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Transcendence of gamma values for $\mathbb{F}_q[T]$

By DINESH S. THAKUR*

Abstract

We prove that many values at proper fractions of the gamma function for $\mathbb{F}_q[T]$ (introduced by Carlitz and Goss) are transcendental over $\mathbb{F}_q(T)$. In particular, $(n-1/b)!$ is transcendental for any integer n and a positive integer $b > 1$, prime to q. Our proof is based on the transcendence criterion of Christol.

Introduction

Gamma functions play an important role in number theory through their connections with cyclotomy, occurrence in the contributions at infinite places to the zeta functions, occurrence of their values at fractions in the expressions for periods of abelian varieties, Lerch's formula, importance of their values at natural numbers (factorials) in combinatorics, and through their occurrence in coefficients and special values of many classical special functions such as hypergeometric functions.

In this paper, we will be concerned with the gamma function (for a polynomial ring over a finite field) interpolated by Goss from the factorials that were introduced by Carlitz. For the relevant history, details and related material on Drinfeld modules (which we do not need here) we refer to [GHR], [T3], and the references there.

Let $\mathbb{F}_q$ be a finite field of characteristic p consisting of q elements. Let T be a variable, $t := 1/T$. Let $A = \mathbb{F}_q[T]$, $K = \mathbb{F}_q(T)$, $K_\infty = \mathbb{F}_q((t))$, and let Ω be the completion of an algebraic closure of K_∞. Then, in many respects, A, K, K_∞, Ω play roles analogous to those of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively, in the theory of function fields.

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We begin by introducing Carlitz's factorial $\Pi(n) \in A$ for $n \in \mathbb{Z}_{\geq 0}$. Let

$$[n] := T^{q^n} - T, \quad D_0 := 1, \; D_n := \prod_{k=1}^{n} [k]^{q^{n-k}} = \prod_{i=0}^{n-1} (T^{q^n} - T^{q^i}).$$

Using the base q expansion of n, we then define the factorial by

$$\Pi(n) := \prod D_i^{n_i}, \text{ for } n = \sum n_i q^i, \; 0 \leq n_i < q.$$

Observing that the 'unit part'

$$\overline{D_i} := \frac{D_i}{T^{\deg D_i}} = 1 - t^{q^i - q^{i-1}} + \text{higher degree terms} \to 1 \text{ as } i \to \infty,$$

Goss interpolated the factorial to $\overline{\Pi}(n) \in K_\infty$ for $n \in \mathbb{Z}_p$ by

$$n! := \overline{\Pi}(n) := \prod \overline{D}_i^{n_i} \text{ for } n = \sum n_i q^i, \; 0 \leq n_i < q. \tag{1}$$

(There is a way to put back the degree part which is removed (see [T3]), but since it does not make any difference for our main results, we avoid the technicalities it introduces. Also, note that, in [T3], we use $n!$ for Π rather than for $\overline{\Pi}$.)

We refer to [T3] and the references there for several reasons why this factorial and gamma (obtained by variable shift of one: $\Gamma(n+1) := n!$) are good analogues of the classical factorial and gamma. We describe some reasons by catch-words: prime factorization, divisibility properties, occurrence in the Taylor coefficients in the relevant exponential (of Carlitz-Drinfeld module), functional equations, interpolations at finite primes and connection with Gauss sums (Gross-Koblitz type formula), connection with periods (Chowla-Selberg type formula).

We will be concerned with the transcendence of $n!$ for n a proper fraction. (Classically, the only such value (up to integral shifts, which we will ignore here and below) known for a long time was $\Gamma(1/2) = (-1/2)! = \sqrt{\pi}$). In [T1], (see also [T2, T3]) we proved that for our factorial $(-1/2)!$ is (the unit part of) $\sqrt{\tilde{\pi}}$, when $p \neq 2$ and where $\tilde{\pi} \in \Omega$ is the period of the Carlitz-Drinfeld exponential (analogue of $2\pi i \in \mathbb{C}$). By work of Wade [W], $\tilde{\pi}$ is known to be transcendental. This implies the transcendence of $(-1/2)!$ if $p \neq 2$. In fact, for any q, the correct analogue for the classical $(-1/2)!$ is $(1/(1-q))!$, because 2 and $q-1$ are the number of roots of unity in $\mathbb{Z}$ and $\mathbb{F}_q[T]$, respectively. In [T1], we related $(a/(1-q))!$ for $0 < a < q$ to $\tilde{\pi}$, and hence these values are transcendental.

In [A], Allouche gave another proof of the transcendence of $\tilde{\pi}$, based on the transcendence criterion of Christol, which was the inspiration for this paper.

In [T1] (see [T2, p. 34]) we proved an analogue of the Chowla-Selberg formula for constant field extensions, expressing periods of Drinfeld modules with complex multiplication in terms of gamma values at some particular fractions. Combining this with their results on transcendence of periods, Jing Yu

[Y] and Thiery [Thi] (by different techniques) proved that $(1/(1-q^2))!$ is transcendental. This should be considered as an analogue of Chudnovsky's ([Chu, p. 8]) result on the transcendence of $\Gamma(1/3)$, $\Gamma(1/4)$ etc. (The analogy relates to the denominators being the number of roots of unity in the field giving the 'complex multiplication' in each case. The field is $\mathbb{F}_{q^2}(T)$ for the function field case and quadratic imaginary classically.)

In his I.C.M. talk (cited above), Chudnovsky calls the problem of transcendence of gamma values at fractions 'the most important and difficult in transcendence theory'. We look at the function field analogue and our main result gives, in particular, the transcendence of $(n - 1/b)!$ for any integer n and any integer $b > 1$, which is prime to p, for the gamma function introduced above.

Main result

We start by recalling the transcendence criterion of Christol (see [C], [CKMR]):

THEOREM 1 (Christol). *The quantity $\sum_{n=0}^{\infty} a_n t^n \in K_\infty$ ($a_n \in \mathbb{F}_q$) is transcendental over K if and only if there are infinitely many distinct subsequences (c_n) of the sequence (a_n) of the form $c_n = a_{q^k n+s}$ with $0 \le s < q^k$.*

We are interested in values of the factorial at rationals r in $\mathbb{Z}_p$. If $r \in \mathbb{Z} \subset \mathbb{Z}_p$, then we have proved ([T2, p. 34]) that $r!$ is transcendental (over K), if $r < 0$ and is in K otherwise. Hence we concentrate on r's which are proper fractions.

By subsection 7.2 of [T2] (see also p. 34 of [T2]), $r!$ depends up to multiplication by a factor algebraic over K only on r modulo $\mathbb{Z}$. So we assume, without loss of generality, that $-1 < r < 0$. Write $r = a/b$, where $a, b \in \mathbb{Z}$ and p does not divide b. Thus, for some $\mu \in \mathbb{N}$ (eg. $\mu = \phi(b)$), b divides $q^\mu - 1$ by Fermat's Little Theorem. Hence we can write $a/b = c/(1-q^\mu)$ for some $0 < c < q^\mu - 1$. Let $c = \sum_{l=0}^{\mu-1} \alpha_l q^l$, with $0 \le \alpha_l < q$. Since $q^l/(1-q^\mu) = \sum q^{l+\mu n}$, it follows from the definitions that

$$(2) \qquad r! = \prod_{l=0}^{\mu-1} \left(\frac{q^l}{1-q^\mu}\right)!^{\alpha_l}.$$

As a corollary to the main result below, we will see that the building blocks $(q^l/(1-q^\mu))!$ are all transcendental.

THEOREM 2. *Let α_l $(0 \le l < \mu)$ be nonnegative integers. If $\mu \sum \alpha_l q^l < q^\mu - 1$, then*

$$G := \prod_{l=0}^{\mu-1} \left(\frac{q^l}{1-q^\mu}\right)!^{\alpha_l} \tag{3}$$

is transcendental over K. More generally, if $\mu \sum \alpha_{l-h} q^l < q^\mu - 1$, for some $0 \le h < \mu$, (we interpret α_{l-h} to be α_k where $0 \le k < \mu$ and $k \equiv l - h \bmod \mu$), then G is transcendental.

Proof. To focus on the main ideas, as a warm-up, we first prove that $(1/(1-q^\mu))!$ is transcendental for each $\mu \in \mathbb{N}$. By (1),

$$\left(\frac{1}{1-q^\mu}\right)! = \prod_{i=1}^{\infty} \overline{D}_{i\mu}. \tag{4}$$

If we put $\overline{[n]} := [n]/T^{q^n} = 1 - t^{q^n - 1}$, then $\overline{D}_n = \prod_{k=1}^{n} \overline{[k]}^{q^{n-k}}$. We bring (4) into a form more suitable for an application of Christol's criterion as follows:

If we denote by $\lceil x \rceil$ the smallest integer not less than x, we have

$$\begin{aligned} P := \left(\frac{1}{1-q^\mu}\right)!^{1-q^\mu} &= \prod_{i=1}^{\infty} \prod_{k=1}^{\mu i} \overline{[k]}^{q^{\mu i - k}(1-q^\mu)} = \prod_{k=1}^{\infty} \overline{[k]}^{\sum_{i=\lceil k/\mu \rceil}^{\infty} q^{\mu i - k}(1-q^\mu)} \\ &= \prod_{k=1}^{\infty} \overline{[k]}^{q^{\lceil k/\mu \rceil \mu - k}} = \prod_{i=1}^{\infty} \prod_{j=0}^{\mu-1} (1 - t^{q^{\mu i} - q^j}). \end{aligned}$$

It is enough to show that P is transcendental. Writing $P = \sum a_n t^n$ we are now ready to use Christol's criterion.

Consider the representations

$$n = \sum (q^{\mu i} - q^j), \quad \text{all terms distinct, } 0 \le j < \mu. \tag{5}$$

It is clear that if such a representation is impossible, then $a_n = 0$, whereas if such a representation is unique (not always the case), then $a_n = \pm 1$.

We will use this to show:

Claim. There are infinitely many subsequences of the form $b_n := a_{q^k n + (q^k - k)} = a_{q^k(n+1)-k}$. (Hence P is transcendental.)

Proof of the claim. Let k be a multiple of $\mu q^{\mu - 1}$.

First note that

$$\sum_{i=1}^{k/\mu - 1} \sum_{j=0}^{\mu-1} (q^{\mu i} - q^j) < \mu \frac{q^k - 1}{q^\mu - 1} < q^k - k \tag{6}$$

for large k, unless $q = 2$ and $\mu = 1$, which has been handled already.

By (6), to get a representation of $q^k(n+1) - k$ of the form $\sum (q^{\mu i} - q^j)$ as in (5), we need to get a contribution of more than $q^k n$, so at least of $q^k(n+1)$

from the $q^{\mu i}$'s in the terms with $i \geq k/\mu$. If they contribute $q^k(n+f)$, with $f \geq 1$, then there being at most $n+f$ terms, the q^j's take away at most $(n+f)q^{\mu-1}$. So the total is at least $(n+f)(q^k - q^{\mu-1}) > (n+1)q^k - k$, if k is large and $f > 1$ and $q^{k-\mu+1} > n+1$ say. So if $q^{k-\mu+1} > n+1$, we need to have $f = 1$ to get a representation.

Now let $n+1 < k/q^{\mu-1}$. Then the resulting sum is at least $q^k(n+1) - (n+1)q^{\mu-1} > q^k(n+1) - k$. Also note that the inequalities still hold if we add the contribution from $i < k/\mu$, if any. Thus no representation is possible.

In other words, for sufficiently large k, b_n is 0 for at least the first $k/q^{\mu-1}-1$ values of n. Since $k/q^{\mu-1} - 1 \to \infty$ as $k \to \infty$, to show that there are infinitely many distinct subsequences (b_n), it is enough to show that infinitely many of these subsequences are not identically zero.

Let $m := k/q^{\mu-1} - 1$ and $n := q^\mu + q^{2\mu} + \cdots + q^{m\mu}$. Then

$$q^k(n+1) - k = (q^k - q^{\mu-1}) + (q^{k+\mu} - q^{\mu-1}) + \cdots + (q^{k+m\mu} - q^{\mu-1})$$

is the unique representation of the form (5), by an argument similar to that above.

In more detail, as above, we are forced to have $f = 1$, unless $q = 2$ and $\mu = 1$, which is already handled. Now $q^k(n+1) = q^k \sum_{i \geq 0} \beta_i q^{\mu i}$, with $0 \leq \beta_i \leq \mu$, implies that β_i is 1, if $i \leq m$ and is 0 otherwise, by the uniqueness of the expansion to the base q^μ, since $\mu < q^\mu$. Hence the positive contribution from $q^{i\mu}$'s matches termwise, so q^j's have to be all $q^{\mu-1}$'s to get the representation. This establishes the uniqueness. So $b_n = \pm 1 \neq 0$, and the proof of the claim and of the transcendence of $(1/(1-q^\mu))!$ is complete.

Now we prove the first part of the theorem. Write $c := \sum \alpha_l q^l$. By (3) and a calculation similar as above, to show that G is transcendental, it is enough to prove the transcendence of

$$\tilde{P} := G^{1-q^\mu} = \prod_{i=1}^{\infty} \prod_{l=0}^{\mu-1} \prod_{j=0}^{\mu-1} (1 - t^{q^{l+\mu i} - q^j})^{\alpha_l}.$$

We write $\tilde{P} = \sum \tilde{a}_n t^n$.

Consider the representations of the form $n = \sum t_{i,j,l}(q^{l+\mu i} - q^j)$ with $t_{i,j,l} \leq \alpha_l$.

If such a representation is impossible, then $\tilde{a}_n = 0$, whereas if such a representation is unique and if each nonzero $t_{i,j,l}$ is α_l, then $\tilde{a}_n = \pm 1$.

For proving the first claim of the theorem, we consider $\tilde{b}_n := \tilde{a}_{q^k(n+1)-k}$ and claim that there are infinitely many such subsequences.

The proof is similar: Let k be a multiple of $w\mu s_\mu$, where $w := \sum \alpha_l$ and $s_\mu := (q^\mu - 1)/(q-1)$.

First note that

$$\sum_{l+\mu i<k} t_{i,j,l}(q^{l+\mu i}-q^j) \leq \sum \alpha_l(q^{l+\mu i}-q^j) \leq \left(\sum_l \alpha_l q^l\right)\mu\frac{q^k-1}{q^\mu-1} < q^k-k,$$

if k is large. Hence, exactly as before, for large enough k, $\tilde{b}_n = 0$, if $n+2 < k/q^{\mu-1}$.

Let $m := k/(ws_\mu)-1$ and $n := c\mu(1+q^\mu+\cdots+q^{m\mu})-1$. Then

$$q^k(n+1)-k = \sum_{l,j}\alpha_l\left(\sum_{i=0}^{m}(q^{k+l+i\mu}-q^j)\right)$$

is the unique representation of the form we want, as we can see as follows:

As before, a straight estimate forces $f=1$, unless $\sum \alpha_l q^l \leq q-1$, the case already handled above. Now,

$$q^k(n+1) = q^k\sum_{i,j,l}(\alpha_l q^l)q^{i\mu} = q^k\sum_{i,j,l} t_{i+k/\mu,j,l}q^{l+i\mu},$$

with $t_{i+k/\mu,j,l} \leq \alpha_l$, implies that $t_{i+k/\mu,j,l}$ is α_l for $0\leq i\leq m$ and is 0 otherwise, by the uniqueness of the expansion to the base q^μ, since $\sum_{j,l} t_{i+k/\mu,j,l}q^l \leq \mu\sum\alpha_l q^l < q^\mu-1$ for any i. Hence the positive contribution matches termwise, and hence the negative contribution has to match termwise to get a representation. This establishes the uniqueness.

Thus $\tilde{b}_n = \pm 1 \neq 0$ and we get infinitely many distinct subsequences of the required form. Application of Theorem 1 then finishes the proof of the first claim.

Finally, we indicate the modifications needed to prove the second claim of the theorem, whose proof follows exactly same method:

Instead of $\tilde{b}_n$ we use $b_n' := q^k(n+1)-(k+h)$. We let $k+h$ be a multiple of $\mu s_\mu w$. The first part of the argument follows from the fact that for large k,

$$\begin{aligned}\sum_{l+\mu i<k} t_{i,j,l}(q^{l+\mu i}-q^j) &< \mu\sum_l \alpha_l q^l \sum_{i=0}^{\lceil (k-l)/\mu\rceil -1} q^{i\mu}\\ &\leq \frac{\mu q^k}{q^\mu-1}\sum \alpha_{l-h}q^l < q^k-k.\end{aligned}$$

Let $m := (k+h)/(ws_\mu)-1$ and $n := c\mu(q^h+q^{h+\mu}+\cdots+q^{h+m\mu})-1$. Then

$$q^k(n+1)-(k+h) = \sum_{l,j}\alpha_l\sum_{i=0}^{m}(q^{l+k+h+i\mu}-q^j)$$

is the unique representation by a similar argument and the theorem is proved. $\square$

COROLLARY 1. (1) *For* $0\leq l<\mu$, $(q^l/(1-q^\mu))!$ *is transcendental.*

(2) *For any integer n and any integer $b > 1$ and prime to p, if $0 < a < b/\phi(b)$ (eg. $a = 1$), then $(n - a/b)!$ is transcendental.*

(3) *If q is a primitive root modulo b and $r \in \mathbb{Z}_p$ is a fraction with exact denominator b, then $r!$ is transcendental.*

Proof. Part (1) follows by taking $h = l$ in the theorem and observing that $\mu < q^\mu - 1$, unless $q = 2$ and $\mu = 1$, the case handled already. As for (2), by the remarks before the theorem, we can assume that $n = 0$. We have to just apply the first claim in the theorem to $\sum \alpha_l q^l = a(q^{\phi(b)} - 1)/b$, with $\mu = \phi(b)$. Part (3) follows by expressing the reduced residues modulo b as powers of q. □

Remarks. (I) There is another gamma function (§4 of [T2]) for $\mathbb{F}_q[T]$ whose domain is Ω (ignoring poles). It was noted in subsection 6.1 of [T2], that for $q = 2$, the values of this gamma function at proper fractions (which are rational functions in T now, rather than rational numbers) are all algebraic multiples of $\tilde{\pi}$, and hence are transcendental. In [T2], only a few special values were shown to be transcendental, for general q, by relating them to the periods of appropriate Drinfeld modules. For general q, by using Anderson's 'soliton' techniques, Sinha has shown in his University of Minnesota thesis (in preparation) that the values at monic proper fractions (of degree less than zero) are essentially periods of appropriate t-motives. (These higher dimensional Drinfeld modules are analogues of Jacobians of Fermat curves.) Hence those values (one can shift the arguments by integers also, by p. 34 of [T2]) turn out to be transcendental by Jing Yu's results on periods.

(II) Several questions remain: (1) Are the values not handled by the theorem or by Sinha (for the other gamma function) transcendental?

(2) What is the situation for the gamma function (and its interpolations at finite primes) for general function fields introduced in [T1]–[T3]?

(3) Certain monomials (see pp. 80–81 of [T3]) in gamma values at fractions were shown to be algebraic. Are the rest transcendental?

(4) What is the situation for the values of the two variable gamma function introduced by Goss (see [T1, §8], and the references there)?

(5) When should the values be algebraically independent?

After seeing this paper, Allouche was able to remove the restrictions on the numerators by applying the Christol's criterion to the logarithmic derivative (with respect to T) of the gamma value. The author has then generalized this result by answering the question (3) above affirmatively. (See 'Transcendence of the Carlitz-Goss gamma function at rational arguments' by J.-P. Allouche (To appear in J. Number Theory) for details.)

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UNIVERSITY OF ARIZONA
E-MAIL ADDRESS: THAKUR@MATH.ARIZONA.EDU

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