

# A new family of character combinations of Hurwitz zeta functions that vanish to second order

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## Abstract

We prove the second order vanishing of a new family of character combinations of Hurwitz zeta functions, thus answering a query of Anderson in [And02]. Along the way, we prove a family of new trigonometric identities.

## 1 Introduction

In [And02] G. Anderson notes that “The relations standing between the Main Formula, the index formulas of Sinnott, Deligne reciprocity, the theory of Fröhlich, the theory of Das, the theory of the group cohomology of the universal ordinary distribution and Stark’s conjecture and its variants deserve to be thoroughly investigated. We have only scratched the surface here. Stark’s conjecture is relevant in view of the well known expansion

$$\zeta(s, x) = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$

of the Hurwitz zeta function at  $s = 0$ .”

This paper is a response to the above comment, in which we tie together the Stark conjectures and Anderson’s result. We prove the second-order vanishing of a character combination of Hurwitz zeta functions whose lead term is known via the Stark conjectures.

## 2 Results

Let  $p$  and  $q$  be distinct primes and set  $N = pq$ . Suppose  $\chi_p$  is a Dirichlet character with  $\text{cond}(\chi)$  dividing  $p$ . We refer to the imprimitive character modulo  $N$  as  $\chi_N$ . That is,  $\chi_N(a) = \chi_p(a)$  for all  $a$  relatively prime to  $N$  and  $\chi_N(a) = 0$  otherwise. Given a set  $S = \{p, q\}$  and a Dirichlet character  $\chi$ , define the incomplete  $L$ -function,  $L_S(s, \chi)$ , to be the ordinary  $L$ -function associated to  $\chi$  with Euler factors for primes in  $S$  removed.

Denote by  $\{x\}$  the fractional part of a real number  $x$ . That is,  $x = [x] + \{x\}$ , where  $[x]$  is the greatest integer in  $x$ . Provided  $x$  is not integral, the version of the Hurwitz zeta function we work with is

$$\zeta(s, \{x\}) = \sum_{n=0}^{\infty} \frac{1}{(\{x\} + n)^s}.$$

### 2.1 A character combination of Hurwitz zeta functions

**Theorem 2.1.** *Assume  $N$ ,  $p$ ,  $q$ ,  $\chi_p$ , and  $\chi_N$  are as above, and that  $\chi_p$  is an even character. The following character combination of Hurwitz zeta functions vanishes to second order at  $s = 0$ :*

$$\begin{aligned} & \sum_{a \bmod N} \chi_N(a) \zeta(s, \{\frac{a}{N}\}) - \sum_{a \bmod p} \chi_p(a) \zeta(s, \{\frac{aq}{N}\}) \\ & + \sum_{a \bmod p} \chi_p(aq) \zeta(s, \{\frac{aq}{N}\}). \end{aligned}$$

*Proof.* It is known that  $L(s, \chi_p)$  vanishes to first order at  $s = 0$  for  $\chi_p$  an even non-trivial character so  $L(s, \chi_p)(1 - q^{-s})$  vanishes to second order. Let  $S = \{p, q\}$ . We calculate that

$$\begin{aligned} & L(s, \chi_p)(1 - q^{-s}) = \\ & L(s, \chi_p)(1 - \chi_p(q)q^{-s}) + L(s, \chi_p)(\chi_p(q)q^{-s} - q^{-s}) = \\ & L_S(s, \chi_N) - q^{-s}L(s, \chi_p) + q^{-s}\chi_p(q)L(s, \chi_p) = \\ & N^{-s} \sum_{a=1}^{N-1} \chi_N(a) \zeta(s, \frac{a}{N}) - N^{-s} \sum_{b=1}^{p-1} \chi_p(b) \zeta(s, \frac{qb}{N}) + N^{-s} \sum_{c=1}^{p-1} \chi_p(qc) \zeta(s, \frac{qc}{N}) = \end{aligned}$$

$$N^{-s} \left[ \sum_{a=1}^{N-1} \chi_N(a) \zeta(s, \frac{a}{N}) - \sum_{b=1}^{p-1} \chi_p(b) \zeta(s, \frac{qb}{N}) + \sum_{c=1}^{p-1} \chi_p(qc) \zeta(s, \frac{qc}{N}) \right].$$

Thus, the latter combination of Hurwitz zeta functions also vanishes to second order at  $s = 0$ . □

## 2.2 Trigonometric relations

**Corollary 2.2.** *Assume  $N, p, q, \chi_p$ , and  $\chi_N$  are as in Theorem 2.1. Assume further that  $\chi_p$  is an even character. Then*

$$\frac{\prod_{\substack{b \bmod p \\ \chi_p(b)=1}} 2 \sin(\{\frac{bq}{N}\} \pi)}{\prod_{\substack{a \bmod N \\ \chi_N(a)=1}} 2 \sin(\{\frac{a}{N}\} \pi) \prod_{\substack{c \bmod p \\ \chi_p(cq)=1}} 2 \sin(\{\frac{cq}{N}\} \pi)} = 1,$$

where each product is over a set of representatives modulo  $\pm 1$ . That is, if  $\sin(x)$  occurs in one of the products then  $\sin(-x)$  does not occur in that product.

*Proof.* By Theorem 2.1,

$$\begin{aligned} & \sum_{a \bmod N} \chi_N(a) \zeta(s, \{\frac{a}{N}\}) - \sum_{a \bmod p} \chi_p(a) \zeta(s, \{\frac{aq}{N}\}) \\ & + \sum_{a \bmod p} \chi_p(aq) \zeta(s, \{\frac{aq}{N}\}) \end{aligned}$$

vanishes to order two; hence, the coefficient of  $s$  is zero. However, by the expansion at  $s = 0$  of the Hurwitz zeta function,

$$\zeta(s, x) = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2),$$

we see that the coefficient of  $s$  is

$$\sum_{a \bmod N} \chi_N(a) \log \frac{\Gamma(\{\frac{a}{N}\})}{\sqrt{2\pi}} - \sum_{b \bmod p} \chi_p(b) \log \frac{\Gamma(\{\frac{bq}{N}\})}{\sqrt{2\pi}} + \sum_{c \bmod p} \chi_p(cq) \log \frac{\Gamma(\{\frac{cq}{N}\})}{\sqrt{2\pi}}.$$

By the orthogonality of the characters, we can isolate the terms where  $\chi_N(a) = \chi_p(b) = \chi_p(cq) = 1$ , to get

$$\begin{aligned} & \sum_{\chi_N(a)=1} \log \frac{\Gamma(\{\frac{a}{N}\})}{\sqrt{2\pi}} - \sum_{\chi_p(b)=1} \log \frac{\Gamma(\{\frac{bq}{N}\})}{\sqrt{2\pi}} + \sum_{\chi_p(cq)=1} \log \frac{\Gamma(\{\frac{cq}{N}\})}{\sqrt{2\pi}} \\ &= \log \frac{\prod_{\chi_N(a)=1} \frac{\Gamma(\{\frac{a}{N}\})}{\sqrt{2\pi}} \prod_{\chi_p(cq)=1} \frac{\Gamma(\{\frac{cq}{N}\})}{\sqrt{2\pi}}}{\prod_{\chi_p(b)=1} \frac{\Gamma(\{\frac{bq}{N}\})}{\sqrt{2\pi}}} = 0. \end{aligned}$$

Because  $\chi_p$  is even,  $\chi_p(x) = 1$  if and only if  $\chi_p(-x) = 1$ ; likewise,  $\chi_N(x) = \chi_N(-x)$ . So the above relation together with the fact that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

give

$$\log \frac{\prod_{\chi_p(b)=1} 2 \sin(\{\frac{bq}{N}\} \pi)}{\prod_{\chi_N(a)=1} 2 \sin(\{\frac{a}{N}\} \pi) \prod_{\chi_p(cq)=1} 2 \sin(\{\frac{cq}{N}\} \pi)} = 0,$$

where the products are now limited by the pairing of the  $x$  and  $-x$  terms. In other words, if  $\sin(x)$  occurs in one of the products then  $\sin(-x)$  does not occur in the same product. This concludes the proof.  $\square$

**Theorem 2.3.** *The first non-vanishing coefficient of*

$$\begin{aligned} & \sum_{a \bmod N} \chi_N(a) \zeta(s, \{\frac{a}{N}\}) - \sum_{a \bmod p} \chi_p(a) \zeta(s, \{\frac{aq}{N}\}) \\ & + \sum_{a \bmod p} \chi_p(aq) \zeta(s, \{\frac{aq}{N}\}). \end{aligned}$$

$$is -\log(q) [\frac{1}{2} \log(p) + \log |(1 - \zeta_p)(1 - \zeta_p^2)^{-1}|].$$

*Proof.* From above, the first non-vanishing coefficient is the coefficient of the lead term of  $\frac{1}{2}(1 - q^{-s})[\zeta(s)(1 - p^{-s}) + L(s, \chi_p)]$ . This lead term is known (because the Stark conjectures are proved in this setting) to be

$$\begin{aligned} & -\log(p)\left(\frac{1}{2}\log(q) + L'(0, \chi_q)\right) = \\ & -\log(q)\left[\frac{1}{2}\log(q) + \log|(1 - \zeta_q)(1 - \zeta_q^{-1})| + \log|(1 - \zeta_q^2)(1 - \zeta_q^{-2})|\right] = \\ & -\log(q)\left[\frac{1}{2}\log(q) + \log|(1 - \zeta_q)(1 - \zeta_q^2)^{-1}|\right], \end{aligned}$$

where  $\zeta_p$  is a primitive  $p$ th root of unity and  $(1 - \zeta_p)(1 - \zeta_p^{-1})$  is a Stark unit in  $\mathbb{Q}(\zeta_p + \zeta_p^{-1}) = \mathbb{Q}(\zeta_p)^+$ , the maximal real subfield of  $\mathbb{Q}(\zeta_p)$ . In other words,  $(1 - \zeta_p)(1 - \zeta_p^{-1})$  is a  $p$ -unit in  $\mathbb{Q}(\zeta_p)$  whose square root generates an extension of  $\mathbb{Q}(\zeta_p)$  that is Abelian over  $\mathbb{Q}$ .  $\square$

### 2.3 Units

Define  $\mathcal{A}$  to be the free abelian group on symbols of the form  $[a]$  for  $[a] \sim [b]$  if and only if  $a - b \in \mathbb{Z}$  and

$$\sin : \mathcal{A} \rightarrow (\mathbb{Q}^{ab})^\times$$

to be the unique homomorphism such that

$$\sin[a] = \begin{cases} 2 \sin(\pi a) = |1 - e(a)| & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0. \end{cases}$$

Let  $\sigma \in G^{ab}$  act on  $\mathcal{A}$  via

$$\sigma([a]) = [b] \iff \sigma \circ e(a) = e(b),$$

where  $e(x) = e^{2\pi i x}$ .

The symbol  $\mathbf{a}_{pq}$  is defined for primes  $p$  and  $q$  with  $2 < p < q$  to be

$$\mathbf{a}_{pq} = \sum_{i=1}^{\frac{p-1}{2}} \left( \left[ \frac{i}{p} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left[ \frac{i}{pq} + \frac{k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left( \left[ \frac{j}{q} \right] - \sum_{l=0}^{\frac{p-1}{2}} \left[ \frac{j}{pq} + \frac{l}{p} \right] \right)$$

and for  $2 = p < q$  to be

$$\mathbf{a}_{pq} = \left( \left[ \frac{1}{4} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left[ \frac{1}{4q} + \frac{k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left( \left[ \frac{j}{q} \right] + \left[ -\frac{1}{2q} + \frac{j}{q} \right] - \left[ \frac{j}{2q} \right] - \left[ -\frac{1}{4q} + \frac{j}{2q} \right] \right).$$

Anderson's result [And02] proves that every quadratic extension over  $k = \mathbb{Q}(\zeta_{pq})$  that is Galois over  $\mathbb{Q}$  is generated either by  ${}^4\sqrt{p}$  or  ${}^4\sqrt{q}$ , or by  $\sqrt{\sin \mathbf{a}_{pq}}$  or one of its conjugates. The work of Das [Das00] and Anderson implies these  $S$ -units generate non-abelian extensions of  $k$ .

### 3 Example

Anderson's result in [And02] includes the construction of algebraic gamma-monomials that generate central, exponent-two extensions of cyclotomic extensions of  $\mathbb{Q}$ , otherwise known as *almost abelian* extensions. Let  $k$  be the cyclotomic field  $\mathbb{Q}(e^{\pi i/2pq})$  for  $p$  and  $q$  distinct odd primes and let  $\alpha \in k$  be such that  $K = k(\sqrt{\alpha})$  is Galois over  $\mathbb{Q}$  (then  $K/\mathbb{Q}$  is an almost abelian extension of fields). Let  $\sigma$  be an element of the Galois group of  $k/\mathbb{Q}$ . By Kummer theory  $\frac{\alpha^\sigma}{\alpha}$  is the square of a unit in  $k$  and, thus, has a square root in  $k$ . For example, take

$$\alpha = \sin(\mathbf{a}_{3,5}) = \frac{\sin(5\pi/15) \sin(2\pi/15)}{\sin(4\pi/15) \sin(3\pi/15)}$$

and  $\sigma : \zeta_{15} \mapsto \zeta_{15}^2$  so

$$\alpha^\sigma = \frac{\sin(10\pi/15) \sin(4\pi/15)}{\sin(8\pi/15) \sin(6\pi/15)}.$$

Then

$$\frac{\alpha^\sigma}{\alpha} = \frac{\frac{\sin(10\pi/15) \sin(4\pi/15)}{\sin(8\pi/15) \sin(6\pi/15)}}{\frac{\sin(5\pi/15) \sin(2\pi/15)}{\sin(4\pi/15) \sin(3\pi/15)}} = \left( \frac{\sin(4\pi/15)}{2 \sin(8\pi/15) \sin(2\pi/15)} \right)^2,$$

where the expression on the right is not just a simplification of the expression on the left. It is found by the algorithm in the appendix.

From this, follow a number of corollaries. First, by dividing both sides of the above expression by the square above, we find a family of new trigonometric identities indexed by products of distinct primes  $pq$ , exemplified by

$pq = 15$ :

$$\frac{4 \sin(2\pi/15) \sin(3\pi/15) \sin(8\pi/15)}{\sin(6\pi/15)} = 1.$$

If we act by multiplication by 8, we see

$$\frac{4 \sin(\pi/15) \sin(4\pi/15) \sin(9\pi/15)}{\sin(3\pi/15)} = 1.$$

Second, in light of the expansion of the Hurwitz zeta function at  $s = 0$ ,

$$\zeta(s, x) = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$

and the fact that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

these identities are equivalent to the fact that certain sums of Hurwitz zeta functions vanish to second order at  $s = 0$ . For example,

$$\begin{aligned} & \zeta(s, \frac{3}{15}) - \zeta(s, \frac{1}{15}) - \zeta(s, \frac{4}{15}) - \zeta(s, \frac{9}{15}) + \zeta(s, \frac{12}{15}) - \zeta(s, \frac{14}{15}) - \zeta(s, \frac{11}{15}) - \zeta(s, \frac{6}{15}) = \\ & s \log \left( \frac{4 \sin(\pi/15) \sin(4\pi/15) \sin(9\pi/15)}{\sin(3\pi/15)} \right) + O(s^2) \end{aligned}$$

vanishes to second order at  $s = 0$ . Therefore, the coefficient of  $s^2$  is of interest and we can calculate that it is  $-\log(3)[\frac{1}{2} \log(5) + \log |(1 - \zeta_5)(1 - \zeta_5^2)^{-1}|]$ !

The vanishing of these Hurwitz zeta functions is connected to the known first-order vanishing of L-functions associated to even Dirichlet characters of conductor dividing odd prime  $p$ . If  $q$  is another odd prime, let  $\chi_p$  be an even character of conductor dividing  $p$  (potentially trivial), and  $\chi_{pq}$  its inflation to the group  $(\mathbb{Z}/pq\mathbb{Z})^\times$ . Thus,  $\chi_{pq}(a) = \chi_p(a)$  except when  $q$  divides  $a$ , in which case  $\chi_{pq}(a) = 0$ . Suppose  $S = \{p, q\}$ , and recall that  $L_S(s, \chi)$  has Euler factors associated to primes in  $S$  removed. Then the second order vanishing of  $L(s, \chi_p)(1 - q^{-s})$  at  $s = 0$  can be expressed in terms of the second order vanishing of Hurwitz zeta functions as in the following example for  $S = \{3, 5\}$ :

$$L(s, \chi_5)(1 - 3^{-s}) =$$

$$15^{-s} \sum_{a=1}^{14} \chi_{15}(a) \zeta\left(s, \frac{a}{15}\right) - 15^{-s} \sum_{b=1}^4 \chi_5(b) \zeta\left(s, \frac{3b}{15}\right) + 15^{-s} \sum_{c=1}^4 \chi_5(3c) \zeta\left(s, \frac{3c}{15}\right).$$

Let  $\zeta(s)$  be the Riemann zeta function. Then the above combination plus  $\zeta(s)(1 - 3^{-s})(1 - 5^{-s})$  is still a function that vanishes to second order, but we have isolated the  $\chi = 1$  terms in each of the above sum. That is, we have isolated the  $a = 1, 4, 11, 14$ ,  $b = 2, 3$ , and  $c = 1, 4$  terms. We can now see that first non-vanishing coefficient from the previous section is, in fact, the lead term of  $\frac{1}{2}(1 - 3^{-s})[\zeta(s)(1 - 5^{-s}) + L(s, \chi_5)]$ . This lead term is known (because the Stark conjectures are proved in this setting) to be

$$\begin{aligned} & -\log(3)\left(\frac{1}{2} \log(5) + L'(0, \chi_5)\right) = \\ & -\log(3)\left[\frac{1}{2} \log(5) + \log |(1 - \zeta_5)(1 - \zeta_5^{-1})| + \log |(1 - \zeta_5^2)(1 - \zeta_5^{-2})|\right] = \\ & -\log(3)\left[\frac{1}{2} \log(5) + \log |(1 - \zeta_5)(1 - \zeta_5^2)^{-1}|\right], \end{aligned}$$

where  $\zeta_5$  is a primitive fifth root of unity and  $(1 - \zeta_5)(1 - \zeta_5^{-1})$  is a Stark unit in  $\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$ , the maximal real subfield of  $\mathbb{Q}(\zeta_5)$ . In other words,  $(1 - \zeta_5)(1 - \zeta_5^{-1})$  is a unit in  $\mathbb{Q}(\sqrt{5})$  whose square root generated an extension that is abelian over  $\mathbb{Q}$ . By comparison,  $\sqrt{\sin(\mathbf{a}_{3,5})}$  generates an extension of  $\mathbb{Q}(\zeta_5)$  that is Galois, but not abelian over  $\mathbb{Q}$ .



## 4 Appendix: explicit square roots

Let  $\mathcal{A}'$  be the subgroup of  $\mathcal{A}$  generated by symbols  $[a]$  for  $a \notin \frac{1}{2}\mathbb{Z}$ . Anderson refers to the Das cocycle,  $\mathbf{c}_\sigma$ , which is an element of  $\mathcal{A}'$ , such that

$$\frac{\sin \mathbf{a}_{pq}^\sigma}{\sin \mathbf{a}_{pq}} = \sin^2 \mathbf{c}_\sigma.$$

We give here an algorithm that gives an explicit expression for the Das cocycle and an interpretation by means of zeta functions.

First, define the symmetric Hurwitz zeta function  $\hat{\zeta}(s, \{x\}) = \zeta(s, \{x\}) + \zeta(s, \{1-x\})$  and its effect on an element of  $\mathcal{A}$ :

$$\begin{aligned} \zeta(s, \mathbf{a}_{pq}) = & \sum_{i=1}^{(p-1)/2} \left[ \hat{\zeta}(s, \{\frac{i}{p}\}) - q^{-s} \sum_{k=0}^{(q-1)/2} \hat{\zeta}(s, \{\frac{i+kp}{N}\}) \right] \\ & - \sum_{j=1}^{(q-1)/2} \left[ \hat{\zeta}(s, \{\frac{j}{q}\}) - p^{-s} \sum_{l=0}^{(p-1)/2} \hat{\zeta}(s, \{\frac{j+ql}{N}\}) \right] \end{aligned}$$

Observe that  $\zeta(s, \mathbf{a}_{pq})$  has  $\log(\sin(\mathbf{a}_{pq}))$  as the coefficient of  $s$  in its expansion at  $s = 0$ .

Let  $N, p, q$  be as in Theorem 2.1. For each  $\sigma \in G_N = \text{Gal}(\mathbb{Q}(\zeta_N)^+/\mathbb{Q})$ , the following algorithm produces the coefficient of  $s$  at  $s = 0$  of  $\zeta(s, \mathbf{a}_{pq}) - \zeta(s, \mathbf{a}_{pq}^\sigma)$  expressed explicitly as a square.

1. Sum over the multiplication-by- $p$  relations starting at  $i = 1, \dots, \frac{p-1}{2}$ , and multiplication-by- $q$  relations starting at  $j = 1, \dots, \frac{q-1}{2}$ .
2. Shift any terms that exceed  $\frac{N-1}{2}$  back into the 1 to  $\frac{N-1}{2}$  range and let  $A$  be the product of these sines.
3. Apply  $\sigma$  to the sum in step 1.
4. Shift this conjugated version of the sum back into the 1 to  $\frac{N-1}{2}$  range and let  $B$  be the product of these sines.

Then the result is  $\sin(\mathbf{a}_{pq})/\sin(\mathbf{a}_{pq}^\sigma) = B^{-2}A^2$ . In terms of zeta functions this says that the coefficient of  $s$  near  $s = 0$  of  $\zeta(s, \mathbf{a}_{pq}) - \zeta(s, \mathbf{a}_{pq}^\sigma)$  is  $2 \log(A/B)$ .

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