This is an open notes prelim.

**Problem 1.** Suppose \( \{X_n\}_{n=1}^\infty \) are iid random variables such that \( \mathbb{E}[|X_1|] = +\infty \).
Show
\[
\lim_{n \to \infty} \frac{S_n}{n} = +\infty \tag{1}
\]
where \( S_n = \sum_{i=1}^n X_i \).
*Hint: Consider the inequality \(|X_n| \leq |S_n| + |S_{n-1}|\) and first see what happens to \( \lim \frac{|X_n|}{n} \). Is \( |X_n|/n \) large fairly regularly?*

**Problem 2.** Suppose \( \{X_n\}_{n=1}^\infty \) are iid Cauchy random variables with density
\[
f(x) = \frac{1}{\pi(1 + x^2)} \quad x \in \mathbb{R}
\]

1. Compute \( \mathbb{E}[|X_1|] \), and find \( \lim_{n \to \infty} S_n/n \).
2. Compute the characteristic function \( \phi(t) \) of \( X_1 \).
   *Hint: Consider using the residue theorem or computing the inverse Fourier transform of \( e^{-|t|} \).*
3. Does \( S_n/n \) have a weak limit?

**Problem 3.** Construct a sequence such that \( X_n \to X \) in distribution but \( X_n \not\to X \) in measure.

Suppose \( F_n(t) \to F(t) \) for all \( t \neq c \), where \( F_n \) is the cumulative distribution function of \( X_n \) and \( F \) is the cdf given by
\[
F(t) = \begin{cases} 
1 & t \geq c \\
0 & t < c
\end{cases}
\]
where \( c \in \mathbb{R} \). Show that \( X_n \to c \) in measure.

**Problem 4.** Let \( Y_1, Y_2, \ldots \) be nonconstant, nonnegative, iid random variables with \( \mathbb{E}Y_m = 1 \).

1. Show that
\[
X_n = \prod_{m \leq n} Y_m
\]
defines a martingale with respect to the filtration \( \mathcal{F}_n = \sigma(Y_1, \ldots, Y_n) \).
2. The martingale convergence theorem tells us that there is an \( X_{\infty} \) such that \( X_n \to X_{\infty} \) \( \mathbb{P} \) a.s. Determine \( X_{\infty} \).
   *Hint: Consider using the law of large numbers.*

**Problem 5.** Let \( p \) be a fixed number in \([1, \infty]\). Let \( X_n \) be a sequence of random variables such that for every \( \epsilon > 0 \), there exists an \( N \) such that for all \( n, m \geq N \),
\[
\mathbb{E}[|X_n - X_m|^p] < \epsilon
\]
Show that there is an \( X \) such that \( X_n \to X \) in probability.