Assume $G$ acts on a finite set $S$ in such a way so there is only one orbit. Let $s \in S$, and let $G_\ast = \{g \in G | gs = s\}$ be the isotropy subgroup of $s$ (sometimes called the stabilizer of $s$).

a) Give a formula for $|S|$, the cardinality of $S$, and prove that your answer is correct.

b) If $s' = gs$ for some $g \in G$, state the relation between $G_\ast$ and $G_{s'}$, and briefly justify your answer.

c) Suppose $\{g \in G | gx = x \forall x \in S\} = \{e\}$. Show if $N$ is any normal subgroup of $G$ that is contained in $G_\ast$ for one particular $s$, then $N = \{e\}$.

2,3) Answer two of the following three problems (A, B, C). Note that each of these problems has multiple parts (if you attempt all three, I will count your best 2).

A i) If $N$ is a normal subgroup of $G$ containing no elements of the commutator subgroup $G^c$ except the identity, show $N \subseteq Z(G)$ where $Z(G)$ denotes the center of $G$.

ii) Suppose $G$ is a group such that $\text{Aut}(G)$ is cyclic. Prove that $G$ must be abelian and justify all of your steps.

B i) Show that if $G$ is an group of order $p^2q$ for distinct primes $p, q$ then at least one of its Sylow subgroups must be normal.

ii) Let $G$ be a finite group, $H$ a normal subgroup, and $P$ a Sylow $p$-subgroup of $G$ for some prime $p$. Prove that $P \cap H$ is a Sylow $p$-subgroup of $H$.

Would this still be true if $H$ were not normal? Prove if true, and if false, explain where your proof would break down and also give a counterexample or explain how you know that a counterexample must exist.

C i) Let $M$ be a $\mathbb{Z}$-module (i.e. an abelian group under addition) and let $S \subseteq M$ be a proper subset.

a) $S$ spans $M$ as a $\mathbb{Z}$-module and no proper subset of $S$ has this property.

b) $S$ is linearly independent over $\mathbb{Z}$ and no set properly containing $S$ has this property.

ii) Let $H, K$, and $J$ be (possibly infinite) abelian groups, and suppose that $J \times H \cong K \times H$. Does this imply that $J \cong K$? Prove it or give a counterexample.

4) Recall that an element $x$ of a ring $A$ is called nilpotent if $x^n = 0$ for some $n \in \mathbb{Z}^+$. 

a) Suppose $A$ is a commutative ring with $1 \neq 0$ satisfying the property that the localization $A_m$ has no nonzero nilpotent elements for any maximal ideal $m$. Prove that $A$ has no nonzero nilpotent elements.

b) Would the analogous result still be true if the phrase “no nonzero nilpotent elements” was replaced by the phrase “no zero-divisors”, (where by definition a zero divisor can not equal 0)? If your answer is yes, give a complete proof, and if your answer is no, tell where your proof in part a breaks down, and also give a counterexample.

5) Assume all rings in this problem are commutative. Use the maximality property of Noetherian rings to prove each of the following.

a) In a Noetherian ring, any ideal contains a product of prime ideals.

b) In a Noetherian ring, any ideal can be written as the intersection of a finite number of irreducible ideals (where an ideal in a commutative ring is said to be irreducible if it can not be written as the intersection of two properly bigger ideals).