MATH 453 FINAL EXAM - SPRING, 2022

Instructions: You may cite results from class or the textbook, but clearly **identify by name** or **state** what you are using.

Note: Problems **#1 - 5** constitute the Geometry Prelim exam.

1. Let N^n and M^m be compact, $F : N \to M$ a surjective submersion. Define an equivalence relation $\sim$ on N by $x \sim x' \iff F(x) = F(x')$. Prove that $N/\sim$ is a smooth manifold and is diffeomorphic to M.

2. Let $G : \mathbb{R}^3 \to \mathbb{R}^3$, $(u, v, w) = G(x, y, z) = (x, xy, xz + yz^2 + z^4)$.

(a) Find the sets of critical points and critical values of G in as explicit forms as you can (don't worry about solving any complicated algebraic equations).

(b) Find G^* of $v\,du + u^2\,dv + dw,\ du \wedge dv + dv \wedge dw$ and $du \wedge dv \wedge dw$.

3. For each of the following three pairs of functions $f_j : N_j \to M_j$ and submanifolds $S_j \subset M_j$, determine whether f_j is transverse to S_j. For any that are, find $f_j^{-1}(S_j)$.

(i) $f_1 : \mathbb{R}^2 \to \mathbf{S}^1$, $f_1(x, y) = e^{i(nx+my)}$ and $S_1 = \{e^{i0} = 1\}$. (Answer might depend on $n, m \in \mathbb{Z}$.) For $n \in \mathbb{N}$, $\mathbf{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

(ii) $f_2 : \mathbb{R}^3 \to \mathbb{R}^2$, $(u, v) = f_2(x, y, z) = (2x + 3y, x - 4z)$ and $S_2 = \{u + v = 1\}$.

(iii) $f_3 : \mathbb{R} \to \mathbb{R}^3$, $f_3(t) = (a + t, b + t, c + t)$ and $S_3 = \mathbf{S}^2$. (Answer might depend on $a, b, c \in \mathbb{R}$.)

4. Suppose $S \subset M^n$ is a regular, codimension k submanifold. Show that if X, $Y \in \Gamma(TM)$ are such that X_p, $Y_p \in T_pS$, $\forall p \in S$, then $[X, Y]$ has the same property: $[X, Y]_p \in T_pS$, $\forall p \in S$.

(continued on back!)

5. Let M be a smooth, oriented, n-dimensional manifold without boundary. For $0 \leq k \leq n$, $\Omega_c^k := \Omega_c^k(M)$ denotes the space of smooth, compactly supported k-forms on M.

Among other things, each Ω_c^k is a vector space over $\mathbb{R}$. Define a bilinear function $B_k : \Omega_c^k \times \Omega_c^{n-k} \to \mathbb{R}$ by

$$B_k(\omega, \tau) = \int_M \omega \wedge \tau.$$

Prove that B_k is nondegenerate, i.e., for all $\omega \in \Omega_c^k$, $\omega \neq 0$, there exists a $\tau \in \Omega_c^{n-k}$ such that $B_k(\omega, \tau) \neq 0$, and for all $\tau \in \Omega_c^{n-k}$, $\tau \neq 0$, there exists a $\omega \in \Omega_c^k$ such that $B_k(\omega, \tau) \neq 0$.

6. If $(M, \partial M)$ is an n-dimensional manifold with $\partial M \neq \emptyset$, then a *boundary defining function* is a $b \in C^\infty(M, \mathbb{R})$ such that

(i) $b = 0$ on ∂M and $b > 0$ on $M \setminus \partial M$; and

(ii) $db_p \neq 0$, $\forall p \in \partial M$.

Prove that any $(M, \partial M)$, $\partial M \neq \emptyset$, has a boundary defining function.