= (\ell_d S_d(3))^3(\ell_d^3 S_{<d}(3))^4 = F_1(d)^3 F_{<3}(d)^4 = F_{3,12}(d),\\ \ell_d^{15} S_{<d} 15 &= \frac{(\ell_d^4 A_{d1})^5 + (\ell_d A_{d0})^4(\ell_d^{16} A_{d2})}{(\ell_d A_{d0})^5} = F_{<15}(d).\end{aligned}$$

Explicitly, we have $F_{<15} = N_{<15}/D_{<15}$, where $D_{<15} = (x^4+x)(X^4+x)^{15}$ and

$$\begin{aligned}N_{<15} = {} & \left(x^{24}+x^{18}+x^9+x^3+1\right)X^{40} + \left(x^8+x^2\right)X^{38} + \left(x^{16}+x^{10}+x^4+x\right)X^{36} + \left(x^8+x^2\right)X^{35}\\ & - X^{34} + \left(x^8+x^2\right)X^{32}Y + \left(x^{32}+x^{26}+x^{20}+x^{17}+x^8y+x^2y\right)X^{32} - X^{31} + xX^{30} - X^{28}Y +\\ & (y+1)X^{28} + xX^{27} + x^2X^{26} + xX^{24}Y + (xy+x)X^{24} + x^2X^{23} + x^3X^{22} + x^2X^{20}Y +\\ & \left(x^2y+x^2\right)X^{20} + x^3X^{19} + x^4X^{18} + x^3X^{16}Y + \left(x^{24}+x^{18}+x^9+x^3y+1\right)X^{16} + x^4X^{15} +\\ & \left(x^8+x^5+x^2\right)X^{14} + x^4X^{12}Y + \left(x^{16}+x^{10}+x^4y+x\right)X^{12} + \left(x^8+x^5+x^2\right)X^{11} + \left(x^6+1\right)X^{10} +\\ & \left(x^8+x^5+x^2\right)X^8Y + \left(x^{32}+x^{26}+x^{20}+x^{17}+x^8y+x^5y+x^5+x^2y\right)X^8 + \left(x^6+1\right)X^7 +\\ & \left(x^7+x\right)X^6 + \left(x^6+1\right)X^4Y + \left(x^{24}+x^{18}+x^9+x^6y+x^6+x^3+y\right)X^4 + \left(x^7+x\right)X^3 +\\ & \left(x^7+x\right)Y + x^{40}+x^{34}+x^{25}+x^{19}+x^{10}+x^7y+x^7+x^4+xy.\end{aligned}$$

On the other hand, $F_{3,12} = N_{3,12}/D_{3,12}$ where $D_{3,12} = (X^4+x)^{15}$ and $N_{3,12}$ is (too large to fit in here)

$$N_{3,12} = (x^{18}+x^6)X^{39} + (x^{16}+x^4)X^{38}Y + (x^{22}+x^{16}y+x^{16}+x^{10}+x^4y+x^4)X^{38} + \cdots$$

Next, We calculate $F_{\leq 3,12}$ such that $F_{\leq 3,12}(d) = \ell_d^{15} S_{\leq d}(3,15)$. By using the identity recursion $F_{\leq 3,12} - (g^{(1)})^{15} F_{\leq 3,12} = F_{3,12}$, we get the Frobenius difference equation

$$(X^4+x)^{15}(Z^{(1)} - N_{3,12}^{(1)}) = (X^{12}+xX^8+X^6+X^3+Y+y+1)^{15} Z,$$

where $Z = (X^4+x)^{15} F_{\leq 3,12}$. We put $Z = \sum_{k=0}^{40} a_k X^k + Y \sum_{m=0}^{36} b_m X^m$. The unique solution obtained is $Z = N_{\leq 3,12}/(x^4+x)$ so that $F_{\leq 3,12} = Z/(X^4+x)^{15} =$

$N_{\le 3,12}/D_{\le 3,12}$ where $D_{\le 3,12} = (x^4+x)(X^4+x)^{15}$ and $N_{\le 3,12}$ is

$$\begin{aligned}
&\left(x^{12}+x^9+x^6+x^3+1\right)X^{40}+\left(x^{20}+x^{17}+x^8+x^5\right)X^{38}+\left(x^{22}+x^{19}+x^{10}+x^7\right)X^{36}+\\
&\left(x^{20}+x^{17}+x^8+x^5\right)X^{35}+\left(x^{24}+x^{18}+x^9+x^3+1\right)X^{34}+\left(x^{20}+x^{17}+x^8+x^5\right)X^{32}Y+\\
&\left(x^{26}+x^{23}+x^{20}y+x^{17}y+x^{14}+x^{11}+x^8y+x^5y\right)X^{32}+\left(x^{24}+x^{18}+x^9+x^3+1\right)X^{31}+\\
&\left(x^{25}+x^{19}+x^{10}\right)X^{30}+\left(x^{24}+x^{18}+x^9+x^3+1\right)X^{28}Y+\\
&\left(x^{24}y+x^{24}+x^{18}y+x^{18}+x^9y+x^9+x^6+x^3y+y+1\right)X^{28}+\left(x^{25}+x^{19}+x^{10}\right)X^{27}+\\
&\left(x^{26}+x^{20}+x^{11}+x^8+x^5\right)X^{26}+\left(x^{25}+x^{19}+x^{10}\right)X^{24}Y+\left(x^{25}y+x^{25}+x^{19}y+x^{19}+x^{10}y+x^7\right)X^{24}+\\
&\left(x^{26}+x^{20}+x^{11}+x^8+x^5\right)X^{23}+\left(x^{27}+x^{21}+x^6\right)X^{22}+\left(x^{26}+x^{20}+x^{11}+x^8+x^5\right)X^{20}Y+\\
&\left(x^{26}y+x^{26}+x^{20}y+x^{20}+x^{14}+x^{11}y+x^8y+x^8+x^5y+x^5\right)X^{20}+\left(x^{27}+x^{21}+x^6\right)X^{19}+\\
&\left(x^{28}+x^{22}+x^{16}+x^{13}+x^7\right)X^{18}+\left(x^{27}+x^{21}+x^6\right)X^{16}Y+\\
&\left(x^{27}y+x^{27}+x^{21}y+x^{21}+x^{18}+x^{15}+x^{12}+x^9+x^6y+x^3+1\right)X^{16}+\left(x^{28}+x^{22}+x^{16}+x^{13}+x^7\right)X^{15}+\\
&\left(x^{29}+x^{23}+x^{17}+x^{14}+x^5\right)X^{14}+\left(x^{28}+x^{22}+x^{16}+x^{13}+x^7\right)X^{12}Y+\\
&\left(x^{28}y+x^{28}+x^{22}y+x^{22}+x^{16}y+x^{16}+x^{13}y+x^{13}+x^{10}+x^7y\right)X^{12}+\left(x^{29}+x^{23}+x^{17}+x^{14}+x^5\right)X^{11}+\\
&\left(x^{30}+x^{24}+x^{18}+x^{15}+x^3+1\right)X^{10}+\left(x^{29}+x^{23}+x^{17}+x^{14}+x^5\right)X^8Y+\\
&\left(x^{29}y+x^{29}+x^{23}y+x^{23}+x^{20}+x^{17}y+x^{14}y+x^{11}+x^8+x^5y\right)X^8+\left(x^{30}+x^{24}+x^{18}+x^{15}+x^3+1\right)X^7+\\
&\left(x^{31}+x^{28}+x^{19}+x^{16}\right)X^6+\left(x^{30}+x^{24}+x^{18}+x^{15}+x^3+1\right)X^4Y+\\
&\left(x^{30}y+x^{27}+x^{24}y+x^{24}+x^{18}y+x^{18}+x^{15}y+x^{15}+x^{12}+x^9+x^3y+x^3+y\right)X^4+\\
&\left(x^{31}+x^{28}+x^{19}+x^{16}\right)X^3+\left(x^{31}+x^{28}+x^{19}+x^{16}\right)Y+x^{34}+x^{31}y+x^{28}y+x^{25}+x^{19}y+x^{19}+x^{16}y.
\end{aligned}$$

The leading term of the numerator of $E = (x^{24}+x^{18}+x^9+x^3+1)F_{\le 3,12} - (x^{12}+x^9+x^6+x^3+1)F_{<15}$ is

$$(x^{44}+x^{41}+x^{38}+x^{35}+x^{32}+x^{20}+x^{17}+x^{14}+x^{11}+x^2)X^{38}$$

so that the degree of $E(d)$ is $-(-80+44q^d)$; since the degree of ℓ_d is $-8(4^d-1)/3$, the degree of $E(d)/\ell_d^{15}$ is $-(-40+4^{d+1})$ and thus as d tends to infinity, the error tends to zero. $\square$

Theorem 8.3. *For $A = \mathbb{F}_2[x,y]/(y^2+y+x^5+x^3+1)$, we have*

$$(x^8+x^6+x^5+x^4+x^3+x+1)\zeta(1,2) = (x^6+x^5+x^3+x+1)\zeta(3).$$

Proof. We will now define several functions in $\mathbb{F}_2(x,y,X,Y)$, where x and X are independent transcendentals and $y^2+y = x^5+x^3+1$ and $Y^2+Y = X^5+X^3+1$. For each function, say h, put $h^{(1)}$ for the function resulting from h after substituting X^2, Y^2 respectively for X, Y, and put $h(d) \in K$ for the function resulting from h after substituting $x^{2^{d-2}}$ and $y^{2^{d-2}}$ for X, Y. Put $F_1 = N_1/D_1$, where $D_1 = (X^8+x)(X^{16}+x+1)$ and

$$\begin{aligned}
N_1 = {}& X^{15}+x^2X^{14}+xX^{13}+\left(x^3+x\right)X^{12}+xX^{11}+X^{10}Y+\left(x^2+x+y\right)X^{10}+(x+1)X^9\\
&+(x+1)X^8Y+\left(x^2+xy+y+1\right)X^8+xX^7+X^6Y+\left(x^2+y\right)X^6+xX^5+xX^4Y\\
&+\left(x^3+xy+x\right)X^4+xX^3+\left(x^3+x^2\right)X^2+xY+x^4+xy+x.
\end{aligned}$$

By Conjecture E in [T92, pa. 194] (which is now a theorem, see [A94]), we get that $\ell_d S_d(1) = F_1(d)$. Next we will calculate $F_{<1}$ in such a way that $F_{<1}(d) =$

$\ell_d S_{<d}(1)$. From the identity $S_{<d+1}(1) = S_{<d}(1) + S_d(1)$ we get

$$\begin{aligned} \ell_{d+1} S_{<d+1}(1) &= \frac{\ell_{d+1}}{\ell_d}(\ell_d S_{<d}(1) + \ell_d S_d(1)) \\ F_{<1}(d+1) &= g(d+1)(F_{<1}(d) + F_1(d)). \end{aligned}$$

The Frobenius equation to be solved is $F_{<1}^{(1)} = g^{(1)}(F_{<1} + F_1)$. Let $Z = (X^8 + x)(X^{16} + x + 1)F_{<1}$. The equation to be solved now is

$$(X^8 + x)(X^{16} + x + 1)Z^{(1)} = (N_g^{(1)})(Z + N_1).$$

We put $Z = \sum_{i=0}^{16} a_i X^i + Y \sum_{j=0}^{10} b_j X^j$, and it is obtained a system of 107 equations with 28 unknowns. The unique solution is $Z = N_{<1}$, so that $F_{<1} = N_{<1}/((X^8 + x)(X^{16} + x + 1))$, where

$$\begin{aligned} N_{<1} = {} & x^2 + x)X^{16} + X^{15} + (x^2+1)X^{14} + xX^{13} + (x^3 + x^2 + x + 1)X^{12} + xX^{11} + X^{10}Y \\ & + (x^2 + y)X^{10} + xX^9 + (x+1)X^8Y + (x^3 + x^2 + xy + y)X^8 + (x+1)X^7 + X^6Y \\ & + (x^2 + x + y + 1)X^6 + X^5 + (x+1)X^4Y + (x^3 + xy + x + y)X^4 + X^3 + (x^3 + x^2)X^2 + Y \\ & + x^4 + x^2 + y. \end{aligned}$$

Next we define $F_{12} = F_1 F_{<1}^2$. Then

$$\ell_d^3 S_d(1,2) = (\ell_d S_d(1))(\ell_d S_{<d}(1))^2 = F_1(d)F_{<1}(d)^2 = F_{12}(d).$$

Explicitly, $F_{12} = N_{12}/D_{12}$, where $D_{12} = (X^8+x)^3(X^{16}+x+1)^3$ and N_{12} is

$$
\begin{aligned}
&\left(x^4+x^2\right)X^{47} + \left(x^6+x^4\right)X^{46} + \left(x^5+x^3+1\right)X^{45} + \left(x^7+x^3+x^2\right)X^{44} + \left(x^5+x^4+x^3+x+1\right)X^{43} \\
&+ \left(x^4+x^2\right)X^{42}Y + \left(x^5+x^4y+x^4+x^2y+x^2+x\right)X^{42} + \left(x^4+x^3\right)X^{41} + \left(x^5+x^4+x^3+x^2+1\right)X^{40}Y \\
&+ \left(x^7+x^6+x^5y+x^5+x^4y+x^4+x^3y+x^3+x^2y+y+1\right)X^{40} + \left(x^6+x^4\right)X^{39} + \left(x^2+x\right)X^{38}Y \\
&+ \left(x^8+x^6+x^3+x^2y+x^2+xy\right)X^{38} + \left(x^7+x^5+x^4+1\right)X^{37} + \left(x^4+x^3+x^2+x\right)X^{36}Y \\
&+ \left(x^9+x^7+x^6+x^4y+x^4+x^3y+x^2y+x^2+xy+x\right)X^{36} + \left(x^7+x^2+x+1\right)X^{35} \\
&+ \left(x^6+x^3+x^2+x\right)X^{34}Y + \left(x^8+x^7+x^6y+x^4+x^3y+x^2y+xy+x\right)X^{34} + \left(x^7+x^4+x^3+x^2\right)X^{33} \\
&+ \left(x^7+x^6+x^5+x^4+x^3+x^2+x+1\right)X^{32}Y \\
&+ \left(x^8+x^7y+x^6y+x^6+x^5y+x^5+x^4y+x^3y+x^2y+x^2+xy+x+y+1\right)X^{32} + \left(x^4+x\right)X^{31} \\
&+ \left(x^6+x^5+x^4+x^3\right)X^{30}Y + \left(x^9+x^6y+x^6+x^5y+x^4y+x^3y\right)X^{30} + \left(x^8+x^6+x^4+x^3+1\right)X^{29} \\
&+ \left(x^7+x^6+x^5+x^2\right)X^{28}Y + \left(x^{10}+x^7y+x^7+x^6y+x^6+x^5y+x^3+x^2y+x^2\right)X^{28} \\
&+ \left(x^8+x^6+x^3+x+1\right)X^{27} + \left(x^7+x^6+x^5+x^4+x^2+x\right)X^{26}Y \\
&+ \left(x^8+x^7y+x^7+x^6y+x^6+x^5y+x^4y+x^2y+xy+x\right)X^{26} + \left(x^8+x^6+x^5+x^4+x^3+x+1\right)X^{25} \\
&+ \left(x^8+x^7+x^5+x^3+x^2+x+1\right)X^{24}Y \\
&+ \left(x^{10}+x^9+x^8y+x^8+x^7y+x^7+x^5y+x^4+x^3y+x^2y+xy+x+y+1\right)X^{24} \\
&+ \left(x^8+x^6+x^3+x^2+1\right)X^{23} + \left(x^7+x^6+x^5+x^2\right)X^{22}Y + \left(x^8+x^7y+x^6y+x^5y+x^5+x^3+x^2y+x\right)X^{22} \\
&+ \left(x^3+x\right)X^{21} + \left(x^8+x^7+x^4+x^3+x^2+x+1\right)X^{20}Y \\
&+ \left(x^9+x^8y+x^7y+x^7+x^5+x^4y+x^3y+x^3+x^2y+xy+x+y+1\right)X^{20} + \left(x^6+x^4+x^2+x\right)X^{19} \\
&+ \left(x^7+x^6+x^4+x\right)X^{18}Y + \left(x^{10}+x^9+x^7y+x^6y+x^6+x^5+x^4y+x^4+x^3+xy\right)X^{18} \\
&+ \left(x^8+x^7+x^5+x^3+x^2+x+1\right)X^{17} + \left(x^7+x^6+x^4+x^3\right)X^{16}Y \\
&+ \left(x^{11}+x^{10}+x^7y+x^7+x^6y+x^4y+x^4+x^3y+x^3\right)X^{16} + \left(x^6+x^5+x^4+x\right)X^{15} \\
&+ \left(x^7+x^3+x^2+x\right)X^{14}Y \\
&+ \left(x^{10}+x^9+x^8+x^7y+x^7+x^6+x^5+x^4+x^3y+x^3+x^2y+xy+x\right)X^{14} \\
&+ \left(x^9+x^8+x^6+x^5+x^4+x^2+1\right)X^{13} + \left(x^8+x^5+1\right)X^{12}Y \\
&+ \left(x^{11}+x^{10}+x^9+x^8y+x^8+x^7+x^5y+x^5+x^4+x^3+x+y+1\right)X^{12} + \left(x^9+x^8+x^4+x^3+1\right)X^{11} \\
&+ \left(x^8+x^6+x^4+x+1\right)X^{10}Y + \left(x^9+x^8y+x^7+x^6y+x^5+x^4y+x^4+xy+y\right)X^{10} \\
&+ \left(x^9+x^8+x^7+x^6+x^5+x^2+x+1\right)X^9 + \left(x^9+x^5+x^2+x\right)X^8Y \\
&+ \left(x^{11}+x^9y+x^9+x^8+x^7+x^6+x^5y+x^2y+x^2+xy\right)X^8 + \left(x^9+x^7+x^6+x^4+x^2\right)X^7 \\
&+ \left(x^8+x^5+x^4+x^3+x^2+x+1\right)X^6Y \\
&+ \left(x^{10}+x^9+x^8y+x^8+x^5y+x^4y+x^4+x^3y+x^3+x^2y+xy+y\right)X^6 \\
&+ \left(x^9+x^6+x^5+x^3+x^2\right)X^5 + \left(x^9+x^7+x^6+x^4+x^3\right)X^4Y \\
&+ \left(x^{11}+x^{10}+x^9y+x^9+x^7y+x^6y+x^6+x^4y+x^3y\right)X^4 + \left(x^9+x^6+x^5\right)X^3 + \left(x^3+x^2\right)X^2Y \\
&+ \left(x^{11}+x^{10}+x^8+x^5+x^3y+x^2y\right)X^2 + \left(x^9+x^6+x^5\right)Y + x^{12}+x^9y+x^8+x^7+x^6y+x^5y+x^5.
\end{aligned}
$$

Now, we calculate $F_{\leq 12}$ such that $F_{\leq 12}(d) = \ell_d^3 S_{\leq d}(1,2)$ by using the identity recursion $F_{\leq 12}^{(1)} - (g^{(1)})^3 F_{\leq 12} = F_{12}^{(1)}$. Let $Z = (X^8+x)^3(X^{16}+x+1)^3 F_{\leq 12}$. The equation to be solved is

$$(X^8+x)^3(X^{16}+x+1)^3(Z^{(1)}-N_{12}^{(1)}) = (N_g^{(1)})^3 Z.$$

Put $Z = \sum_{i=0}^{48} a_i X^i + Y \sum_{j=0}^{40} b_j X^j$. This gives a system of 329 equations in 90 unknowns. The unique solution implies that $F_{\leq 12} = N_{\leq 12}/D_{\leq 12}$, where $D_{\leq 12} = (x^2+x)(X^8+x)^3(X^{16}+x+1)^3)$, and $N_{\leq 12}$ is

$$\begin{aligned}
&\left(x^6+x^5+x^3+x+1\right)X^{48}+\left(x^6+x^5+x^4+x^3\right)X^{46}+\left(x^8+x^7+x^4+x^3\right)X^{44}\\
&+\left(x^7+x^6+x^5+x^4+x^2+x\right)X^{42}+\left(x^6+x^5+x^4+x^3\right)X^{41}+\left(x^9+x^7+x^6+x^4+x^3+x+1\right)X^{40}\\
&+\left(x^6+x^5+x^4+x^3\right)X^{39}+\left(x^7+x^6+x^5+x\right)X^{38}+\left(x^7+x^3+x^2+x\right)X^{37}+\left(x^6+x^5+x^4+x^3\right)X^{36}Y\\
&+\left(x^8+x^7+x^6y+x^5y+x^5+x^4y+x^3y+x\right)X^{36}+\left(x^7+x^3+x^2+x\right)X^{35}\\
&+\left(x^7+x^6+x^4+x^3+x^2+x\right)X^{34}+\left(x^6+x^5+x^3+x^2\right)X^{33}+\left(x^7+x^3+x^2+x\right)X^{32}Y\\
&+\left(x^9+x^7y+x^7+x^6+x^5+x^4+x^3y+x^2y+xy+x+1\right)X^{32}+\left(x^6+x^5+x^3+x^2\right)X^{31}+\left(x^8+x\right)X^{30}\\
&+\left(x^7+x^5+x^4+x\right)X^{29}+\left(x^6+x^5+x^3+x^2\right)X^{28}Y\\
&+\left(x^{10}+x^9+x^8+x^6y+x^5y+x^3y+x^3+x^2y+x^2+x+1\right)X^{28}+\left(x^7+x^5+x^4+x\right)X^{27}\\
&+\left(x^9+x^8+x^6+x^5+x^4+x^3\right)X^{26}+\left(x^8+x^7+x^5+x^4\right)X^{25}+\left(x^7+x^5+x^4+x\right)X^{24}Y\\
&+\left(x^{11}+x^{10}+x^9+x^7y+x^5y+x^4y+x^3+x^2+xy\right)X^{24}+\left(x^8+x^7+x^5+x^4\right)X^{23}\\
&+\left(x^9+x^6+x^5+x^4+x^3+x\right)X^{22}+\left(x^9+x^7+x^6+x^3\right)X^{21}+\left(x^8+x^7+x^5+x^4\right)X^{20}Y\\
&+\left(x^9+x^8y+x^8+x^7y+x^7+x^6+x^5y+x^5+x^4y+x^3+x^2+1\right)X^{20}+\left(x^9+x^7+x^6+x^3\right)X^{19}\\
&+\left(x^8+x^6+x^5+x^3+x^2+x+1\right)X^{18}+\left(x^7+x^5+x^4+x\right)X^{17}+\left(x^9+x^7+x^6+x^3\right)X^{16}Y\\
&+\left(x^{11}+x^{10}+x^9y+x^8+x^7y+x^6y+x^4+x^3y+x^2+x\right)X^{16}+\left(x^7+x^5+x^4+x\right)X^{15}\\
&+\left(x^9+x^7+x^3+x+1\right)X^{14}+\left(x^8+x^7+x^6+x^4+x^3+x^2+1\right)X^{13}+\left(x^7+x^5+x^4+x\right)X^{12}Y\\
&+\left(x^{11}+x^{10}+x^9+x^7y+x^7+x^5y+x^4y+x^3+xy\right)X^{12}+\left(x^8+x^7+x^6+x^4+x^3+x^2+1\right)X^{11}\\
&+\left(x^{10}+x^9+x^7+x^4+x^3+x^2+x\right)X^{10}+\left(x^9+x^8+x^6+x^4+x^2+x\right)X^9\\
&+\left(x^8+x^7+x^6+x^4+x^3+x^2+1\right)X^8Y\\
&+\left(x^{12}+x^{11}+x^9+x^8y+x^8+x^7y+x^6y+x^6+x^5+x^4y+x^4+x^3y+x^3+x^2y+y\right)X^8\\
&+\left(x^9+x^8+x^6+x^4+x^2+x\right)X^7+\left(x^{10}+x^8+x^7+x^6+x^3\right)X^6+\left(x^{10}+x^8+x^7+x^6+x^3\right)X^5\\
&+\left(x^9+x^8+x^6+x^4+x^2+x\right)X^4Y+\left(x^9y+x^9+x^8y+x^8+x^6y+x^5+x^4y+x^4+x^2y+xy\right)X^4\\
&+\left(x^{10}+x^8+x^7+x^6+x^3\right)X^3+\left(x^{10}+x^8+x^7+x^6+x^3\right)Y\\
&+x^{12}+x^{11}+x^{10}y+x^{10}+x^8y+x^7y+x^7+x^6y+x^3y.
\end{aligned}$$

We first calculate F_3, such that $F_3(d) = \ell_d^3 S_d(3)$. Since

$$F_{<1}(d) = \ell_d S_{<d}(1) = \frac{\ell_d^2 A_{d1}}{\ell_d A_{d0}},$$

it follows that $\ell_d^2 A_{d1} = (\ell_d A_{d0})(\ell_d S_{<d}(1)) = F_1(d) F_{<1}(d)$.

$$\ell_d^3 S_d(3) = \ell_d A_{d0}(\ell_d^2 A_{d1} + (\ell_d A_{d0})^2) = F_1(d)(F_1(d)F_{<1}(d) + F_1(d)^2).$$

We then define $F_3 = F_1^2 F_{<1} + F_1^3$.

Finally, $F_{<3}$ is calculated such that $F_{<3}(d) = \ell_d^3 S_{<d}(3)$. In a similar way as $F_{<1}$ was obtained, we obtain the equation

$$F_{<3}^{(1)} = (g^{(1)})^3(F_{<3} + F_3).$$

Making the change of variable $Z = (X^8 + x)^3(X^{16} + x + 1)^3 F_{<3}$ we obtain $(X^8 + x)^3(X^{16} + x + 1)^3 Z = (N_g^{(1)})^3(Z + N_3)$. We put $Z = \sum_{i=0}^{48} a_i X^i + Y \sum_{j=0}^{45} b_j X^j$ and obtaing a system of 334 equations with 95 unknowns. Then $F_{<3} = N_{<3}/D_{<3}$, where $D_{<3} = ((x^2 + x)(X^8 + x)^3(X^{16} + x + 1)^3$ and $N_{<3}$ is

$$\begin{aligned}
&\left(x^8+x^6+x^5+x^4+x^3+x+1\right)X^{48}+\left(x^4+x^2\right)X^{46}+\left(x^8+x^6+x^4+x\right)X^{44}\\
&+\left(x^5+x^3+x^2+x\right)X^{42}+\left(x^4+x^2\right)X^{41}+\left(x^{10}+x^7+x^3+x^2+1\right)X^{40}+\left(x^4+x^2\right)X^{39}\\
&+\left(x^8+x^6+x^2+x\right)X^{38}+\left(x^5+x^4+x^3+x\right)X^{37}+\left(x^4+x^2\right)X^{36}Y\\
&+\left(x^{10}+x^9+x^8+x^7+x^5+x^4y+x^4+x^2y\right)X^{36}+\left(x^5+x^4+x^3+x\right)X^{35}\\
&+\left(x^9+x^8+x^7+x^6+x^5+x^3+x^2+x\right)X^{34}+\left(x^8+x^6+x^5+x^4+x^3+x^2\right)X^{33}\\
&+\left(x^5+x^4+x^3+x\right)X^{32}Y+\left(x^8+x^7+x^6+x^5y+x^4y+x^4+x^3y+x^3+x^2+xy+1\right)X^{32}\\
&+\left(x^8+x^6+x^5+x^4+x^3+x^2\right)X^{31}+\left(x^4+x\right)X^{30}+\left(x^9+x^7+x^4+x\right)X^{29}\\
&+\left(x^8+x^6+x^5+x^4+x^3+x^2\right)X^{28}Y\\
&+\left(x^{11}+x^8y+x^7+x^6y+x^6+x^5y+x^5+x^4y+x^4+x^3y+x^3+x^2y+1\right)X^{28}\\
&+\left(x^9+x^7+x^4+x\right)X^{27}+\left(x^{10}+x^9+x^8+x^5+x^3+x^2\right)X^{26}+\left(x^9+x^7\right)X^{25}\\
&+\left(x^9+x^7+x^4+x\right)X^{24}Y\\
&+\left(x^{12}+x^9y+x^9+x^8+x^7y+x^7+x^6+x^5+x^4y+x^4+x^3+x^2+xy\right)X^{24}+\left(x^9+x^7\right)X^{23}\\
&+\left(x^{10}+x^9+x^5+x^4+x^3+x\right)X^{22}+\left(x^{10}+x^8+x^7+x^5+x^3+x^2\right)X^{21}+\left(x^9+x^7\right)X^{20}Y\\
&+\left(x^{11}+x^9y+x^9+x^7y+x^5+x^4+x+1\right)X^{20}+\left(x^{10}+x^8+x^7+x^5+x^3+x^2\right)X^{19}\\
&+\left(x^{10}+x+1\right)X^{18}+\left(x^9+x^8+x^7+x^4+x^2+x\right)X^{17}+\left(x^{10}+x^8+x^7+x^5+x^3+x^2\right)X^{16}Y\\
&+\left(x^{10}y+x^8y+x^7y+x^7+x^5y+x^5+x^4+x^3y+x^3+x^2y\right)X^{16}+\left(x^9+x^8+x^7+x^4+x^2+x\right)X^{15}\\
&+\left(x^6+x^5+x^4+x^2+1\right)X^{14}+\left(x^{10}+x^9+x^8+x^7+x^4+x^2+1\right)X^{13}\\
&+\left(x^9+x^8+x^7+x^4+x^2+x\right)X^{12}Y\\
&+\left(x^{12}+x^{11}+x^{10}+x^9y+x^8y+x^8+x^7y+x^7+x^5+x^4y+x^4+x^2y+x^2+xy+x\right)X^{12}\\
&+\left(x^{10}+x^9+x^8+x^7+x^4+x^2+1\right)X^{11}+\left(x^{11}+x^8+x^7+x^2+x\right)X^{10}\\
&+\left(x^{10}+x^9+x^8+x^7+x^6+x^5\right)X^9+\left(x^{10}+x^9+x^8+x^7+x^4+x^2+1\right)X^8Y\\
&+\left(x^{13}+x^{10}y+x^9y+x^9+x^8y+x^8+x^7y+x^7+x^6+x^4y+x^3+x^2y+y\right)X^8\\
&+\left(x^{10}+x^9+x^8+x^7+x^6+x^5\right)X^7+\left(x^{11}+x^{10}+x^9+x^6+x^5+x^2+x\right)X^6\\
&+\left(x^{11}+x^{10}+x^9+x^6+x^5+x^2+x\right)X^5+\left(x^{10}+x^9+x^8+x^7+x^6+x^5\right)X^4Y\\
&+\left(x^{10}y+x^{10}+x^9y+x^9+x^8y+x^8+x^7y+x^7+x^6y+x^5y\right)X^4\\
&+\left(x^{11}+x^{10}+x^9+x^6+x^5+x^2+x\right)X^3+\left(x^{11}+x^{10}+x^9+x^6+x^5+x^2+x\right)Y\\
&+x^{11}y+x^{11}+x^{10}y+x^9y+x^9+x^7+x^6y+x^6+x^5y+x^5+x^4+x^2y+xy.
\end{aligned}$$

Notice that denominators of $F_{\leq 12}$ and $F_{<3}$ match. The leading term of $(x^8 + x^6 + x^5 + x^4 + x^3 + x + 1)N_{\leq 12} - (x^6 + x^5 + x^3 + x + 1)N_{<3}$ is $(x^{14} + x^{13} + x^{11} +$

$x^9+x^7+x^6+x^4+x^2)X^{46}$, so that the degree of $E(d)=(x^8+x^6+x^5+x^4+x^3+x+1)F_{\leq 12}(d)-(x^6+x^5+x^3+x+1)F_{<3}(d)$ for $d\geq 3$ is $-(-24+52\times 2^{d-2})$; since the degree of ℓ_d is $-(2^{d+2}-2)$ the degree of $E(d)/\ell_d^3$ is $-(-12+2^d)$ $(d\geq 2)$; therefore, $(x^8+x^6+x^5+x^4+x^3+x+1)S_{\leq d}(1,2)-(x^6+x^5+x^3+x+1)S_{<3}(d)$ tends to zero as d tends to infinity. $\square$

Finally, we take this opportunity to record corrections in normalizations in [T93][Pa. 561], when the degree δ of the infinite place is more than one: In 0.3.7, replace $2\overline{\infty}$ there by $\overline{\infty}^{(-2)}+\overline{\infty}^{(-1)}$ and normalize the differential ω by requiring the residue of $\omega^{(1)}/f$ at $\overline{\infty}^{(-1)}$ be 1. In 0.3.5 recipe, we just drop the implied claim that the resulting Drinfeld module ρ is sgn-normalized (when $\delta>1$). We are not aware of any mistakes in the literature resulting from this, as the usage (as in this paper) so far seems to be limited to the case $\delta=1$. We also record that for the Hopf algebra alluded to in [T17][Pa. 1006], associativity and co-associativity is still conjectural with a lot of computational evidence, and in [T17][Thm. 10.1] the entry $A[v^n]$ should be $A[\zeta_{v^n}]$.

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