J. Ramanujan Math. Soc. **15** No. 1 (2000) pp. 43–52

Hypergeometric Functions for Function Fields II

Dinesh S. Thakur (thakur@math.arizona.edu)

Department of Mathematics, University of Arizona, Tucson, AZ 85721, U.S.A.

Communicated by V. Kumar Murty

Received October 5, 1999; revised December 20, 1999

Abstract. We develop the theory of hypergeometric functions introduced in [6] further by explaining the non-commutative and differential equations aspects and various analogies satisfied by it. In particular, we give analogue of Kummer's solutions at ∞, by defining a suitable analogue of $(a)_{-n}$. The analogies do not always work as expected naively.

0. Introduction

This paper is a sequel to [6], where we introduced and studied hypergeometric functions in the setting of function fields over finite fields. We showed that they satisfy analogues of Gauss differential equations, have integral representations, satisfy good transformation formulae, have interesting continued fractions and that analogues of various special functions and orthogonal polynomials occur as their specializations. In fact, there were two analogues: ${}_rF_s$ with the parameters a, b, c etc in a characteristic zero domain and ${}_r\mathcal{F}_s$ with parameters in a characteristic p domain. (We just recall here that there are also two analogues of cyclotomic theory, gamma functions etc.)

1991 Mathematics Subject Classification. 11G09, 11R58, 33C19, 33D15, 33E30.

Keywords and phrases. Drinfeld modules, Hypergeometric functions, q-analogues.

Supported in part by NSF grants.

To make this paper reasonably self-contained, we will recall the notation and some basic analogies, but leaving more analogies and motivation, including the connection to the theory of Drinfeld modules, to [6]. We will use the same notation, except we write d_{τ} for d_F there.

The q-analogues of the classical hypergeometric series are obtained when n is replaced by q^n in a certain fashion. At this naive level, the $A := \mathbb{F}_q[t]$ analogue seems to be $[n] := t^{q^n} - t$ or just t^{q^n}. Let us define the basic building blocks:

$$d_0 := l_0 := 1, \quad d_n := [n]d_{n-1}^q, \quad l_n := -[n]l_{n-1} \quad n \geq 1. \tag{1}$$

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}, \quad l(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{l_i}. \tag{2}$$

We note here that $e(z)$ and $l(z)$ are the exponential and the logarithm of the Carlitz module respectively and d_i is the Carlitz factorial of q^i. But (see 2.3 of [6]) note that $d_i = [i]([i]-[1])\cdots([i]-[i-1])$ and hence can be regarded as 'factorial' of i. This naive analogy is justified in [1] where the factorials are associated to a subset X of a Dedekind ring R, and d_i turns out to be the value at i of the factorial associated to $X = \{t^{q^j} : j \geq 0\}$ or $X = \{[j] : j \geq 0\}$ and $R = \mathbb{F}_q[t]$ (as well as the value at q^i for $X = R = \mathbb{F}_q[t]$).

Finally we recall the definition of the first analogue of the hypergeometric series: Let

$$(a)_n := \begin{cases} d_{n+a-1}^{q^{-(a-1)}} & \text{if } a \geq 1 \\ 1/l_{-a-n}^{q^n} & \text{if } n \leq -a \geq 0 \\ 0 & \text{if } n > -a \geq 0 \end{cases} \tag{3}$$

and for $a_i, b_i \in \mathbb{Z}$ (for which it makes sense, see 2.2 of [6]), we define the first analogue by

$${}_rF_s := {}_rF_s(a_1, \cdots, a_r; b_1, \cdots, b_s; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n d_n} z^{q^n}. \tag{4}$$

Let us also introduce the second analogue, with which we deal only in 3.2. Let Ω be the completion of an algebraic closure of $\mathbb{F}_q((1/t))$ (our analogue of the field of complex numbers).

$$(z)_i := e_i(z) := \prod_{\substack{a \in A \\ \deg a < i}} (z - a). \tag{5}$$

For $a_i, b_i \in \Omega$ for which it makes sense [6], we define the second analogue by

$${}_r\mathcal{F}_s := {}_r\mathcal{F}_s(a_1, \cdots, a_r; b_1, \cdots, b_s; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n d_n} z^{q^n}. \tag{6}$$

We will concentrate on $F := {}_2F_1$ and $\mathcal{F} := {}_2\mathcal{F}_1$ and leave it to the reader to see what generalizes to ${}_rF_s$ and ${}_r\mathcal{F}_s$ in a similar fashion.

Finally, we also need to recall the following analogue of the binomial coefficients and the Carlitz module:

$$\left\{ \begin{matrix} z \\ q^i \end{matrix} \right\} := \sum_{k=0}^{i} \frac{z^{q^k}}{d_k l_{i-k}^{q^k}}. \tag{7}$$

$$C_a(z) := \sum_{i=0}^{\deg a} \left\{ \begin{matrix} a \\ q^i \end{matrix} \right\} z^{q^i}, \quad a \in A. \tag{8}$$

In the first section, we make explicit the non-commutative and differential aspects implicit in [6], in the second section, we analyze the convergence and other solutions of our analogue of the Gauss differential equation, in particular, we define $(a)_{-n}$ and give analogue of Kummer solutions at ∞. In the last section, we work out some more properties and mention some open problems.

1. Functions and Differential Operators

1.1. We will be concerned with $\mathbb{F}_q$-linear polynomials or power series in z, with coefficients in Ω. These form a non-commutative ring under composition and we identify them with $\Omega\{\tau\}$ or $\Omega\{\{\tau\}\}$, the (non-commutative) polynomial or power series rings respectively, where the commutation relation is $\tau w = w^q \tau$ for $w \in \Omega$. Sometimes we also use τ^{-1} with obvious meaning: $\tau^{-1} w = w^{q^{-1}} \tau^{-1}$.

1.2. The (non-commutative) ring of differential operators for us is $\mathcal{D} := \Omega\{\tau, d_\tau\}$, with commutation relations

$$\tau w = w^q \tau, \quad d_\tau w = w^{q^{-1}} d_\tau, \quad d_\tau \tau - \tau d_\tau = -[-1].$$

1.3. The action of $\mathcal{D}$ on the functions is the obvious one, except perhaps for $d_\tau(\sum a_n \tau^n) := \sum a_n^{1/q}[n]^{1/q}\tau^{n-1}$. We note that $d_\tau(f+g) = d_\tau(f) + d_\tau(g)$ and $d_\tau(wf) = w^{1/q} d_\tau(f)$ and $d_\tau(fw) = d_\tau(f)w$, for $w \in \Omega$. Recall that $\Delta = \tau d_\tau$ and $\Delta_a = \Delta - [-a]$ (for $a \in \mathbb{Z}$). Note that Δ is a derivation. In fact, Δ is just a commutator with t (or any generator $t+\alpha$, $\alpha \in \mathbb{F}_q^*$, of $\mathbb{F}_q[t]$ over $\mathbb{F}_q$). The operators Δ_a's commute among themselves and with the constants. We have $\Delta_a \tau = \tau \Delta_{a+1}$ and so $d_\tau^w = \tau^{-w}\Delta_0 \cdots \Delta_{1-w}$.

The motivation (and analogies) for all this is explained in [6]. We just note that Δ_a is an analogue of $zd/dz + a$, d_τ is an analogue of d/dz.

2. Solutions of Gauss Equation

2.1. In [6], we showed that $y = {}_2F_1(a, b; c; z)$ satisfies the Gauss differential equation (19) of [6]:

$$\Delta_a \Delta_b y = d_\tau \Delta_{c-1} y \tag{9}$$

or equivalently,

$$((1-\tau)\tau d_\tau^2 + (-[-c] + ([-1]^q + [-b] + [-c])\tau)d_\tau - [-a][-b])y = 0.$$

Now we study solutions of (1) more systematically. First we seek the solutions of the form $y = \sum c_n z^{q^{n+g}}$. Equating the coefficients of $z^{q^{g-1}}$, we get the indicial equation $[g+c-1]^{q^{-c}}[g]^{q^{-1}} = 0$ analogous to (1.3.3) of [5], which shows that $g = 0$ or $g = 1-c$. Further,

$$c_{n+1} = \frac{[n+g+a]^{q^{-a+1}}[n+g+b]^{q^{-b+1}}}{[n+g+c]^{q^{-c+1}}[n+g+1]} c_n^q. \tag{10}$$

In 3.1 of [6], we gave two solutions $F(a, b; c; z)$ and $F(1+a-c, 1+b-c; 2-c; z)^{q^{1-c}}$ corresponding two g's which correspond to a particular choice of c_0. As noted there and in 1.3, d_τ is right linear, but not left linear, so together with solution $y(z)$ we get solutions $y(\theta z)$ depending on choice of c_0, in contrast to the classical case where we get $\theta y(z)$. (Of course, for special cases such as $y(z) = z^{q^n}$, eg. with $a = -n$, $c = 1 - n$ both are equivalent.) In particular, if we think of these solutions as functions (rather than non-commutative formal power series in τ) the radius of convergence is affected by the choice of c_0.

2.2. *Radius of convergence*: $F(a, b, c; z)$ makes sense in some region if $c > 0$ (so one of the two solutions in 2.1 is always defined), when it is an honest power series if $a, b > 0$ and a polynomial (terminating case) otherwise, or if $c \leq a \leq 0$ or $c \leq b \leq 0$, when it is a polynomial (with standard conventions, when $c = b$, there are two ways to interpret the series: either as a series truncating at the appropriate place or as ${}_1F_0$. Both are good options). We now state the radius of convergence (in the non-terminating case) both ∞-adically and $\wp$-adically. At ∞, the results follow from the explicit formula above by the calculation of degrees of coefficients. At $\wp$, a prime of A of degree d, we have only to note that $\wp$ divides (to the exponent one then) $[n]$ iff d divides n. We omit the straight-forward calculations.

2.2.1. At ∞: For F the radius of convergence is $q^{c+1-a-b}$, so that for other solutions $F(\theta z)$ the radius is $q^{c+1-a-b+\deg\theta}$. In particular, if we choose $c_0 = 1$ in (2), then we get the radius to be 1.

2.2.2. At $\wp$: For F the radius of convergence is 1 for any $\wp$, so that for other solutions $F(\theta z)$ it is $q^{\mathrm{val}_\wp \theta}$. In particular, if we choose $c_0 = 1$ in (2), then we get the radius to be 1 for $\wp$ of large enough (depending on a, b, c) degree.

2.3. *Kummer solutions*: Kummer gave six different series solutions to Gauss equation, two each at 0, 1 and ∞. (See 1.3 of [5]). We now look for analogues of those. In 2.1, we have mentioned the solution at $z = 0$. Solutions around $z = 1$ or $z = \infty$ involve typically power series in $(z-1)$ or $1/z$. We note that $(z-1)^{q^n}$ or z^{-q^n} are not linear functions. We do not know how to make a good sense of Kummer solution at 1, but to look at 'solutions around infinity' we rather look at τ^{-n} i.e we seek a solution of Gauss equation of the form $y = \sum c_n z^{q^{-n+g}}$. It is easy to verify that parallel to Kummer's solutions $(-z)^{-a}{}_2F_1(a, 1+a-c; 1+a-b; 1/z)$ and $(-z)^{-b}{}_2F_1(b, 1+b-c; 1+b-a; 1/z)$ we have the solutions $\overline{F}(a, 1+a-c; 1+a-b, z)^{q^{-a}}$ and $\overline{F}(b, 1+b-c; 1+b-a; z)^{q^{-b}}$ with

$$\overline{F}(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_{-n}(b)_{-n}}{(c)_{-n}d_{-n}} z^{q^{-n}} \tag{11}$$

where for $n \geq 0, a \in \mathbb{Z}$ we have put

$$(a)_{-n} := \begin{cases} l_{n+a-1}^{q^{-n}} & \text{if } a \geq 1 \\ 1/d_{-a-n}^{q^a} & \text{if } n \leq -a \geq 0 \\ 0 & \text{if } n > -a \geq 0 \end{cases} \tag{12}$$

and $d_{-n} := (1)_{-n} = l_n^{q^{-n}}$. We note some immediate consequences of these definitions:

$$(a)_{-n-1} = [-n-a]^{q^{a-1}}(a)_{-n}^{q^{-1}} \tag{13}$$

$$(a)_{-n} = (a+1)_{-n+1}^{q^{-1}} \text{ for } a \neq 0, \tag{14}$$

$$(a+1)_{-n} := [-n-a]^{q^a}(a)_{-n} \text{ for } a \neq 0. \tag{15}$$

Secondly, if $a, b, c > 0$, these are only formal solutions, since straight valuation calculation shows that the series converges only for $z = 0$, both at ∞ or $\wp$. On the other hand, we get honest solutions in the terminating cases. But note that in that case, we do not get any new solution than ones listed in 2.1.

2.4. The notation $(a)_{-n}$ clashes with the earlier notation $(a)_n$ when $n = 0$ and that means if we want to consider a bilateral series, we need to choose c_0 appropriately to get the same term for $n = 0$.

3. More Properties

3.1. *Contiguous relations and orthogonal polynomials*: Classically, many orthogonal polynomials can be obtained by simple specializations of the hypergeometric series. In 3.5 of [6], we showed that analogues by

Carlitz [2, 3] of binomial coefficients, Jacobi and Legendre polynomials etc. come as simple specializations of our hypergeometric functions. Carlitz has given several reasons for such analogies in [2, 3]. But we now examine for them one crucial property that holds for classical orthogonal polynomials: they satisfy a three term recursion relation of the type $p_{n+1}(x) = (a_n x + b_n)p_n(x) + c_n p_{n-1}(x)$ and conversely sequence of polynomials satisfying such recursion (with mild sign conditions on a_i, b_i, c_i) gives orthogonal polynomials for some inner product by Favard's theorem. By analogy, we would expect a relation of the form $p_{n+1}(z) = (a_n\tau + b_n)p_n(z) + c_n p_{n-1}(z)$ for our 'linear' orthogonal polynomials p_n of degree q^n. In fact, for any reasonable notion of inner product for which τ is self-adjoint, we are forced to this by imitating the classical proof.

Now by (27) of [6], the binomial coefficients do indeed satisfy such relation, even with $c_n = 0$. But by (29) and (30) of [6], the particular form for the specializations there show that the three term relations there are just contiguous relations (of type 1.4.1 of [5]) for F, namely the relations as in 3.3 of [6] between F, $F(a+1)$ and $F(a-1)$. A straight calculation gives the contiguous relation of the form $A(1-\tau)F(a+1) = (B\tau + C)F(a) + DF(a-1)$, which translates for $P_m := P_m^{(n,k)}$ of (29) of [6] as $P_{n+1} = (a_n\tau + b_n)P_n + c_n(1-\tau)P_{n-1}$ with $c_n \neq 0$.

3.2. *Integral formula*: Classically, we have well-known integral formula of Euler for F:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt.$$

In essence, it is obtained via the integral formula (a consequence of Cauchy's residue theorem for appropriate contour C)

$$(f \circ g)(z) = \frac{1}{2\pi i}\int_C f(w)g(z/w)\frac{dw}{w}$$

for the Hadamard convolution

$$(f \circ g)(z) = \sum a_n b_n z^n, \text{if } f(z) = \sum a_n z^n, g(z) = \sum b_n z^n$$

because $F(a,b;c;z)$ is Hadamard convolution (term by term product) of $\sum(a)_n/n!z^n = (1-z)^{-a}$ and $\sum(b)_n/(c)_n z^n$ which is essentially up to multiplying by z^{-c} can be realized as a shifted version of the first series.

For our second analogue, we do have ((32) of [6]) $\sum(a)_n/(1)_n z^{q^n} = e(a(l(z))$, an analogue of $e(a\log(1-z)) = (1-z)^a$, but we do not know yet how to interpret the second series as a shifted version. But in the special case $b=c$, the series is just $\sum z^{q^n} = (1-\tau)^{-1}(z)$ in contrast to the classical $\sum z^n = (1-z)^{-1}$. In this case, Hadamard integral just reduces to the Cauchy integral. We do get similar integral formula for $F(a,b;b;z)$ case, because the first series being linear, we may use $\sum z^n = (1-z)^{-1}$ in place of our $\sum z^{q^n}$.

For the first analogue, we get Hadamard convolution integrals for $F(-m,b;b;z)$ and $F(-m,-n;1;z)$, by realizing them as convolutions of binomial coefficients ((27) of [6]) rather than of full binomial series!

Now instead of looking at the negative parameters, let us look at the positive parameters: Let us first compare $\sum(a)_n/(1)_n\tau^n$ with $(1-\tau)^{-a}$ in analogy with the classical case. We have already noted that for $a=1$ we do get $(1-\tau)^{-1}$. By (13) and (17) of [6], we have

$$\sum(a)_n/(1)_n\tau^n = \Delta_{a-1}\cdots\Delta_1\sum\tau^n$$

whose analogue would be just $(zd/dz+a-1)\cdots(zd/dz+1)(1-z)^{-1} = (a-1)!(1-z)^{-a}$. Similarly, $\sum(b)_n/(c)_n\tau^n$ can be realized by (12) of [6], which implies that $\tau^{c-1}(c)_n = (1)_{n+c-1}\tau^{c-1}$, as $\tau^{1-c}\sum(b-c+1)_n/(1)_n\tau$ essentially (plus a correction term which goes away anyway taking Hadamard convolution, just as in the classical case). This would be a good analogue of integral formula then, except that because of our non-commutative rules the sum under consideration is not really '$(a-1)!(1-\tau)^{-1}$'. For example, for $a=2$, $\Delta_1(1-\tau)^{-1} = (1-\tau)^{-1})(-[-1])(1-\tau)^{-1}$.

3.3. *Summation formula*: The zeros of F satisfied interesting analogy in 3.7 of [6]. Now (32) of [6] specializes to $e(a(l(1))) = C_a(1)$, for $a \in A$ and this is zero iff $q=2$ and a is a multiple of t^2+t.

3.4. Continued fractions: In addition to the generalized continued fractions for the hypergeometric functions in 3.6 of [6], we refer to Section 6 of [7] for evaluation of some interesting simple continued fractions in terms of hypergeometric functions, in spirit of Lehmer's results on classical hypergeometric functions.

3.5. *Open questions*: The theory and literature on hypergeometric and q-hypergeometric functions is vast, so there are indeed a huge number of open questions. But we have focused on only the simplest properties and it seems that the main open question is that of geometric interpretation. Since historically hypergeometric functions and especially their q-analogues also got backed by solid mathematical structures (such as quantum groups) much later, we are hopeful that a good geometric understanding of the hypergeometric functions, integral formulas etc. will emerge.

Another important issue is what happens when the two solutions in series around zero coincide (i.e., when $c = 1$). Classically one gets a logarithm function from the bigger space of functions than series around zero. What should be a good space to look at, when there is only one $\mathbb{F}_q$-linear series solution? Are Kummer solutions at other points to be found in such spaces? May be the formal solutions we found are related to actual solutions in bigger function spaces, analogous to the classical Stokes line phenomenon.

Finally, is there an interesting p-adic (or $\wp$-adic) theory analogous to Dwork's theory [4]?

Acknowledgement

I am grateful to Greg Anderson, John Brillhart, Pierre Deligne and Bernard Dwork for helpful conversations explaining various points of the classical hypergeometric function theory.

References

[1] M. Bhargava, *P*-orderings and polynomial functions on arbitrary subsets of Dedekind rings, *J. Reine Angew. Math* **490** (1997), 101–127.

[2] L. Carlitz On a certain functions connected with polynomials in a Galois field, *Duke Math. J.* **1** (1935), 137–168.

[3] ——, Some special functions over $GF(q, x)$, *Duke Math. J.* **27** (1960), 139–158.

[4] B. Dwork, *Generalized Hypergeometric Functions*, Oxford University Press, New York (1990).

[5] L. Slater *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge (1966).

[6] D. Thakur Hypergeometric series for function fields, *Finite fields and their applications* **1** (Carlitz Special Issue) (1995), 219–231.

[7] D. Thakur Exponential and continued fractions, *J. Number Theory* **59** (1996), 248–261.