A NOTE ON GAPS BETWEEN ZEROS OF THE ZETA FUNCTION

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Recently Montgomery and Odlyzko [1] showed, assuming the Riemann Hypothesis, that infinitely often consecutive zeros of the zeta function differ by at least 1.9799 times the average spacing and infinitely often they differ by at most 0.5179 times the average spacing. We improve their result by a better choice of a certain weight function which occurs in their proof and we give explicit bounds for what is possible by this method. In addition, we indicate a method of proof which is considerably simpler than that of Montgomery and Odlyzko.

Let

$$\lambda = \limsup\,(\gamma' - \gamma)\frac{\log \gamma}{2\pi};$$

$$\mu = \liminf\,(\gamma' - \gamma)\frac{\log \gamma}{2\pi},$$

where $\gamma \leqslant \gamma'$ are consecutive ordinates of zeros of $\zeta(s)$. Note that the average of $\gamma' - \gamma$ is $2\pi/\log \gamma$.

THEOREM. *If the Riemann Hypothesis is true, then* $\lambda > 2.337$ *and* $\mu < 0.5172$.

We indicate the modifications in [1] that lead to our theorem. Let

$$h(c) = c - \operatorname{Re} \sum_{\substack{n,k \\ nk \leqslant K}} a_k \overline{a_{nk}}\, g(n)\Lambda(n) n^{-1/2} \Big/ \sum_{k \leqslant K} |a_n|^2$$

where

$$g(n) = \frac{2\sin\left(\pi c(\log n)/\log T\right)}{\pi \log n}$$

so that $|g(n)| \leqslant \dfrac{2c}{\log T}$ and $K = T^{1-\delta}$ for some small $\delta > 0$. By (20) and (21) of [1], if $h(c) < 1$ for some choice of c, a_k, and $\delta > 0$, then $\lambda \geqslant c$, and if $h(c) > 1$ for some choice of c, a_k, and $\delta > 0$, then $\mu \leqslant c$.

To obtain the bound for λ we choose $a_k = d_r(k)k^{-1/2}$ where d_r is multiplicative and for a prime p,

$$d_r(p^m) = \frac{\Gamma(m+r)}{\Gamma(r)m!}.$$

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We now assume that $r \geqslant 1$ so that $d_r(mn) \leqslant d_r(m)d_r(n)$ for all m and n. Then

$$\sum_{\substack{n,k\\ nk \leqslant K}} a_k \overline{a_{nk}} g(n)\Lambda(n)n^{-1/2}$$

$$= \frac{2}{\pi} \sum_{\substack{k,p\\ kp \leqslant K}} \frac{d_r(k)d_r(kp)}{kp} \sin\left(\pi c(\log p)/\log T\right) + O\left(\frac{1}{\log T}\sum_{k \leqslant K} \frac{d_r(k)^2}{k}\right)$$

$$= \frac{2r}{\pi} \sum_{p \leqslant K} \frac{\sin\left(\pi c(\log p)/\log T\right)}{p} \sum_{k \leqslant K/p} \frac{d_r(k)^2}{k} + O\left(\frac{1}{\log T}\sum_{k \leqslant K} \frac{d_r(k)^2}{k}\right)$$

since the term for which n is either a power of a prime higher than the first or a prime divisor of k may be included in the O-term. It is not difficult to show that

$$\sum_{k \leqslant x} \frac{d_r(k)^2}{k} = C_r(\log x)^{r^2} + O\left((\log T)^{r^2-1}\right)$$

for fixed $r \geqslant 1$, uniformly for $x \leqslant T$, where C_r is a certain constant. Hence by the prime number theorem and Stieltjes integration,

$$h(c) = c - 2r\int_0^1 \frac{\sin \pi c\theta(1-\delta)}{\pi\theta}(1-\theta)^{r^2}d\theta + O(1/\log T).$$

By an easy computation, $h(2.337) < 1$ if $r = 2.2$ and δ is sufficiently small.

We obtain the bound for μ in a similar way, only with $a_k = (-1)^{\Omega(k)}d_r(k)k^{-1/2}$, where $\Omega(k)$ is the total number of prime factors of k. With this choice,

$$h(c) = c + 2r\int_0^1 \frac{\sin \pi c\theta(1-\delta)}{\pi\theta}(1-\theta)^{r^2}d\theta + O(1/\log T).$$

Now, if $r = 1.1$ and δ is sufficiently small, then $h(0.5172) > 1$. This proves the theorem.

It is conjectured that $\mu = 0$ and $\lambda = \infty$. These are not attainable by this method. In fact, for any choice of a_k we can show that $h(c) < 1$ if $c < \frac{1}{2}$ and $h(c) > 1$ if $c \geqslant 6.2$. (The latter bound is probably not very good.) We let $a_k = k^{-1/2}b_k$. Then, since $|b_k b_{nk}| \leqslant (|b_k|^2 + |b_{nk}|^2)/2$ and $|\sin x| \leqslant |x|$, we have

$$|h(c)| \leqslant c + \frac{c}{\log T}\sum_{k \leqslant K} \frac{|b_k|^2}{k}\left(\sum_{n \leqslant K/k} \frac{\Lambda(n)}{n} + \sum_{n|k} \Lambda(n)\right)\Big/ \sum_{k \leqslant K} \frac{|b_k|^2}{k}$$

$$\leqslant 2c + O(1/\log T).$$

Thus, $h(c) < 1$ if $c < \frac{1}{2}$. Next, by Cauchy's inequality,

$$h(c) \geqslant c - \left(\frac{2}{\pi} \sum_{nk \leqslant K} \frac{|b_k|^2}{k} \frac{\Lambda(n)}{n \log n} |\sin \pi c (\log n)/\log T|\right)^{1/2} \left(\frac{c}{\log T} \sum_{k \leqslant K} \frac{|b_k|^2}{k} \sum_{n|k} \Lambda(n)\right)^{1/2} \Big/ \sum \frac{|b_k|^2}{k}$$

$$\geqslant c - \left(\frac{2}{\pi} \sum_{n \leqslant T} \frac{\Lambda(n)}{n \log n} |\sin \pi c (\log n)/\log T|\right)^{1/2} c^{1/2}$$

$$= c - \left(\frac{2c}{\pi} \int_{2-}^{T} \frac{|\sin \pi c (\log u)/\log T|}{u \log u} \, d\left(\sum_{n \leqslant u} \Lambda(n)\right)\right)^{1/2}$$

$$= c - \left(\frac{2c}{\pi} \left(\int_0^1 \frac{|\sin \pi c v|}{v} \, dv + O(1/\log T)\right)\right)^{1/2}$$

by the prime number theorem. Now it is easy to see that

$$\int_0^1 \frac{|\sin \pi c v|}{v} \, dv \leqslant \frac{2}{\pi} \log |c| + A \, .$$

The value $A = 2.23$ is admissible here. This implies that $h(c) > 1$ if $c \geqslant 6.2$.

Finally, we note that there is an easy method to obtain (20) and (21) of [1] which led to $h(c)$ above. Let

$$A(t) = \sum_{k \leqslant K} a_k k^{-it}$$

and let

$$M_1 = \int_T^{2T} |A(t)|^2 dt \, ; \qquad M_2(c) = \int_{-\pi c/\log T}^{\pi c/\log T} \sum_{T \leqslant \gamma \leqslant 2T} |A(\gamma + \alpha)|^2 d\alpha \, .$$

Clearly, $M_2(c)$ is monotonically increasing and $M_2(\mu) \leqslant M_1 \leqslant M_2(\lambda)$. Therefore, if $M_2(c) < M_1$ for some choice of a_k and $\delta > 0$, then $c < \lambda$. Similarly, if $M_2(c) > M_1$, then $c > \mu$. The means M_1 and M_2 are easily evaluated; the first by term-by-term integration, the second by Cauchy's theorem applied to

$$\int \frac{\zeta'}{\zeta} (s - i\alpha) A\left(-i(s - \tfrac{1}{2})\right) A\left(i(s - \tfrac{1}{2})\right) ds \, ,$$

where the path of integration is a rectangle with vertices $a + iT$, $a + 2iT$, $1 - a + 2iT$, $1 - a + iT$ where $a = 1 + 1/\log T$. Then we find that

$$M_2/M_1 = h(c) + O(1/\log T) \, .$$

Added in proof, November 1983. The constant 6.2 that appears in the preceding analysis may be replaced by 3.74. Also, by a more complicated argument, we can show that $\lambda > 2.68$ in the theorem.

Reference

1. H. L. MONTGOMERY and A. M. ODLYZKO, 'Gaps between zeros of the zeta function', to appear.

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