$$= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s} \int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty} \frac{x^{w} - x^{2w}}{w^2 n^w} \cdot dw$$

$$= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s} \left(\int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty} \left(\frac{x}{n}\right)^w \frac{dw}{w^2} - \int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty} \left(\frac{x^2}{n}\right)^w \frac{dw}{w^2} \right)$$

$$= -\sum_{n\le x} \Lambda(n)\chi(n)n^{-s}\left(\log \frac{x}{n} - \log \frac{x^2}{n}\right) - \sum_{x<n\le x^2} \Lambda(n)\chi(n)n^{-s}\left(-\log \frac{x^2}{n}\right)$$

$$= \log x \sum_{n\le x^2} \Lambda_x(n)\chi(n)n^{-s} .$$

Now consider the residues obtained by pulling the line of integration to the left. The residue at $z = s$ is $-\log x \frac{L'}{L}(s,\chi)$. That at $z = 1$ is $-E(\chi) \dfrac{x^{1-s} - x^{2(1-s)}}{(1-s)^2}$.

The residues at the nontrivial zeros ρ are $\dfrac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2}$

and at the trivial zeroes they are $\dfrac{x^{-2r-\alpha-s} - x^{2(-2r-\alpha-s)}}{(-2r-\alpha-s)^2}$.

The result now follows. □

Before we can state the next lemma we need to introduce some more notation. For each zero $\beta+i\gamma$ of each L-function (mod q) with $\beta > \frac{1}{2}$, we remove the segment $(\frac{1}{2}+i\gamma, \beta+i\gamma]$ from the half-plane $\sigma > \frac{1}{2}$. We also remove the segment $(\frac{1}{2},1]$. Call the resulting slit half-plane

$\mathcal{D}_q$. We may obviously choose a single-valued analytic branch of $\log L(s,\chi)$ for each $\chi \pmod q$ on $\mathcal{D}_q$ so that

$$\lim_{\sigma\to\infty} \log L(s,\chi) = 0 .$$

It is well known that

$$\log L(s,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s \log n} \qquad (\sigma > 1) .$$

For $\sigma_0 > 1$ we put $\delta = \sigma_0 - 1$; then

$$\log L(s,\chi) = \sum_{n<N^\delta} \frac{\Lambda(n)\chi(n)}{n^s \log n} + O(N^{-\delta^2})$$

uniformly for $\sigma > \sigma_0$. The next lemma extends this formula (with modifications) to the right half of the critical strip.

Lemma 3.9. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and set $\delta = \sigma_1 - \frac{1}{2}$. For each $T \geq 0$ there is a subset $\mathcal{J}(T) \subseteq [0,T]$ with measure

$$J(T) = T(1 + o(1))$$

such that when T is large enough,

$$(20) \quad \max_{s\in C}\left|\log L(s+i\tau,\chi) - \sum_{n<T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(n)\chi(n)}{n^{s+i\tau}\log n}\right| \ll T^{-\delta^2/16} .$$

for all $\tau \in \mathcal{J}(T)$ and $\chi \pmod q$. The implied constant depends on q and σ_1.

Proof: Let $A = \max_{s \in C} |s|$. For each zero $\beta + i\gamma$ of each L-function (mod q) with $\beta \geq \frac{1}{2} + \frac{\delta}{2}$ we remove from $[T^{1-\delta/4}, T]$ those τ satisfying $|\gamma - \tau| \leq T^{\delta/4} + A$. In this way we remove

$$\ll \sum_{\chi \pmod q} N(\tfrac{1}{2}+\tfrac{\delta}{2}, T, \chi)$$

intervals each having measure $\ll T^{\delta/4}$. Since by Lemma 3.1 this is

$$\ll (qT)^{1-\delta/2} \log^{14} qT ,$$

the set of τ removed has measure

$$\ll T^{1-\delta/4} \log^{14} T$$

for fixed q. Letting $\mathcal{J}(T)$ be the set of τ remaining in $[T^{1-\delta/4}, T]$ and letting $J(T)$ be its measure, we find that

$$\begin{aligned} J(T) &= T - T^{1-\delta/4} + O(T^{1-\delta/4} \log^{14} T) \\ &= T(1 + o(1)) . \end{aligned}$$

Note that if $\tau \in \mathcal{J}(T)$, then $C+i\tau \subseteq \mathcal{D}_q$ and if $\sigma+it \in C$, then

(21) $$\min_{\gamma} |\gamma-t-\tau| > T^{\delta/4} .$$

We now prove (20). Let $\tau \in \mathcal{J}(T)$ and $s = \sigma+it \in C$. Integrating both sides of (19) along the half-line $[\sigma+it+i\tau, \infty+it+i\tau)$, we see that

$$\log L(s+i\tau,\chi) - \sum_{n<x^2} \frac{\Lambda_x(n)\chi(n)}{n^{s+i\tau}\log n}$$

$$<< \frac{E(\chi)}{\log x} \int_{\sigma}^{\infty} \frac{x^{2(1-u)} + x^{1-u}}{(1-u)^2+(\tau+t)^2} du$$

$$+ \frac{1}{\log x} \int_{\sigma}^{\infty} \sum_{r=0}^{\infty} \frac{x^{-2r-\alpha-u} + x^{-2(2r+\alpha+u)}}{(2r+\alpha+u)^2 + (\tau+t)^2} du$$

$$+ \frac{1}{\log x} \int_{\sigma}^{\infty} \sum_{\beta+i\gamma} \frac{x^{\beta-u} + x^{2(\beta-u)}}{(\beta-u)^2 + (\gamma-\tau-t)^2} du .$$

Hence

(22) $$\log L(s+i\tau, \chi) - \sum_{n<x^2} \frac{\Lambda_x(n)\chi(n)}{n^{s+i\tau}\log n}$$

$$<< \frac{1}{\log x}(R_1(s+i\tau,\chi) + R_2(s+i\tau,\chi) + R_3(s+i\tau,\chi)) .$$

We now estimate the R_j, $1 \leq j \leq 3$ assuming that $x > 1$

and that T is so large that $\tau+t > 0$ for all $\tau \in \gamma(T)$ and $\sigma+it \in C$. For R_1 we have

$$R_1(s+i\tau,\chi) \ll \frac{1}{(\tau+t)^2} \int_\sigma^\infty x^{2(1-u)} + x^{1-u}\, du$$

$$\ll \frac{1}{(\tau+t)^2} \left(\int_\sigma^1 x^{2(1-u)}\, du + \int_1^\infty x^{1-u}\, du \right)$$

$$\ll \frac{1}{(\tau+t)^2} \cdot \frac{x^{2(1-\sigma)}}{\log x} .$$

Since $\sigma > \sigma_1 = \frac{1}{2}+\delta$, this is

$$R_1(s+i\tau, \chi) \ll \frac{x^{1-2\delta}}{(\tau+t)^2 \log x} . \tag{23}$$

For R_2 we have

$$R_2(s+i\tau, \chi) \ll \frac{1}{(\tau+t)^2} \int_\sigma^\infty \left(\sum_{r=0}^\infty x^{-2r-\sigma-u} + {}^{-2(2r+\sigma+u)} \right) du$$

$$\ll \frac{1}{(\tau+t)^2} \int_\sigma^\infty \left(\sum_{r=0}^\infty x^{-2r-\sigma-u} \right) du$$

$$\ll \frac{1}{(\tau+t)^2} \left(\sum_{r=0}^\infty x^{-2r} \right) \int_\sigma^\infty x^{-u}\, du$$

$$\ll \frac{1}{(\tau+t)^2} \frac{x^{-\sigma}}{\log x} .$$

Thus, since $\sigma > \frac{1}{2}+\delta$ for $s \in C$,

(24) $$R_2(s+i\tau, \chi) << \frac{x^{-1/2-\delta}}{(\tau+t)^2 \log x} .$$

To treat R_3 we write

(25) $$R_3(s+i\tau,\chi) = R_{31}(s+i\tau,\chi) + R_{32}(s+i\tau,\chi) ,$$

where

$$R_{31}(s+i\tau,\chi) = \int_\sigma^\infty \sum_{|\gamma-t-\tau|\leq T^{\delta/4}} \frac{x^{\beta-u} + x^{2(\beta-u)}}{(\beta-u)^2 + (\gamma-t-\tau)^2} du$$

and

$$R_{32}(s+i\tau,\chi) = \int_\sigma^\infty \sum_{|\gamma-t-\tau|>T^{\delta/4}} \frac{x^{\beta-u} + x^{2(\beta-u)}}{(\beta-u)^2 + (\gamma-t-\tau)^2} du .$$

In view of (21), if $\beta+i\gamma$ is a zero with $|\gamma-t-\tau| \leq T^{\delta/4}$, then $\beta < \frac{1}{2}+\frac{\delta}{2}$. Also, for $u \geq \sigma$, $\beta-u < -\frac{\delta}{2}$. Thus

$$R_{31}(s+i\tau,\chi) << \int_\sigma^\infty x^{\beta-u}du \sum_{|\gamma-t-\tau|\leq T^{\delta/4}} \frac{1}{\delta^2 + (\gamma-t-\tau)^2}$$
$$<< \frac{x^{-\delta/2}}{\log x}\left(\sum_{|\gamma-t-\tau|\leq 1} \frac{1}{\delta^2} + \sum_{1<|\gamma-t-\tau|\leq T^{\delta/4}} \frac{1}{(\gamma-t-\tau)^2}\right) .$$

By Lemma 3.2 and the assumption that $t+\tau > 0$ we have

$$\sum_{|\gamma-t-\tau|\le 1} \frac{1}{\delta^2} << \frac{1}{\delta^2} \log q(|t+\tau| + 2) << \log(t+\tau+2)$$

and

$$\sum_{1<|\gamma-t-\tau|\le T^{\delta/4}} \frac{1}{(\gamma-t-\tau)^2}$$

$$<< \sum_{j=1}^{[T^{\delta/4}]} \frac{\log q(|t+\tau+j| +2)}{j^2} + \sum_{j=1}^{[T^{\delta/4}]} \frac{\log q(|t+\tau-j| +2)}{j^2}$$

$$<< \log(t+\tau+T^{\delta/4} + 2) \ .$$

Hence

$$(26) \qquad R_{31}(s+i\tau,\chi) << \frac{x^{-\delta/2}}{\log x} \log(t+\tau+T^{\delta/4} +2) \ .$$

Since $\beta < 1$ for every zero $\beta+i\gamma$, we find that

$$R_{32}(s+i\tau,\chi) << \int_\sigma^\infty (x^{1-u} + x^{2(1-u)})\,du \sum_{|\gamma-t-\tau|>T^{\delta/4}} \frac{1}{(\gamma-t-\tau)^2} \ .$$

The integral is

$$<< \frac{x^{1-2\delta}}{\log x}$$

as $\sigma > \sigma_1 = \frac{1}{2}+\delta$. For the sum we obtain by means of Lemma 3.2 and the assumption $t+\tau > 0$ the estimate

$$\sum_{|\gamma-t-\tau|>T^{\delta/4}} \frac{1}{(\gamma-t-\tau)^2} << \sum_{j>[T^{\delta/4}]} \frac{\log q(|\tau+t+j|+2)}{j^2}$$

$$+ \sum_{j>[T^{\delta/4}]} \frac{\log q(|\tau+t-j|+2)}{j^2}$$

$$<< \sum_{j>[T^{\delta/4}]} \frac{\log(\tau+t+j+2)}{j^2}$$

$$<< \int_{T^{\delta/4}}^{\infty} \frac{\log(\tau+t+x+2)}{x^2}\, dx$$

$$= \frac{-\log(\tau+t+x+2)}{x}\Big|_{T^{\delta/4}}^{\infty} + \int_{T^{\delta/4}}^{\infty} \frac{dx}{x(\tau+t+x+2)}$$

$$<< \frac{\log(\tau+t+T^{\delta/4}+2)}{T^{\delta/4}}$$

Thus

$$(27) \qquad R_{32}(s+i\tau,\chi) << \frac{x^{1-2\delta}}{\log x} \frac{\log(\tau+t+T^{\delta/4}+2)}{T^{\delta/4}} .$$

Combining (25) through (27), we deduce that

$$R_3(s+i\tau,\chi) << \frac{\log(\tau+t+T^{\delta/4}+2)}{\log x}\left(x^{-\delta/2} + \frac{x^{1-2\delta}}{T^{\delta/4}}\right) .$$

This estimate along with (22), (23), and (24) leads to

$$(28)\quad \log L(s+i\tau,\chi) - \sum_{n<x^2} \frac{\Lambda_x(n)\chi(n)}{n^{s+i\tau}\log n}$$

$$<< \frac{1}{(\log x)^2}\left(\frac{x^{1-2\delta}}{(\tau+t)^2} + \frac{x^{-1/2-\delta}}{(\tau+t)^2}\right.$$

$$\left. + \left(x^{-\delta/2} + \frac{x^{1-2\delta}}{T^{\delta/4}}\right)\log(\tau+t+T^{\delta/4}+2)\right) .$$

If we retrace our steps it is not difficult to see that the implied constant depends only on q and δ or q and σ_1. Now choosing $x = T^{\delta/8}$ and taking T so large that

$$T^{1-\delta/4} << \tau+t << T$$

for $\tau \in \mathcal{J}(T)$ and $\sigma+it \in C$, we find that the right-hand side of (28) is

$$<< \frac{1}{\delta^2(\log T)^2}\left(T^{3\delta/8-1-\delta^2 4} + T^{3\delta/16-\delta^2/8-1} + T^{-\delta^2/16}\log T\right.$$

$$\left. + T^{-\delta/8-\delta^2/4}\log T\right)$$

$$<< T^{-\delta^2/16} .$$

Thus (28) gives the estimate

$$\log L(s+i\tau,\chi) - \sum_{n<T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(n)\chi(n)}{n^{s+i\tau}\log n} << T^{-\delta^2/16}$$

uniformly for $s \in C$, $\tau \in \mathcal{J}(T)$ and $\chi \pmod q$. This completes the proof of the lemma. □

Lemma 3.10. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and set $\delta = \sigma_1 - \frac{1}{2}$. Suppose that $\rho > \mu \geq e$. There exist entire functions $\ell_\mu(s,\chi)$ ($\chi \pmod q$) such that if $0 \leq \theta_p < 1$ for each prime $p|q$ or $\mu < p \leq \rho$, and if d is small enough, where $0 < d < \frac{1}{2}$, then there exists a $\tau \in \mathbb{R}$ with

$$\left\|\frac{-\tau \log p}{2\pi} - \theta_p\right\| \leq d \qquad (p|q)$$

and

$$\max_{\chi \pmod q} \max_{s \in C} \left|\log L(s+i\tau,\chi) - \ell_\mu(s,\chi) - \sum_{\mu<p<\rho} \frac{e(\theta_p)\chi(p)}{p^s}\right|$$

$$\ll \mu^{-\delta}.$$

The implied constant depends on σ_1, σ_2, q, and C.

<u>Proof</u>: Let $N = [8 \log \mu]$ and take T so large that $T^{\delta/8} \geq \max(\mu^N, \rho)$. With $\Lambda_x(n)$ as in Lemma 3.8, we have

$$(29)\qquad \sum_{n<T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(n)\chi(n)}{n^{s+i\tau}\log n} = \sum_{p\le\mu}\sum_{k=1}^{N} \frac{\chi(p^k)}{p^{k(s+i\tau)}}$$

$$+ \sum_{\mu<p\le\rho} \frac{\chi(p)}{p^{s+i\tau}}$$

$$+ \sum_{\rho<p\le T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(p)\chi(p)}{p^{s+i\tau}\log p}$$

$$+ \sum_{p\le\mu}\sum_{k=N+1}^{\infty} \frac{\Lambda_{T^{\delta/8}}(p^k)\chi(p^k)}{p^{k(s+i\tau)}\log p^k}$$

$$+ \sum_{\mu<p\le T^{\delta/4}}\sum_{k=2}^{\infty} \frac{\Lambda_{T^{\delta/8}}(p^k)\chi(p^k)}{p^{k(s+i\tau)}\log p^k} .$$

We assume $s \in C$. Since $|\Lambda_{T^{\delta/8}}(n)\chi(n)/\log n| \le 1$ we see that

$$\sum_{p\le\mu}\sum_{k=N+1} \frac{\Lambda_{T^{\delta/8}}(p^k)\chi(p^k)}{p^{k(s+i\tau)}\log p^k} << \sum_{p\le\mu}\sum_{k=N+1}^{\infty} \frac{1}{p^{\sigma k}}$$

$$<< \sum_{p\le\mu} \frac{1}{p^{\sigma(N+1)}} << \mu 2^{-\sigma(N+1)} .$$

Since $\sigma > \frac{1}{2}$ for $s \in C$ and $N = [8 \log \mu]$, this is

$$<< \mu(e^{-4\log 2})^{\log\mu} = \mu^{1-4\log 2} .$$

As $\log 2 > .69$ this is

$$\ll \mu^{-1} .$$

Also we find that

$$\sum_{\mu<p\le T^{\delta/4}} \sum_{k=2}^{\infty} \frac{\Lambda_{T^{\delta/8}}(p^k)\chi(p^k)}{p^{k(s+i\tau)}\log p^k} \ll \sum_{\mu<p\le T^{\delta/4}} \sum_{k=2}^{\infty} \frac{1}{p^{\sigma k}}$$

$$\ll \sum_{\mu<p} \frac{1}{p^{2\sigma}} \ll \sum_{\mu<n} \frac{1}{n^{2\sigma}} \ll \frac{\mu^{1-2\sigma}}{2\sigma-1}$$

$$\ll \mu^{-2\delta} ,$$

since $\sigma > \sigma_1 = \frac{1}{2}+\delta$ for $s \in C$. Now define the entire functions $\ell_\mu(s,\chi)$ by

$$\ell_\mu(s,\chi) = \sum_{p\le\mu} \sum_{k=1}^{N} \frac{\chi(p^k)}{kp^{ks}} \qquad (\chi \pmod q) .$$

Using this definition, the above estimates, and the estimate

$$\mu^{-1} < \mu^{-2\delta}$$

in (29) leads to

$$(30)\quad \sum_{n<T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(n)\chi(n)}{n^{s+i\tau}\log n} - \ell_{\mu}(s+i\tau,\chi) - \sum_{\mu<p\le\rho} \frac{\chi(p)}{p^{s+i\tau}}$$

$$= \sum_{\rho<p\le T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(p)\chi(p)}{p^{s+i\tau}\log p} + O(\mu^{-2\delta})$$

uniformly for $s \in C$, $\tau \in R$, and $\chi \pmod q$; the constant in the O-term depends on δ and thus on σ_1 . Letting $\mathcal{J}(T)$ be the set in Lemma 3.9 and assuming T is large enough, we obtain from (20) and (30) that

$$(31)\quad \log L(s+i\tau,\chi) - \ell_{\mu}(s+i\tau,\chi) - \sum_{\mu<p\le\rho} \frac{\chi(p)}{p^{s+i\tau}}$$

$$= \sum_{\rho<p\le T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(p)\chi(p)}{p^{s+i\tau}\log p} + O(\mu^{-2\delta}) + O(T^{-\delta^2/16})$$

uniformly for $s \in C$, $\tau \in \mathcal{J}(T)$, and for each $\chi \pmod q$. Here the O-terms depend on q and σ_1 .

Let Λ be the sequence of logarithms of the primes in Lemma 3.7 and for the $b_{\lambda}^{(n)}(T)$ take $\Lambda_{T^{\delta/8}}(p)\chi(p)/\log p$, a different character χ corresponding to each n . Then the elements of Λ are linearly independent over $\mathbb{Q}$, $|\Lambda_{T^{\delta/8}}(p)\chi(p)/\log p| \le 1$, and

$$\delta(T) = \min_{p \neq p' \leq T} |\log p - \log p'|$$

$$\geq |\log(T-2) - \log T| > \frac{2}{T} .$$

Also $N_\Lambda(u) << e^u$. Thus the conditions of Lemma 3.7 are met. We conclude that there is a $\tau \in \mathcal{J}(T)$ such that

(32) $$\left\| - \frac{\tau \log p}{2\pi} \right\| \leq d \quad \text{if} \quad p \leq \mu \ \text{and} \ p \nmid q ,$$

(33) $$\left\| - \frac{\tau \log p}{2\pi} - \theta_p \right\| \leq d \quad \text{if} \quad \mu < p \leq \rho \ \text{or} \ p | q ,$$

and

(34) $$\max_{\chi (\mathrm{mod}\ q)} \max_{s \in C} \left| \sum_{\rho < p \leq T^{\delta/4}} \frac{\Lambda_{T^{\delta/8}}(p)\chi(p)}{p^{s+i\tau} \log p} \right| << \rho^{1/2-\sigma_1} = \rho^{-\delta} .$$

Here the implied constant in $<<$ depends on σ_1 , σ_2 , C , and q . Since $\ell_\mu(s,\chi)$ is independent of any p dividing q , we conclude from (32) that if d is small enough,

(35) $$\max_{\chi (\mathrm{mod}\ q)} \max_{s \in C} \left| \ell_\mu(s+i\tau,\chi) - \ell_\mu(s,\chi) \right| \leq \mu^{-\delta} .$$

Furthermore we find from (33) that if d is small enough,

(36) $$\max_{\chi (\mathrm{mod}\ q)} \max_{s \in C} \left| \sum_{\mu < p \leq \rho} \frac{\chi(p)}{p^{s+i\tau}} - \sum_{\mu < p \leq \rho} \frac{\chi(p)e(\theta_p)}{p^s} \right| \leq \mu^{-\delta}.$$

By (31), (34), (35), and (36) we see that if T is sufficiently large and d is sufficiently small, there is a $\tau \in \mathbb{R}$ with

$$\max_{\chi(\text{mod } q)} \max_{s \in C} \left| \log L(s+i\tau,\chi) - \ell_\mu(s,\chi) - \sum_{\mu<p\leq\rho} \frac{e(\theta_p)\chi(p)}{p^s} \right|$$

$$<< \mu^{-2\delta} + T^{-\delta^2/16} + \mu^{-\delta} + \rho^{-\delta}$$

$$<< \mu^{-\delta} + T^{-\delta^2/16} .$$

Since we are assuming $T^{\delta/8} \geq \mu^N$ and $N = [8 \log \mu] \geq [8 \log e] = 8$, we have

$$T^{-\delta^2/16} \leq \mu^{-4\delta} < \mu^{-\delta} .$$

Thus the above is

$$<< \mu^{-\delta} .$$

Note that the implied constant depends on σ_1, σ_2 , q , and C . Finally, by (33), the same τ satisfies

$$\left\| - \frac{\tau \log p}{2\pi} - \theta_p \right\| \leq d \qquad\qquad (p|q) .$$

This completes the proof of the lemma. □

§4. Proof of Theorem 3.1

Since the simply connected compact set C is in the strip $\frac{1}{2} < \sigma < 1$, there exist σ_1, σ_2 such that C lies in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$. Let $\delta = \sigma_1 - \frac{1}{2}$, let $\ell_\mu(s,\chi)$ ($\chi \pmod q$) be the entire functions of Lemma 3.10, and assume $\mu \geq \max(q,e)$. For $1 \leq a \leq q$, $(a,q) = 1$, we define

$$F_a(s) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)(f_\chi(s) - \ell_\mu(s,\chi)) ,$$

where the $f_\chi(s)$ are the functions given in Theorem 3.1. Clearly each $F_a(s)$ is continuous on C and analytic in the interior of C. Using the relations

$$\sum_{a=1}^{q}{}^{*} \chi'(a)\overline{\chi}(a) = \begin{cases} \phi(q) & \text{if } \chi = \chi' \\ 0 & \text{if } \chi \neq \chi' , \end{cases}$$

we find

$$\sum_{a=1}^{q}{}^{*} \chi'(a)F_a(s)$$

$$= \sum_{a=1}^{q}{}^{*} \chi'(a)\left(\frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)(f_\chi(s) - \ell_\mu(s,\chi))\right)$$

$$= \sum_{\chi \pmod q} (f_\chi(s) - \ell_\mu(s,\chi))\left(\frac{1}{\phi(q)} \sum_{a=1}^{q}{}^{*} \chi'(a)\overline{\chi}(a)\right)$$

$$= f_{\chi'}(s) - \ell_\mu(s,\chi') ,$$

or

$$(37) \quad f_\chi(s) - \ell_\mu(s,\chi) = \sum_{a=1}^{q}{}^{*} \chi(a)F_a(s) \qquad (\chi \pmod q) .$$

Let $\Lambda_a = \{\log p | p \equiv a \pmod q\}$ for $1 \le a \le q$, $(a,q) = 1$. By Lemma 3.3,

$$N_{\Lambda_a}(x) = \frac{1}{\phi(q)} \text{li } e^x + O(e^{x-c\sqrt{x}}) .$$

Obviously

$$N_{\Lambda_a}(x) << e^x .$$

Furthermore

$$N_{\Lambda_a}(x + \frac{c_1}{x^2}) - N_{\Lambda_a}(x) = \frac{1}{\phi(q)} \int_{e^x}^{e^{x+\frac{c_1}{x^2}}} \frac{dt}{\log t} + O(e^{x+\frac{c_1}{x^2}-c\sqrt{x}})$$

$$>> \frac{e^{x+\frac{c_1}{x^2}} - e^x}{\log(e^{x+\frac{c_1}{x^2}})}$$

$$>> \frac{e^x}{x^3} .$$

Similarly

$$N_{\Lambda_a}(x) - N_{\Lambda_a}\left(x-\frac{c_1}{x^2}\right) >> \frac{e^x}{x^3} .$$

Thus each $N_{\Lambda_a}(x)$ satisfies the hypotheses of Lemma 2.2. Hence there exists a number $\rho > \mu$ and numbers $0 \leq \theta_p < 1$ such that

$$(38) \qquad \max_{s\varepsilon C}\left| F_a(s) - \sum_{\substack{\mu<p\leq\rho \\ p\equiv a \pmod q}} \frac{e(\theta_p)}{p^s} \right| << \mu^{-1/2}$$

for each a with $1 \leq a \leq q$ and $(a, q) = 1$. By (37) and (38) we see that

$$f_\chi(s) - \ell_\mu(s,\chi) = \sum_{a=1}^{q}{}^{*} \chi(a)\left(\sum_{\substack{\mu<p\leq\rho \\ p\equiv a \pmod q}} \frac{e(\theta_p)}{p^s} + O(\mu^{-1/2}) \right)$$

$$= \sum_{\mu<p\leq\rho} \frac{\chi(p)e(\theta_p)}{p^s} + O(q\mu^{-1/2}) .$$

for each $\chi \pmod q$. Thus

$$(39) \quad \max_{\chi \pmod q} \max_{s\varepsilon C}\left| f_\chi(s) - \ell_\mu(s,\chi) - \sum_{\mu<p\leq\rho} \frac{\chi(p)e(\theta_p)}{p^s} \right| << \mu^{-1/2} .$$

We remark that the assumption $\mu \geq q$ is necessary so that the θ_p $(p|q)$ given in the statement of Theorem 3.1 are independent of the θ_p in (39). Now by Lemma 3.10, if $0 < d < \frac{1}{2}$ and d is small enough, there is a $\tau \in R$ such that

(40) $$\left\| -\frac{\tau \log p}{2\pi} - \theta_p \right\| \leq d \qquad (p|q)$$

and

(41) $$\max_{\chi (\mathrm{mod}\ q)} \max_{s\in C} \left| \log L(s+i\tau,\chi) - \ell_\mu(s,\chi) - \sum_{\mu<p\leq\rho} \frac{\chi(p)e(\theta_p)}{p^s} \right| << \mu^{-\delta} .$$

Combining (39) and (41) we obtain

$$\max_{\chi (\mathrm{mod}\ q)} \max_{s\in C} \left| \log L(s+i\tau,\chi) - f_\chi(s) \right| << \mu^{-\delta} + \mu^{-1/2} << \mu^{-\delta} .$$

This in turn leads to

(42) $$L(s+i\tau,\chi) = e^{f_\chi(s)} \cdot e^{\log L(s+i\tau,\chi)-f_\chi(s)} = e^{f_\chi(s)}(1 + O(\mu^{-\delta}))$$

$$= e^{f_\chi(s)} + O((\max_{s\varepsilon C} e^{|f_\chi(s)|})\mu^{-\delta})$$

$$= e^{f_\chi(s)} + O(\mu^{-\delta})$$

uniformly for $s \varepsilon C$ and for χ (mod q). Finally, if in addition to our other hypotheses on d we assume that $d < \varepsilon$, where ε is as in Theorem 3.1, and if μ is taken large enough to begin with, then there is a $\tau \varepsilon R$ with

$$\left\| - \frac{\tau \log p}{p\pi} - \theta_p \right\| < \varepsilon \qquad (p|q)$$

and

$$\max_{s\varepsilon C} |L(s+i\tau,\chi) - e^{f_\chi(s)}| < \varepsilon \qquad (\chi \pmod q) ,$$

because of (40) and (42). This completes the proof of Theorem 3.1. □

CHAPTER IV

UNIVERSALITY OF THE HURWITZ ZETA-FUNCTIONS

§1. Statement of Results

Throughout this chapter we let α denote a real number with $0 < \alpha \leq 1$. For any such α the Hurwitz zeta-function, $\zeta(s,\alpha)$, is defined by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s} \qquad (\sigma > 1) .$$

These functions may all be continued to the entire complex plane and are analytic except for a simple pole at $s = 1$.

In this chapter we prove the following universality theorem.

Theorem 4.1. Let C be a compact simply connected set in the strip $\frac{1}{2} < \sigma < 1$. Let α be rational or transcendental, $\alpha \neq \frac{1}{2}$, $\alpha \neq 1$. If $f(s)$ is continuous in C and analytic in the interior of C , then for any $\varepsilon > 0$ there exists a $\tau \in R$ such that

$$\max_{s \in C} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon .$$

Notice that we are able to approximate $f(s)$ rather than just $e^{f(s)}$ as in our previous universality theorems. Theorems of the previous type for $\alpha = \frac{1}{2}$ and $\alpha = 1$ follow immediately from Theorem 3.1 since $\zeta(s,1) = \zeta(s)$ and $\zeta(s,\frac{1}{2}) = 2^s L(s,\chi)$, where χ is the (unique) character (mod 2).

In 1936 H. Davenport and H. Heilbronn [5] considered the question of whether $\zeta(s,\alpha)$ has zeros in the half plane $\sigma > 1$. If $\alpha = \frac{1}{2}$ or 1, we find $\zeta(s,\frac{1}{2}) = (1-2^s)\zeta(s)$ and $\zeta(s,1) = \zeta(s)$. Since $\zeta(s)$ has a convergent Euler product in $\sigma > 1$, $\zeta(s)$ has no zeros in $\sigma > 1$. Therefore neither does $\zeta(s,\frac{1}{2})$ or $\zeta(s,1)$. However, they showed that for all other rational and transcendental values of α, $\zeta(s,\alpha)$ has infinitely many zeros in $\sigma > 1$. This left open the question for α algebraic and irrational. Some twenty-five years later, J.W.S. Cassels [2] settled this case in the affirmative.

S.M. Voronin [22] recently announced an extension of the result of Davenport and Heilbronn. Specifically, he asserts that for α rational and not equal to $\frac{1}{2}$ or 1, $\zeta(s,\alpha)$ has infinitely many zeros in the strip $\frac{1}{2} < \sigma < 1$. From Theorem 4.1 we deduce

Theorem 4.2. Let α be rational or transcendental, $\alpha \neq \frac{1}{2}$, $\alpha \neq 1$. The real parts of the zeros of $\zeta(s,\alpha)$ are dense in $[\frac{1}{2}, 1]$.

To prove Theorem 4.2, let C be a closed disc in the strip $\frac{1}{2} < \sigma < 1$ and let $f(s)$ be analytic in the interior of C and continuous on C. Also assume $f(s)$ has zeros inside C but not on the boundary of C, ∂C. If $m = \min_{s\in\partial C} |f(s)|$, then we find by Theorem 4.1 that there is a $\tau \in \mathbb{R}$ such that

$$\max_{s\in\partial C} |\zeta(s+i\tau,\alpha) - f(s)| < m = \min_{s\in\partial C} |f(s)| .$$

This and Ronché's theorem imply that $\zeta(s+i\tau,\alpha)$ has the same number of zeros in C as $f(s)$. Theorem 4.2 now follows.

We expect Theorems 4.1 and 4.2 to remain true when α is an algebraic irrational, but we are unable to prove that this is so. It is not hard to appreciate the difficulty in proving Theorem 4.1 for such α. When α is rational we may express $\zeta(s,\alpha)$ in terms of L-functions and then apply Theorem 3.1 to obtain Theorem 4.1. When α is transcendental our proof relies on the fact that the numbers $\log(m+\alpha)$ $(m = 0,1,\ldots)$ are linearly independent over $\mathbb{Q}$. Neither of these approaches work when α is an algebraic irrational. All we can say in this case is that more than half the numbers of the set $\{\log(m+\alpha)\}_{m=0}^{M}$ are linearly independent over $\mathbb{Q}$ when M is large. This fact was discovered by Cassels [3] (see also Worley [27]) and used by him to settle the problem of the existence of zeros of $\zeta(s,\alpha)$ in $\sigma > 1$ when α is an algebraic irrational. It seems likely that this observation may be of use in our problem too.

§2. <u>Proof of Theorem 4.1 for Rational</u> α

Let $\alpha = \frac{a}{q}$ where $(a,q) = 1$. Since $\alpha \neq \frac{1}{2}$ and $\alpha \neq 1$, we have $q \geq 3$. Our starting point is the identity

$$(1) \qquad \zeta(s,\tfrac{a}{q}) = \frac{q^s}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a) L(s,\chi)$$

valid for all $s \in \mathbb{C}$. To prove (1) we use the formula

$$\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a)\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod q \\ 0 & \text{if } n \not\equiv a \pmod q \end{cases}$$

For $\sigma > 1$ the right-hand side of (1) is

$$\frac{q^s}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{q^s}{n^s} \left(\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a)\chi(n) \right)$$

$$= \sum_{\substack{n=1 \\ n \equiv a (\mathrm{mod}\ q)}}^{\infty} \frac{q^s}{n^s}$$

$$= \sum_{k=0}^{\infty} \frac{q^s}{(kq+a)^s}$$

$$= \zeta(s,\tfrac{a}{q}) \; .$$

This establishes (1) in $\sigma > 1$. Except for a simple pole at $s = 1$, the right hand side of (1) is analytic in the entire complex plane. Thus, by analytic continuation, (1) defines $\zeta(s,\frac{a}{q})$ for all s .

Let $f(s)$ and C be as in the statement of Theorem 4.1. Since $\phi(q) \geq 2$ for $q \geq 3$, we can find a constant c such that neither $f(s)q^{-s}+c$ nor $f(s)q^{-s} - c(\phi(q) - 1)$ vanishes on C . For such a c , and for χ_1 an arbitrary but fixed character (mod q), define

$$f(s,\chi) = \begin{cases} \chi(a)(f(s)q^{-s}+c) & \text{if } \chi \neq \chi_1 \\ \chi_1(a)(f(s)q^{-s} - c(\phi(q)-1)) & \text{if } \chi = \chi_1 \; . \end{cases}$$

Then we have

$$(2) \qquad f(s) = \frac{q^s}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\; q)} \overline{\chi}(a) f(s,\chi) \; .$$

Furthermore, each $f(s,\chi)$ is continuous and nonvanishing on C and analytic on the interior of C . Thus we may select a branch of $\log f(s,\chi)$ for each χ (mod q) which is continuous on C and analytic in the interior of C. Since C is simply connected and compact, we may apply Theorem 3.1 with $f_\chi(s)$ of the theorem equal to $\log f(s,\chi)$ to deduce that if $\varepsilon_1 > 0$, there is a $\tau \in \mathbb{R}$ such that

$$(3) \qquad \| - \tau \log p \| < \varepsilon_1 \qquad (p|q)$$

and

$$(4) \qquad \max_{s\in C} |L(s+i\tau,\chi) - f(s,\chi)| < \varepsilon_1 \qquad (\chi \pmod q).$$

If ε_1 is small enough and $\varepsilon > 0$, (3) and (4) lead to

$$(5) \qquad \max_{s\in C} |q^{i\tau} L(s+i\tau,\chi) - f(s,\chi)| < \frac{\varepsilon}{q} \qquad (\chi \pmod q).$$

For τ as in (5) we find from (1), (2), and (5) that

$$\max_{s\in C} |\zeta(s+i\tau,\tfrac{a}{q}) - f(s)|$$

$$= \max_{s\in C} \left| \frac{q^{s+i\tau}}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a) L(s+i\tau,\chi) - \frac{q^s}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a) f(s,\chi) \right|$$

$$= \max_{s\in C} \left| \frac{q^s}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a)\, q^{i\tau} L(s+i\tau,\chi) - f(s,\chi) \right|$$

$$< \frac{1}{\phi(q)} \max_{s\in C} |q^s| \sum_{\chi \pmod q} \max_{s\in C} \left| q^{i\tau} L(s+i\tau,\chi) - f(s,\chi) \right|$$

$$< \frac{q}{\phi(q)} \sum_{\chi \pmod q} \frac{\varepsilon}{q} = \varepsilon .$$

This proves the theorem for α rational. □

§3. Proof of Theorem 4.1 for Transcendental α

We begin with a lemma.

Lemma 4.1. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and set $\delta = \sigma_1 - \frac{1}{2}$. If $0 < \alpha \leq 1$ and T is large enough, then

$$\max_{s \varepsilon C} \left| \zeta(s+i\tau,\alpha) - \sum_{m=0}^{T} \frac{1}{(m+\alpha)^{\delta+i\tau}} \right| << T^{-\delta} \tag{6}$$

for all $\tau \varepsilon [T^{1/2}, T]$; the implicit constant is absolute.

Proof: Let $z = x+iy$. If $\frac{1}{2} \leq x \leq 1$ and $|y| \leq \pi T$, then by a well known formula (see Davenport [4; p. 173])

$$\zeta(z,\alpha) = \sum_{m=0}^{T} \frac{1}{(m+\alpha)^{z}} - \frac{T^{1-z}}{1-z} + O(T^{-x}) .$$

Replacing z by $s+i\tau$ where $s \varepsilon C$ and $\tau \varepsilon [T^{1/2}, T]$ and noting that if T is large enough $|\tau+t| \leq \pi T$, we obtain

$$\zeta(s+i\tau,\alpha) = \sum_{m=0}^{T} \frac{1}{(m+\alpha)^{s+i\tau}} + O(T^{1/2-\sigma_1}) + O(T^{-\sigma_1}) .$$

This proves the lemma since $\frac{1}{2}-\sigma_1 > -\sigma_1$.

Let $\Lambda = \{\log(m+\alpha)\}_{m=0}^{\infty}$. Then Λ is monotonically increasing and its counting function is

$$N_\Lambda(x) = \sum_{\log(m+\alpha)\le x} 1 = \sum_{m+\alpha\le e^x} 1 = e^x + O(1) .$$

It follows that if $c > 0$,

$$|N_\Lambda(x+\frac{c}{x^2}) - N_\Lambda(x)| >> \frac{e^x}{x^2} .$$

Therefore $N_\Lambda(x)$ satisfies the hypotheses of Lemma 2.2. Fix $\mu > 0$ and let $f(s)$ and C be as in Theorem 4.1. We can find numbers σ_1 and σ_2 so that C is in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$. Clearly

$$f(s) - \sum_{0\le m\le\mu} \frac{1}{(m+\alpha)^s}$$

is continuous on C and analytic in the interior of C . Thus, by Lemma 2.2, there are real numbers ρ and θ_m $(\mu < m \le \rho)$ such that

$$(7) \quad \max_{s\epsilon C}\left| f(s) - \sum_{0\le m\le\mu} \frac{1}{(m+\alpha)^s} - \sum_{\mu<m\le\rho} \frac{e(\theta_m)}{(m+\alpha)^s}\right| << \mu^{-1/2} .$$

Here the implicit constant depends on σ_1 , σ_2 , Λ , and C .

Next we take $N = 1$ and

$$b^{(1)}_{\log(m+\alpha)}(T) = \begin{cases} 1 & \text{if } m \le T \\ 0 & \text{if } m > T \end{cases}$$

in Lemma 3.7. For $T \ge 0$ let $\mathcal{J}(T) = [T^{1/2}, T]$. The measure of $\mathcal{J}(T)$ is obviously

$$J(T) = T - T^{1/2} = T(1 + o(1)) .$$

Also

$$\delta(T) = \min_{\rho<\log(m+\alpha)\neq\log(m'+\alpha)\le T} |\log(m+\alpha) - \log(m'+\alpha)|$$

$$\gg \frac{1}{T} .$$

It remains to show that Λ is linearly independent over $\mathbb{Q}$ when α is transcendental. Suppose there is a relation of the form

$$\sum_{m=0}^{M} a_m \log(m+\alpha) = 0$$

with the a_m integers. Then

$$\prod_{m=0}^{M} (m+\alpha)^{a_m} = 1.$$

This cannot be an identity in α since it does not hold for

$\alpha = -m$, m being any integer $\leq M$ for which $a_m \neq 0$. Thus α satisfies a polynomial with integer coefficients. This contradicts the assumption that α is transcendental. We now have from Lemma 3.7 that if $0 < d < \frac{1}{2}$ and T is large, there is a $\tau \in \mathcal{J}(T)$ such that

$$(8) \qquad \left\| \frac{-\tau \log(m+\alpha)}{2\pi} \right\| \leq d \qquad \text{for} \qquad 0 \leq m \leq \mu ,$$

$$(9) \qquad \left\| \frac{-\tau \log(m+\alpha)}{2\pi} - \theta_m \right\| \leq d \qquad \text{for} \qquad \mu < m \leq \rho ,$$

and

$$(10) \qquad \max_{s \in C} \left| \sum_{\rho < m \leq T} \frac{1}{(m+\alpha)^{s+i\tau}} \right| << \rho^{-\delta} ,$$

where $\delta = \sigma_1 - \frac{1}{2}$ and the θ_m in (9) are the same as those in (7). The constant in (10) depends only on σ_1 , σ_2 , Λ , and C . If d is small enough, we obtain from (8) and (9) that

$$\max_{s \in C} \left| \sum_{0 \leq m \leq \mu} \frac{1}{(m+\alpha)^s} + \sum_{\mu < m \leq \rho} \frac{e(\theta_m)}{(m+\alpha)^s} - \sum_{0 \leq m \leq \rho} \frac{1}{(m+\alpha)^{s+i\tau}} \right| \leq \rho^{-\delta} .$$

This, (7), and (10) yield

(11) $$\max_{s\in C}\left| f(s) - \sum_{0\leq m\leq T} \frac{1}{(m+\alpha)^{s+i\tau}}\right| \ll \rho^{-\delta} + \mu^{-1/2} .$$

By our choice of $\gamma(T)$, the estimate (6) of Lemma 4.1 is valid. Combining it with (11) leads to

$$\max_{s\in C}| f(s) - \zeta(s+i\tau,\alpha)| \ll \rho^{-\delta} + \mu^{-1/2} + T^{-\delta} .$$

Since $\mu < \rho < T$, the result follows. This completes the proof of Theorem 4.1. □

CHAPTER V

A q-ANALOGUE OF THE UNIVERSALITY OF $\zeta(s)$

§1. Statement of the Result

Our aim in this chapter is to prove the following q-analogue of Voronin's universality theorem for $\xi(s)$ (see Theorem D, Section II.1).

Theorem 5.1. Suppose that C is a simply connected compact set in the strip $\frac{1}{2} < \sigma < 1$, and that $f(s)$ is continuous on C and analytic in the interior of C. If $\varepsilon > 0$ and q is sufficiently large, there exists a character $\chi \pmod q$ such that

$$\max_{s \in C} |L(s,\chi) - e^{f(s)}| < \varepsilon .$$

The proof of Theorem 5.1 parallels that of Theorem 3.1 to some extent. However, the present proof is simpler as we are not concerned with a simultaneous result.

§2. Some Lemmas

For our first lemma see Prachar [16; Chap. 1, Satz 5.1].

Lemma 5.1. If $q \geq 3$, then

$$\phi(q) \gg \frac{q}{\log \log q} .$$

The following is an easy consequence of the properties of characters.

Lemma 5.2. For complex a_p and $\rho > 0$ we have

$$\sum_{\chi \pmod q} \left| \sum_{\rho < p \le q} \frac{a_p \chi(p)}{p^s} \right|^2 = \phi(q) \sum_{\rho < p \le q} \frac{|a_p|^2}{p^{2\sigma}} .$$

The next three lemmas are essentially q-analogues of Lemmas 3.5, 3.6, and 3.7.

Lemma 5.3. Suppose that $\mathcal{P}$ is a set of K primes $(K \ge 0)$ and that for each $p \in \mathcal{P}$, θ_p is a fixed number with $0 \le \theta_p < 1$. If $0 < d < \frac{1}{2}$ and $(q, \prod_{p \in \mathcal{P}} p) = 1$ (we set $\prod_{p \in \mathcal{P}} p = 1$ if $\mathcal{P}$ is empty), we let $I_d(q)$ be the number of characters $\chi \pmod q$ satisfying

$$\left\| \frac{\arg \chi(p)}{2\pi} - \theta_p \right\| \le d , \qquad p \in \mathcal{P}.$$

Then

$$\lim_{\substack{q \to \infty \\ (q, \prod_{p \in \mathcal{P}} p) = 1}} \frac{I_d(q)}{\phi(q)} = (2d)^K .$$

Proof: Throughout the proof we assume $(q, \prod_{p \in \mathcal{P}} p) = 1$. If $K = 0$ the result is obvious so we assume $K \ge 1$. We first show that for $0 < d < \frac{1}{2}$ and q large enough, there is a $\chi \pmod q$ such that

$$\left\| \frac{\arg \chi(p)}{2\pi} - \theta_p \right\| \le d , \qquad p \in \mathcal{P} .$$

To do this it clearly suffices to show that for any $\varepsilon > 0$ and q sufficiently large, there exists a $\chi \pmod q$ such that

(1) $$\left|1 + \sum_{p\in\mathcal{P}} \chi(p)e(-\theta_p)\right| \geq K+1-\varepsilon .$$

To this end we consider the expression

$$\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \left|1 + \sum_{p\in\mathcal{P}} \chi(p)e(-\theta_p)\right|^{2m} ,$$

where m is a positive integer to be specified later. By the multinomial theorem this is

$$\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \left| \sum_{\substack{\nu_0+\dots+\nu_K=m \\ \nu_k\geq 0}} \frac{m!}{\nu_0!\dots\nu_K!} \chi(p_1^{\nu_1} \dots p_K^{\nu_K}) e(-\sum_{k=1}^{K} \nu_k\theta_k)\right|^2$$

$$\sum_{\substack{\nu_0+\dots+\nu_K=m \\ \nu_k\geq 0}} \sum_{\substack{\nu_0'+\dots+\nu_K'=m \\ \nu_k'\geq 0}} \frac{m!}{\nu_0!\dots\nu_K!} \frac{m!}{\nu_0'!\dots\nu_K'!} e\left(\sum_{k=1}^{K} (\nu_k-\nu_k')\theta_k\right)$$

$$\cdot \left(\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \chi\left(\prod_{k=1}^{K} p_k^{\nu_k}\right) \bar{\chi}\left(\prod_{k=1}^{K} p_k^{\nu_k'}\right)\right) ,$$

where $p_1,\dots,p_K$ are all the primes in $\mathcal{P}$. Now if q is large enough

$$\prod_{k=1}^{K} p_k^{\nu_k} \equiv \prod_{k=1}^{K} p_k^{\nu_k'} \qquad (\mathrm{mod}\ q)$$

only when $\nu_k = \nu_k'$, $1 \leq k \leq K$; so by the formula

$$\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \chi(a)\overline{\chi}(b) = \begin{cases} 1 & \text{if} \quad a \equiv b \pmod q \\ 0 & \text{if} \quad a \not\equiv b \pmod q \end{cases} ,$$

the above is

$$= \sum_{\substack{\nu_0+\dots+\nu_K=m \\ \nu_k \geq 0}} \left(\frac{m!}{\nu_0!\dots\nu_K!} \right)^2$$

$$\geq \left(\sum_{\substack{\nu_0+\dots+\nu_K=m \\ \nu_k \geq 0}} \frac{m!}{\nu_0!\dots\nu_K!} \right)^2 \Bigg/ \sum_{\substack{\nu_0+\dots+\nu_K=m \\ \nu_k \geq 0}} 1$$

$$= (K+1)^{2m} \Bigg/ \sum_{\substack{\nu_0+\dots+\nu_K=m \\ \nu_k \geq 0}} 1 ,$$

provided q is sufficiently large. The number of solutions in nonnegative integers of the equation

$$\nu_0 + \dots + \nu_K = m$$

is at most $(m+1)^K$. Using this estimate we find that

$$\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \left| 1 + \sum_{p \in \mathcal{P}} \chi(p) e(-\theta_p) \right|^{2m} \geq \frac{(K+1)^{2m}}{(m+1)^K}$$

if q is sufficiently large. Since the left-hand side is

$$\leq \max_{\chi (\mathrm{mod}\ q)} \left|1 + \sum_{p \varepsilon \mathcal{P}} \chi(p) e(-\theta_p)\right|^{2m} ,$$

we have on taking $2m^{th}$ roots.

$$\max_{\chi (\mathrm{mod}\ q)} \left|1 + \sum_{p \varepsilon \mathcal{P}} \chi(p) e(-\theta_p)\right| \geq (K+1)/(m+1)^{\frac{K}{2m}} .$$

Taking m large enough as a function of K, the right hand side can be made

$$\geq K+1-\varepsilon .$$

This proves (1).

Now let C_1 and C_2 be two equal cubes in the K-dimensional unit cube with sides parallel to the axes, and with centers at the points $\left(\frac{\arg \chi_1(p_1)}{2\pi}, \ldots, \frac{\arg \chi_1(p_K)}{2\pi}\right)$ and $\left(\frac{\arg \chi_2(p_1)}{2\pi}, \ldots, \frac{\arg \chi_2(p_K)}{2\pi}\right)$, where χ_1 and χ_2 are characters to the same modules q. If $I_j(q)$ $(j=1,2)$ is the number of characters χ (mod q) for which the point $\left(\frac{\arg \chi(p_1)}{2\pi}, \ldots, \frac{\arg \chi(p_K)}{2\pi}\right)$ is in C_j, then

$$\lim_{\substack{q\to\infty \\ (q, \prod_{p\varepsilon\mathcal{P}} p)=1}} \frac{I_1(q)}{I_2(q)} = 1 .$$

For $\left(\frac{\arg \chi(p_1)}{2\pi}, \ldots, \frac{\arg \chi(p_k)}{2\pi}\right)$ will lie inside C_2 if and only if

$$\left(\frac{\arg \chi(p_1)\chi_2(p_1)\overline{\chi_1}(p_1)}{2\pi}, \ldots, \frac{\arg \chi(p_K)\chi_2(p_K)\chi_1(p_K)}{2\pi}\right)$$

lies inside C_1, and $\chi\chi_2\overline{\chi}_1$ is a character (mod q). It follows that

$$\lim_{\substack{q\to\infty \\ (q,\prod_{p\in\rho} p)=1}} \frac{I_j(q)}{\phi(q)} = \text{volume}(C_j) \qquad (j=1,2) .$$

The result follows from this. □

Lemma 5.4. Let C be a compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$. Let $\Lambda_x(n)$ be as in Lemma 3.8, let χ be a character (mod q), let $\rho \geq 1$, and set

$$S(s,\chi) = \sum_{\rho<p\leq q} \frac{\Lambda_{q^{\delta/4}}(p)\chi(p)}{p^s \log p} .$$

Fix $0 \leq \theta_p < 1$ for each $p \leq \rho$, and for $0 < d < \frac{1}{2}$ let $\mathcal{Q}_d(q)$ be the set of characters χ (mod q) satisfying

$$\left\|\frac{\arg \chi(p)}{2\pi} - \theta_p\right\| \leq d , \quad p \leq \rho , \; p \nmid q .$$

If $K(q)$ is the number of $p \leq \rho$ with $p \nmid q$, then for q

sufficiently large

$$\sum_{\chi \in \mathcal{S}_d(q)} \max_{s \in C} |S(s,\chi)|^2 \ll (2d)^{K(q)} \phi(q) \rho^{1-2\sigma_1} .$$

The implicit constant depends on σ_1 , σ_2 , and C .

<u>Proof</u>: Fix K , $0 \le K \le \pi(\rho)$ and let $\mathcal{P}_K$ be a set of K primes $\le \rho$. We associate with $\mathcal{P}_K$ the sequence $Q(\mathcal{P}_K)$ of all $q \ge 1$ with

$$(q, \prod_{p \in \mathcal{P}_K} p) = 1 , \prod_{\substack{p<\rho \\ p \notin \mathcal{P}_K}} p | q .$$

For $q \in Q(\mathcal{P}_K)$, $\mathcal{S}_d(q)$ is simply the set of χ (mod q) with

$$\left\| \frac{\arg \chi(p)}{2\pi} - \theta_p \right\| \le d , \ p \in \mathcal{P}_K ,$$

and to prove the lemma it suffices to show that for any choice of K and $\mathcal{P}_K$, if q is large enough, $q \in Q(\mathcal{P}_K)$, then

$$\sum_{\chi \in \mathcal{S}_d(q)} \max_{s \in C} |S(s,\chi)|^2 \ll (2d)^{K} \phi(q) \rho^{1-2\sigma_1} , \tag{2}$$

where the constant depends on σ_1 , σ_2 , and C . To this end let U be the interior of the rectangle with vertices $\sigma_1 \pm iA$, $\sigma_2 \pm iA$, where $A = \max_{s \in C} |s|$. Set

$$\delta = \min_{z\varepsilon\partial U} \min_{s\varepsilon C}|s-z| \ .$$

By Cauchy's integral formula

$$\max_{s\varepsilon C}|S(s,\chi)|^2 = \max_{s\varepsilon C}\left|\frac{1}{2\pi i}\int_{\partial U} \frac{S(z,\chi)}{z-s}\,dz\right|^2$$

$$\leq \frac{1}{(2\pi\delta)^2}\int_{\partial U} |S(z,\chi)|^2 \,|dz| \int_{\partial U} |dz| \ .$$

Thus

$$\sum_{\chi\varepsilon\mathcal{J}_d(q)} \max_{s\varepsilon C}|S(s,\chi)|^2$$

$$\leq \frac{1}{(2\pi\delta)^2}\int_{\partial U} |dz| \int_{\partial U}\left(\sum_{\chi\varepsilon\mathcal{J}_d(q)} |S(z,\chi)|^2\right)|dz|$$

$$\leq \frac{1}{(2\pi\delta)^2}\left(\int_{\partial U} |dz|\right)^2 \max_{z\varepsilon\partial U}\left(\sum_{\chi\varepsilon\mathcal{J}_d(q)} |S(z,\chi)|^2\right)$$

$$= \left(\frac{2A+(\sigma_2-\sigma_1)}{\pi\delta}\right)^2 \max_{z\varepsilon\partial U}\left(\sum_{\chi\varepsilon\mathcal{J}_d(q)} |S(z,\chi)|^2\right) .$$

Since $0 < \sigma_2-\sigma_1 < \frac{1}{2}$, we may write this as

$$(3) \qquad \sum_{\chi\varepsilon\mathcal{J}_d(q)} \max_{s\varepsilon C}|S(s,\chi)|^2 \ll \max_{z\varepsilon\partial U}\left(\sum_{\chi\varepsilon\mathcal{J}_d(q)} |S(z,\chi)|^2\right) ,$$

where the constant depends only on δ and A. We now distinguish between two cases.

First assume $K = 0$. Then $\mathcal{L}_d(q)$ is the set of all characters (mod q). Writing $z = x+iy$ and applying Lemma 5.2, the sum on the right hand side of (3) is

$$= \phi(q) \sum_{\rho<p\leq q} \frac{|\Lambda_{q^{\delta/4}}(p)\chi(p)/\log p|^2}{p^{2x}} .$$

Since $|\Lambda_{q^{\delta/4}}(p)\chi(p)/\log p| \leq 1$ and $x \geq \sigma_1$ for $z \in \partial U$, this is

$$\leq \phi(q) \sum_{\rho<p\leq q} \frac{1}{p^{2\sigma_1}} \leq \phi(q) \sum_{\rho<n} \frac{1}{n^{2\sigma_1}}$$

$$<< \phi(q)\rho^{1-2\sigma_1} ,$$

where the constant depends on σ_1. Using this in (3) and noting that δ and A depend on σ_1, σ_2, and C, we obtain (2).

Now assume $K \geq 1$. By Cauchy's inequality we may write the sum on the right hand side of (3) as

(4) $$\sum_{\chi\in\mathcal{L}_d(q)} |S(z,\chi)|^2$$

$$\leq 2\sum_{\chi\in\mathcal{L}_d(q)} |S_1(z,\chi)|^2 + 2\sum_{\chi\in\mathcal{L}_d(q)} |S_2(z,\chi)|^2 ,$$