1a) Let \( K \) be an extension of fields where \( K \) is algebraic over \( E \) and \( E \) is algebraic over \( k \). Prove that \( K \) must be algebraic over \( k \). You may use facts about towers of finite extensions without proof if you want.

b) Let \( f(x) = x^3 + ax + b \in k[x] \) where \( \text{char}(k) \neq 2, 3 \), and let \( \Delta = -4a^3 - 27b^2 \) be the discriminant of \( f(x) \) (so that \( \Delta = \delta^2 \) where \( \delta = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \) for \( \alpha_1, \alpha_2, \alpha_3 \) the roots of \( f(x) \)). Explain how one can use \( \Delta \) to determine the Galois group of \( f(x) \) over \( k \) when \( f(x) \) is irreducible over \( k \), giving a complete proof to justify your answer. Does your argument go through in characteristic 2? Why or why not?

2) Let \( k \) be a field of characteristic \( p \neq 0 \). Prove each of the following.

a) Let \( K \) be a cyclic extension of \( k \) of degree \( p \). Then \( K = k(\alpha) \) for some \( \alpha \in K \) that is a root of a polynomial \( x^p - x - c \) for some \( c \in k \). [Hint: You might want to start by explaining why \( \text{Tr}(-1) = 0 \) where \( \text{Tr} \) represents the Trace from \( K \) to \( k \), and then applying the additive form of Hilbert’s Theorem 90, which you can use without proof.]

b) Conversely, for any \( c \in k \), the polynomial \( x^p - x - c \) either has one root in \( k \), in which case, all its roots are in \( k \), or it is irreducible. Moreover, in the latter case, \( k(\alpha) \) is Galois and cyclic of degree \( p \) over \( k \).

3, 4, 5) For purposes of the prelims, I will count your best 3 of the following 4 problems (A, B, C, or D, some of which have multiple parts, all of which were homework questions or are very related to ones that were). For purposes of the class final, all 4 will count, but you do not have to complete all 4 problems, as the scores will be curved.

Ai) Suppose \( \text{char}(K) = p \), and let \( a \in K \). If \( a \) has no \( p^{\text{th}} \) root in \( K \), then \( x^{p^n} - a \) is irreducible for all integers \( n \geq 1 \).

ii) Suppose \( \text{char}(K) = p \). Let \( \alpha \) be algebraic over \( K \). Show that \( \alpha \) is separable over \( K \) if and only if \( K(\alpha) = K(\alpha^{p^n}) \) for all positive integers \( n \).

B) Let \( E \) be an algebraic extension of \( k \) such that every nonconstant polynomial \( f(x) \in k[x] \) has at least one root in \( E \). Prove that \( E \) is algebraically closed.

Ci) Let \( G \) be a finite cyclic group. Prove there exists a Galois extension of \( \mathbb{Q} \) with Galois group \( G \).

ii) Prove the same result if \( G \) is a finite abelian (not necessarily cyclic) group.

Di) Let \( K/F \) be an extension of finite fields. Show that the norm \( N_{K/F}^{\times} \) is surjective (as a map from \( K^\times \) to \( F^\times \)). [Hint: As one step of your argument, explain why Hilbert’s Theorem 90, which you can use without proof, can be applied to \( K/F \).]

ii) A polynomial is called reciprocal if whenever \( \alpha \) is a root, so is \( \frac{1}{\alpha} \). Suppose that \( f(x) \) is a polynomial with coefficients in a subfield \( k \) of the real numbers, and \( f(x) \) is irreducible over \( k \). Suppose, moreover, that \( f(x) \) has a non-real complex root of absolute value 1. Show that \( f(x) \) is reciprocal of even degree.