General Topology (MTH 440)

Fall 2022

Final/Qualifying exam*, December 19, 8:00-11:00 am

Problem I. *Let X be a compact, and HAUSDORFF topological space, and let C, and C' be disjoint closed subsets of X. Prove that there are disjoint open subsets U, and U' of X containing C, and C', respectively.*

Problem II. *Let X be a topological space, and let $(C_n)_{n=1}^{\infty}$ be a sequence of connected subspaces of X such that for every n the sets C_n, and C_{n+2} intersect. Prove that $\bigcup_{n=1}^{\infty} C_n$ is either connected, or has precisely 2 connected components.*

Problem III. *Given real numbers a and b satisfying $27b^2 + 4a^3 \neq 0$, denote by $F_{a,b}$ the subset of $\mathbb{R}^2$ defined by*

$$F_{a,b} := \left\{(x, y) \in \mathbb{R}^2 \colon y^2 \leq x^3 + ax + b\right\}.$$

1. *Prove that for all a, and b the set $F_{a,b}$ is a manifold with boundary of dimension 2;*
2. *Find all a, and b for which $F_{a,b}$ is connected.*

Problem IV. *Consider the set of probability vectors Δ in $\mathbb{R}^3$, defined by*

$$\Delta := \{(x, y, z) \colon x \geq 0, y \geq 0, z \geq 0, x + y + z = 1\}.$$

Let X be a manifold, and let A, B, and C be pairwise disjoint closed subsets of X. Prove that there is a smooth function $f \colon X \to \Delta$, such that

$$f = (1, 0, 0) \text{ on } A, f = (0, 1, 0) \text{ on } B, \text{ and } f = (0, 0, 1) \text{ on } C.$$

Problem V. *Let $R \colon \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth function defined by*

$$R(\theta, t) := (\cos(2\theta)(1 + t\sin\theta), \sin(2\theta)(1 + t\sin\theta), t\cos\theta),$$

and put $M := R\left(\mathbb{R} \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$.

1. *Prove that M is a smooth manifold with boundary that is homeomorphic to a MÖBIUS band;*
2. *Denote by $\mathbf{0}$ the origin in $\mathbb{R}^3$, and put*

   $$M_0 := \operatorname{Int}(M) \times \{\mathbf{0}\}, \text{ and } S_0 := S^1 \times \{0\} \times \{\mathbf{0}\}.$$

   Note that $S_0 \subseteq M_0$, and that M_0, and S_0 are both submanifolds of the normal bundle of $\operatorname{Int}(M)$. Compute the mod 2 intersection number $I_2(S_0, M_0)$ inside the normal bundle of $\operatorname{Int}(M)$.

*"Master pass" = 2 essentially correct solutions; "Ph.D. pass" = at least 3 essentially correct solutions.