

Math 453  
Final Exam  
4:00-7:00 pm, May 8, 2020

EXAM ID NUMBER : \_\_\_\_\_

- The exam consists of 5 questions.
- Please read the questions carefully.
- **Show all your work in legibly written, well-organized mathematical sentences.**
- GOOD LUCK !!!

1. (20 pts) Define an atlas on the cylindrical surface

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, 0 < z < h\},$$

where  $h, r \in \mathbb{R}^+$ .

2. (20 pts) **a)** State the Inverse Function Theorem.

**b)** Let  $f : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$  be a function defined by  $f(x, y) = (x, y^3 + xy)$ . Is  $f$  a diffeomorphism?

3. (20 pts) Consider the vector fields on  $\mathbb{R}^2$ :

$$X = x \frac{\partial}{\partial x} + 4xy \frac{\partial}{\partial y}, \quad Y = 2y \frac{\partial}{\partial y},$$

and let  $\omega$  be the differential form on  $\mathbb{R}^2$  given by  $\omega = (x^2 + y)dx + (2x + y^2)dy$ . Show that  $\omega$  satisfies the relation  $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ , between the bracket product and the exterior differential.

4. a) State the Frobenius Theorem.

b) Consider the vector fields on  $\mathbb{R}^2$ :

$$X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad Y = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

Do they span an involutive distribution? Is this distribution (completely) integrable?

5. (20 pts) Calculate the de Rham cohomology groups,  $H_{DR}^k(S^2)$ , of the two-sphere  $S^2$  for all  $k$ . Justify your answers. (you can use the Mayer-Vietoris sequence, but you don't have to).