Algebra 1 Prelims: August 26, 2022

1,2) Answer 2 of the following 3 problems (***A, B, C***). Note that Problems A and C have multiple parts *(If you attempt to do all three, I will count the ones you do best).*

Ai) If G is a simple nonabelian group and $e \neq x \in G$, prove that x must have at least 3 conjugates (including itself).

ii) Assume a group G acts both transitively and faithfully on a set S. Show that G_s, the stabilizer of s in G, contains no nontrivial normal subgroups of G for any $s \in S$.

iii) If G is a group that has a subgroup H of ***index*** $d > 1$ such that H contains no nontrivial normal subgroups of G, then $|G|$ divides $d!$ *(d factorial).*

B) Prove that there are exactly two nonabelian groups of order 8, up to isomorphism.

Ci) Show that $\mathbb{Q}/\mathbb{Z}$ is a torsion group that has exactly 1 subgroup of each order $n \geq 1$, and that each such subgroup must be cyclic.

ii) Deduce from this result that $\mathbb{Q}/\mathbb{Z}$ can ***<u>not</u>*** be written as the direct sum or direct product of cyclic groups *(either of a finite number of such groups, or of an infinite number).*

iii) Let H, K, and J be *(possibly infinite)* abelian groups, and suppose that $J \times H \approx K \times H$. Does this imply that $J \approx K$? Prove it or give a counterexample.

3i) Let G be a group of order $385 = 5 \cdot 7 \cdot 11$. Let H be a Sylow 7 subgroup of G. Must H be contained in the center of G? If so, give a careful proof. If not, carefully explain why not.

ii) Answer the same question for K if K is a Sylow 11 subgroup of G.

iii) Let G be a finite group, H a normal subgroup, and P a Sylow p-subgroup of G for some prime p. Prove that $P \cap H$ is a Sylow p-subgroup of H.

iv) Would the conclusion of Part *iii* still be true if H were not normal? Prove it if true, and if false, explain where your proof would break down and also give a counterexample or explain how you know that a counterexample must exist.

4a) Let A be a nonzero commutative ring with multiplicative identity which has the property that for each $x \in A$, there exists an integer $n \geq 2$ such that $x^n = x$, where the integer n may vary with x. If p is a prime ideal in A, prove that the ring A_p is a field, where A_p denotes the localization of A with respect to the set $S = A - p$.

b) Let A be an integral domain. Show that $\bigcap_m A_m = A$ (where the A_m are all viewed as being subrings of the quotient field K of A and the intersection is taken over all maximal ideals m in A). *Warning. You can't simply argue that if an element $\frac{a}{b} \in K$ belongs to A_m, then $b \notin m$, as that is false.*

5a) Let A be a local Noetherian integral domain with maximal ideal m in which $m^n = m^{n+1}$ for some positive integer n. Deduce that A must be a field. Explain where you are using each of the hypotheses of the problem in your answer, and if you rely on any well known theorems or lemmas, be specific as to what these say.

b) An ideal in a commutative ring is said to be irreducible if it can not be written as the intersection of two properly bigger ideals. Use the maximality property of Noetherian rings to show that any ideal in a Noetherian commutative ring can be written as the intersection of a finite number of irreducible ideals.