Algebra II Prelims/Math 437 Final
May 12, 2014

1a) Let α and β be algebraic over F. Let $f(x) = \mathrm{Irr}(\alpha, F, x)$ and $g(x) = \mathrm{Irr}(\beta, F, x)$ denote the minimal polynomials of α and β respectively over the field F, and suppose the degrees of $f(x)$ and $g(x)$ are relatively prime. Prove that $g(x)$ is irreducible in the polynomial ring $F(\alpha)[x]$.

b) Let E be an algebraic extension of F. Show that every subring of E containing F is actually a field. Is this necessarily true if E is not algebraic over F? Prove it or give a counterexample.

2) Let K be the splitting field for $x^4 - 2$ over $\mathbb{Q}$.

a) Describe the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ in terms of a well known group or direct product of well known groups and justify your answer.

b) Give each of the elements of your group a name and describe the action of each one on a set of generators for K over $\mathbb{Q}$. Use this information to illustrate the complete correspondence between subgroups of the Galois group and intermediate fields. Include a brief explanation of your reasoning.

3***a)*** Define the n^{th} cyclotomic polynomial $\Phi_n(x)$ over an arbitrary field k where the characteristic of k is either 0 or a prime p not dividing n.

b) Prove that $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.

c) When will $\Phi_n(x)$ be irreducible over $\mathbb{F}_p$ if p is a prime not dividing n?

4) Let $n \in \mathbb{Z}^+$ and assume that the characteristic of k is either 0 or relatively prime to n. Assume that $\zeta_n \in k$ where ζ_n denotes a primitive n^{th} root of 1.

a) Let α be a root of $x^n - a$ for some fixed $a \in k$. Show that $k(\alpha)$ is Galois over k with a Galois group that is cyclic and $[k(\alpha){:}\,k]$ dividing n. Give an example to show that $[k(\alpha){:}\,k]$ need not equal n.

b) Let K/k be a cyclic extension of degree n. Show that $K = k(\alpha)$ where α is a root of an irreducible polynomial of the form $x^n - a$ for some $a \in k$. *[Hint: You might want to start by noting that $N(\zeta_n^{-1}) = 1$ where N represents the norm from K to k and then applying Hilbert's Theorem 90. You can use Hilbert's Theorem 90 without proof if you want.]*

5) Choose ***one*** of Problems ***A*** or ***B***.

A) Let F be an intermediate field between K and k where K/k is a finite Galois extension with Galois group G.

i) Let $H = \{\sigma \in G \mid \sigma(F) = F\}$. Show that H equals the normalizer of $J = \mathrm{Gal}(K/F)$ in $G = \mathrm{Gal}(K/k)$.

ii) Let $E = K^H$. Show that E is the smallest subfield of F containing k such that F/E is Galois.

Bi) Show that if a and b are elements of a finite field k, then $f(x)$ must have a root in k, where

$$f(x) = (x^3 + ax + b)(x^2 + 4a^3 + 27b^2).$$

You can assume the characteristic does not equal 2 or 3 if you want.

ii) Let K be a field in characteristic 0 with the property that every cubic polynomial with coefficients in K has at least one root in K. Let $f(x)$ be an irreducible 4^{th} degree polynomial over K whose discriminant Δ is a square in the field K. Find the Galois group of $f(x)$ over K and completely justify your answer. *(Recall the definition of the discriminant of a polynomial. If $g(x) = \prod_{i=1}^{n} (x - \alpha_i)$, then the discriminant Δ of $g(x)$ equals δ^2 where $\delta = \prod_{i<j} (\alpha_i - \alpha_j)$.)*

Complex analysis 467 – Final exam

May 8, 2014

Please prove the following, justifying all statements.

1. Give two proofs of the fundamental theorem of algebra, with the first proof using Liouville's theorem, and the second using Rouché's theorem.

2. Let f be an entire function and suppose there exists constants $C > 0$ and D such that

$$|f(z)| \leq C|z|^n + D \quad \text{for all } z \in \mathbb{C}.$$

Use Cauchy's estimates to prove that f is a polynomial of degree at most n.

3. Prove that

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

making use of the contour defined by the line segment $[-R, R]$ on the real axis and a large half-circle centered at the origin in the upper half-plane. (see picture)

4. Argue using Hadamard's factorization theorem that $e^z + z^2 + 1 = 0$ has infinitely many solutions in $\mathbb{C}$.

5. Let Ω be a bounded simply connected domain in $\mathbb{C}$, and let α and β be two distinct points in Ω. Let ϕ_1 and ϕ_2 be two conformal maps from $\Omega \to \Omega$. Suppose $\phi_1(\alpha) = \phi_2(\alpha)$ and $\phi_1(\beta) = \phi_2(\beta)$. Prove that $\phi_1 = \phi_2$. [Hint: one approach uses Schwarz's lemma.] (Side note: this is also true without the bounded condition.)

Math 453
Final Exam
May 3, 2014
9:00 - 12:00

Name : ________________________________

- The exam consists of 5 questions.

- Please read the questions carefully.

- **Show all your work in legibly written, well-organized mathematical sentences.**

- GOOD LUCK !!!

1. (20 pts) **a)** State the Regular Value Theorem.

b) Prove that the subset H of the Euclidean space $\mathbb{R}^3$ of all the points (x, y, z) of $\mathbb{R}^3$ satisfying $x^3 + y^3 + z^3 - 2xyz = 1$ admits a C^∞ 2-manifold structure.

2. (20 pts) For the following vector fields X and Y and differential forms α and β on $\mathbb{R}^3$, calculate the Lie bracket $[X, Y]$ and the Lie derivatives $L_X\alpha$ and $L_{[X,Y]}(\alpha \wedge \beta)$:

$X = x\dfrac{\partial}{\partial x} - z^2\dfrac{\partial}{\partial y}$ and $Y = z\dfrac{\partial}{\partial y} + x^3\dfrac{\partial}{\partial z}$, $\alpha = e^x dx + ydy + zdz$, $\beta = dx \wedge dy \wedge dz$.

a) $[X, Y] =$

b) $L_X\alpha =$

c) $L_{[X,Y]}(\alpha \wedge \beta) =$

3. (20 pts) **a)** Determine whether the two-form $\omega = z dx \wedge dy$ is exact in $\mathbb{R}^3$.

b) Let M denote the embedded submanifold of $\mathbb{R}^3$ given by $M = \{z - x^2 - y^2 = 1\}$. Determine whether the restriction of ω to M is exact.

4. (20 pts) Let $\mathbb{C}^*$ be the punctured complex plane, $\mathbb{C} - \{(0,0)\}$. Let $z = x + iy$ be the usual complex coordinate on $\mathbb{C}$. Let

$$\eta = Re(\frac{dz}{2\pi iz})$$

be a one form in $\mathbb{C}^*$. (Here $dz = dx + idy$).

a) Calculate $\int_C \eta$ for the circle of radius r centered at the origin.

b) Show that the η is a generator for the de Rham cohomology $H^1(\mathbb{C}^*)$.

5. (20 pts) Show that the Laplacian $\Delta = dd^* + d^*d$ has the following properties :

a) Δ is self-adjoint, that is $\langle \Delta\omega, \eta\rangle = \langle \omega, \Delta\eta\rangle$.

b) A necessary and sufficient condition for $\Delta\omega = 0$ is that $d\omega = 0$ and $d^*\omega = 0$.

c) If ω is a harmonic form, so is $*\omega$.