1a) Define the \( n^{th} \) cyclotomic polynomial \( \Phi_n(x) \) over an arbitrary field \( k \) where the characteristic of \( k \) is either 0 or a prime \( p \) not dividing \( n \).

b) Prove that \( \Phi_n(x) \) is irreducible over \( \mathbb{Q} \).

c) Give an example if possible, and briefly explain why your example works. If no such example exists, briefly explain why this is so.

i) an integer \( n \geq 5 \) satisfying the property that \( \Phi_n(x) \) is irreducible over \( \mathbb{F}_p \) for all primes \( p \) not dividing \( n \).

ii) an integer \( n \geq 5 \) satisfying the property that \( \Phi_n(x) \) is reducible over \( \mathbb{F}_p \) for all primes \( p \) not dividing \( n \).

2) This relates to material in the book, not to a HW problem. To answer this question, you can use the additive form of Hilbert’s Theorem 90 without proof.

Let \( k \) be a field in characteristic \( p \neq 0 \). Prove each of the following.

a) Let \( K \) be a cyclic extension of \( k \) of degree \( p \). Then \( K = k(\alpha) \) for some \( \alpha \in K \) that is a root of a polynomial \( x^p - x - a \) for some \( a \in k \).

b) Conversely, for any \( a \in k \), the polynomial \( x^p - x - a \) either has one root in \( k \), in which case, all its roots are in \( k \), or it is irreducible. Moreover, in the latter case, \( k(\alpha) \) is Galois and cyclic of degree \( p \) over \( k \).

3,4,5) For purposes of the prelims, your best 3 of the following 4 problems (A, B, C, or D, some of which have multiple parts, all of which were homework questions or are very related to ones that were) will count. For purposes of the class final, all 4 will count, but you do not have to complete all 4 problems, as the scores will be curved.

Ai) Let \( L \) be an extension of a field \( k \), and let \( \alpha, \beta \in L \) be algebraic over \( k \). Suppose that \( k(\alpha) \) is a separable extension of \( k \) and that \( k(\beta) \) is a purely inseparable extension of \( k \).

a) Show that if \( F \) is an intermediate field between \( L \) and \( k \), then \( F(\alpha) \) is a separable extension of \( F \).

b) Show that if \( E \) is an intermediate field between \( L \) and \( k \), then \( E(\beta) \) is a purely inseparable extension of \( E \).

c) Show that \( k(\alpha + \beta) = k(\alpha, \beta) \)

ii) Give an example of a purely inseparable extension of fields of degree bigger than 1.

Bi) Suppose \( E \) and \( L \) are finite extensions of a field \( k \) contained in the same algebraic closure of \( k \). For each of the following, provide a brief proof if the statement is true, and a counterexample if the statement is false.

a) \([EL: E]\) must divide \([L: k]\)

b) If \([EL: E] = [L: k]\), then \( E \cap L = k \)

c) If \( E \cap L = k \), then \([EL: E] = [L: k]\).

ii) Would your answers to part i change any under the added hypothesis that \( L \) is Galois over \( k \)? Briefly explain why this is so.

C) Suppose that \( E \) is an \textit{algebraic} extension of \( k \) for which every non-constant polynomial in \( k[x] \) has at least one root in \( E \). Prove that \( E \) is algebraically closed.

Di) Let \( K \) be a field with no nontrivial abelian Galois extensions. Suppose that \( n \) is a positive integer and either \( \text{char}(K) = 0 \) or \( n \) is relatively prime to \( \text{char}(K) \). Prove that every element of \( K \) is an \( n^{th} \) power in \( K \).

ii) Is the statement still true if \( n \) is not relatively prime to \( \text{char}(K) \)? Prove it or give a counterexample.