Math 467: Theory of Analytic Functions

Final (Prelim is first 5 problems)

May, 2022

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Instructions:

1. Read the notes below:

   - **Using any notes, books, online resources, or contacting any other people during this exam is prohibited.**

2. Read the following Academic Honesty Statement and sign:

   I affirm that I will not use any unauthorized resourced, or give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:____________________

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points) Suppose f is an entire function such that for any $a \in \mathbb{C}$ at least one coefficient in the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ is zero, i.e. for any a there exists a non-negative integer n such that $f^{(n)}(a) = 0$. Prove that f must be a polynomial.

2. (10 points) Let f be a complex function. Prove that if f and f^2 are harmonic, then either f or $\bar{f}$ must be holomorphic.

3. (10 points) Prove that in an infinite dimensional Hilbert space a basis (not a Hilbert basis) cannot be an orthonormal family. In your argument do not use results outside of what was proven in class.

4. (10 points) Let f be a holomorphic function in a region containing the unit disk centered at the origin and on the boundary of the disk it satisfies the inequality $|f(z)| < 1$. Determine the number of fixed points f can possibly have in the unit disk.

5. (10 points) Evaluate $\int_{-\infty}^{\infty} \frac{\cos t}{t-i} dt$. Justify your answer.

6. (10 points) Which of the following are Schwartz functions? Fully justify your answer in each case.

1. $f(x) = (1 + |x|^2)^{-N}$ where N is a positive integer.
2. $g(x) = e^{-\pi|x|^2}$.
3. $h(x) = e^{-\pi|x|^2} \sin(e^{\pi|x|^2})$.