1. Let \((X, S, \mu)\) be a measure space such that \(\mu(X) < \infty\). Prove that if \(\mathcal{A}\) is a family of disjoint sets in \(S\) such that \(\mu(A) > 0\) for all \(A \in \mathcal{A}\), then \(\mathcal{A}\) is a countable set.

2. Let \((X, S, \mu)\) be a measure space and consider \((f_n) \subset L^1(X)\) to be a sequence of functions converging pointwise a.e. to \(f \in L^1(X)\). Show that

\[
\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0
\]

if and only if

\[
\lim_{n \to \infty} \int_X |f_n| \, d\mu = \int_X |f| \, d\mu.
\]

3. If \(h : \mathbb{R} \to \mathbb{R}\) is a Lebesgue measurable function, then its associated Hardy-Littlewood maximal function \(h^* : \mathbb{R} \to [0, \infty]\) is defined by

\[
h^*(b) = \sup_{t > 0} \frac{1}{2t} \int_{[b-t, b+t]} |h(x)| \, d\lambda(x), \quad (\forall) \ b \in \mathbb{R}.
\]

Prove that

\[
\lambda(\{b \in \mathbb{R}; \ h^*(b) = \infty\}) = 0
\]

for all \(h \in L^1(\mathbb{R})\).

4. Let \(\lambda\) denote the Lebesgue measure on \([0, 1]\).
   i) Show that

\[
\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, d\lambda(y) \, d\lambda(x) = \frac{\pi}{4},
\]

\[
\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, d\lambda(x) \, d\lambda(y) = -\frac{\pi}{4}.
\]

   ii) Argue why the previous two equalities violate neither Tonelli’s theorem nor Fubini’s theorem.

5. Let \(f : [0, 1] \to \mathbb{R}\) be a Lebesgue measurable function. Prove that

\[
\lim_{p \to \infty} \|f\|_{L^p([0,1])} = \|f\|_{L^\infty([0,1])}.
\]