MATH 471 FINAL EXAM
DECEMBER 20, 2022

Instructions: Print and sign your name on each green book that you use, and indicate which problems are in which book. Please write as neatly as possible. **Each problem should start on a new page.**

You may also use results from calculus to evaluate or estimate integrals, and you may cite results from undergraduate real analysis or topology. You may do the same for results from this class, **unless** you are being asked to **'prove directly'** the result in question. If citing a known result, **identify it clearly**, by name or simply stating the property you are using.

1. (i) Define the *outer measure* of a general subset $E \subseteq \mathbb{R}$.

(ii) Let $\mathcal{A} = \{E \subseteq \mathbb{R} \,:\, |E| = 0 \text{ or } |R \setminus E| = 0\,\}$.
Prove that $\mathcal{A}$ is a σ-algebra.

(iii) Is the restriction of outer measure $|\cdot|$ to $\mathcal{A}$ a measure on $\mathcal{A}$? Prove or find a counterexample.

2. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}_+}$ be $\mathcal{S}$-measurable.

(i) Show that $B := \{x \in X \,:\, f(x) < \infty\}$ and $X \setminus B$ belong to $\mathcal{S}$.

(ii) Show that, if $\int_X f \, d\mu < \infty$, then $\mu(X \setminus B) = 0$.

(iii) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable, $\overline{\mathbb{R}_+}$-valued functions on $\mathbb{R}$, with $||f_n||_{L^1(\mathbb{R})} \leq n^{-2}$ for all n. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers, $\mathbb{Q}$. Prove that $\sum_{n=1}^{\infty} f_n(x - r_n)$ converges for a.e. $x \in \mathbb{R}$.

3. Let $\{g_n\}_{n=1}^{\infty}$ be measurable, $\overline{\mathbb{R}}$-valued functions on $(X, \mathcal{S}, \mu)$.

(i) Prove that $h := \limsup_{n\to\infty} g_n$ is $\mathcal{S}$-measurable.

(ii) Prove that $Y := \{x \in X \,:\, \lim_{n\to\infty} g_n(x) \text{ exists }\}$ is measurable and that $g := \lim g_n$ is measurable (with respect to $\mathcal{S}$ restricted to Y).

(continued on back)

4. Let λ denote the restriction of Lebesgue measure on $\mathbb{R}$ to $(1, \infty) \subset \mathbb{R}$. For $f \in \mathcal{L}^2((1,\infty), \lambda)$, let

$$G(y) := \int_1^\infty \frac{f(x)}{x+y}\, d\lambda(x), \quad y \geq 1.$$

Prove that G is well-defined, bounded and continuous on $(1, \infty)$.

5. Let V be a Banach space and V^* its continuous dual. Suppose $\{T_n\}_{n=1}^\infty \subset V^*$ is a sequence of bounded linear functionals on V such that, for every $f \in V$, $\lim_{n\to\infty} T_n f$ exists (the limit being taken with respect to the norm on $\mathbb{F}$.)

Prove directly that there exists a bounded linear functional $T \in V^*$ such that $|T_n f - Tf| \to 0$ as $n \to \infty$ for all f.