Prelims
May, 2013

Algebra II

1) Show that if α is algebraic over a field k, then the mutiplicity of α in its minimal polynomial $f(x) = \mathrm{irr}(\alpha, k, x)$ must be 1 if the characteristic is 0, and p^μ for some nonnegative integer μ if the characterisitic is $p > 0$. In the latter case, deduce that α^{p^μ} must be separable over k and

$$[k(\alpha) : k] = p^\mu [k(\alpha) : k]_s \tag{1}$$

Hint: Equation (1) follows easily from everything else. There are several different ways to see this.

2) Let F be an intermediate field between K and k where K/k is a finite Galois extension with Galois group G.

a) Let $H = \{\sigma \in G \mid \sigma(F) = F\}$. Show that H equals the normalizer of $J = \mathrm{Gal}(K/F)$ in $G = \mathrm{Gal}(K/k)$.

b) Let $E = K^H$. Show that E is the smallest subfield of F containing k such that F/E is Galois.

3a) Give an example of a Galois extension of $\mathbb{Q}$ whose Galois group is isomorphic to $\mathbb{Z}_3 \times S_3$ if possible (justifying each one of your claims). If not possible, briefly explain why not.

b) Prove that there are an infinite number of such extensions if true. If false, briefly explain why not.

c) Give an example of a Galois extension of a finite field whose Galois group is isomorphic to $\mathbb{Z}_3 \times S_3$ if possible (justifying each one of your claims). If not possible, briefly explain why not.

4a) Let $n \in \mathbb{Z}^+$ and assume that the characteristic of k is either 0 or relatively prime to n. Let $\zeta = \zeta_n$ be a primitive n^{th} root of 1, and let α be a root of $x^n - a$ for some fixed $a \in k$. Show that if $\zeta \in k$, then $k(\alpha)$ is Galois over k with a Galois group that is cyclic and $[k(\alpha) : k]$ dividing n. Give an example to show that $[k(\alpha) : k]$ need not equal n.

b) Let $k = \mathbb{F}_p^a$ be the algebraic closure of $\mathbb{F}_p$ and let $K = k(t)$ for some element t transcendental over k. Does there exist an extension of K that is purely inseparable of degree p over K? If so, produce a specific example of such an extension being careful to prove every one of your claims. If such an extension does not exist, then say so.

c) Let K be as in part b above. Does there exist an extension of K that is Galois cyclic of degree p? If so, produce a specific example of such an extension being careful to prove every one of your claims. If such an extension does not exist, then say so.

5) Give an example of each of the following if possible. If not possible, briefly explain why not. In each case, completely justify your reasoning. *For purposes of the prelims, you must have at least 4 of these correct to count as having the problem correct.*

a) an extension K/k of finite degree that has an infinite number of intermediate fields

b)a prime $p \neq 7$ such that ζ_7 has degree $\varphi(7) = 6$ over $\mathbb{F}_p$

c) a prime $p \neq 7$ such that ζ_7 has degree 2 over $\mathbb{F}_p$

d) a prime $p \neq 3, 5$ such that ζ_{15} has degree $\varphi(15) = 8$ over $\mathbb{F}_p$

e) an irreducible cubic polynomial in $\mathbb{Q}[x]$ with precisely one real root whose discriminant Δ is a square in $\mathbb{Q}$

f) an irreducible cubic polynomial over a finite field k whose discriminant Δ is ***NOT*** a square in k.

Complex Analysis

1. Suppose that u and u^2 are harmonic on a region (an open connected set) $\Omega \subset \mathbb{C}$. Prove that u is constant on Ω. (You may wish to use the fact that any two points in a region may be connected by a curve consisting of line segments parallel to the coordinate axis.)

2. Find a conformal map $f(z)$ of the region $\Omega = \{z : |z| < 1 \text{ and } \Im z > 1/2\}$ which maps Ω one-to-one and onto the unit disc $\{z : |z| < 1\}$. Illustrate the various stages of your mapping and argue that the map does what you claim.

3. Suppose that $a_1, a_2, \ldots, a_n$ are distinct point in $\mathbb{C}$ with $|a_k| < 1$. Put

$$f(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}.$$

   a) What is the modulus of $f(z)$ on $|z| = 1$? Why?

   b) Select a number b in $|z| < 1$ and show that $f(z) = b$ has n solutions in $|z| < 1$ counting multiplicities.

4. Use the calculus of residues to evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^2} dx.$$

5. Let $f(z)$ be holomorphic on the disc $|z| \leq R$ with $R > 1$, and suppose that $f(z)$ has a root at $z = i$ and at $z = -i$. Let $M = \max_{|z|=R} |f(z)|$. Prove that

$$|f(z)| \leq \frac{M|z^2 + 1|}{R^2 - 1}.$$

Geometry, problems and answers

1.

(a)(6) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x,y) = (x^2+y^2, xy)$. Carefully compute the pull back $F^*(-vdu + udv)$. List the properties of d and the pull back which are used in this computation.

Answers:

A quick list of facts that make calculating with forms easier: F^* commutes with d and distributes over $\wedge$. d is an antiderivation, i.e. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. $F^*(f) = f \circ F$.

$$\begin{aligned} F^*(-vdu + udv) &= -F^*vF^*(du) + F^*(u)F^*(dv) \\ &= -(v \circ F)d(F^*u) + (u \circ F)dF^*v \\ &= -xyd(x^2+y^2) + (x^2+y^2)d(xy) \\ &= -xy(2xdx + 2ydy) + (x^2+y^2)(ydx + xdy) \\ &= (y^3 - x^2y)dx + (x^3 - xy^2)dy \end{aligned}$$

(b)(7) Let α be a non-zero 1 co-vector and let γ be a k-covector on $\mathbb{R}^n$ $(n > k)$. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $k-1$ convector β on $\mathbb{R}^n$.

Answers:

Let $\gamma = \alpha \wedge \beta$ then $\alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0$ because $\alpha \wedge \alpha = 0$ for any one form.

Let $\alpha \wedge \gamma = 0$. Let $\alpha = \alpha^1, \alpha^2, \ldots \alpha^n$ be a basis for one forms on $\mathbb{R}^n$ (any set of independent vectors can be extended to a basis). Then $\gamma = \sum a_I \alpha^I$ for sets $I = \{i_1 \ldots i_k\}$ because such wedge products form a basis for $\Lambda^k \mathbb{R}^n$. Considering $0 = \alpha \wedge \gamma = \sum a_I \alpha^1 \wedge \alpha^I$ we see that $a_I = 0$ for all I that do not contain 1 since such $\alpha^1 \wedge \alpha^I$ are independent basis vectors of $\Lambda^{k+1}\mathbb{R}^n$. From the remaining terms we can factor out α^1. The $k-1$ form β is given by $\beta = \sum a_I \alpha^{I-\{1\}}$ summed over all of the sets I which contain 1.

(c)(7) What axioms describe a derivation of smooth functions at $p \in \mathbb{R}^n$? Show that the map from derivations δ to directional derivatives (D_v) at p is bijective and linear.

Answers:

A derivation δ is a linear functional from the space of smooth functions at p to $\mathbb{R}$. In addition δ satisfies the product rule: $\delta(fg) = f(p)\delta g + g(p)\delta f$. These axioms are enough to guarantee that δ takes constants to 0 and that δ is a local operator.

The directional derivative $D_v f = \frac{d}{dt} f(p + tv) \mid_{t=0}$ is clearly a derivation and the inclusion map is linear and 1 to 1 since directional derivatives are different on coordinate functions.

It remains to show that every derivation is a directional derivative for some $v \in \mathbb{R}^n$. To determine v apply δ to the coordinate functions x^i. Given $\delta x^i = a^i$ set $v = a^i \frac{\partial}{\partial x^i}$. δ and D_v agree on coordinate functions.

Use the Taylor formula to show that if the directional derivative and the derivation agree on coordinate functions they agree on all smooth functions. Since $f(x) = f(p) + f_i \cdot (x^i - p^i) + f_{ij} \cdot (x^i - p^i)(x^j - p^j) +$ higher order terms, we see that δ and D_v are completely determined by the values they take when applied to the coordinate functions.

2.

The Lie group $G = SO(2,1)$ consists of the invertible matrices which preserve a certain matrix:

$$G = \{A \mid A^T J A = J\} \text{ where } J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a)(7) Describe the elements in the tangent space to G at the identity element (e or I). Describe the relation between these elements and the lie algebra of G, giving a definition of the lie algebra of G.

Answers:

To determine the restrictions on the tangent space at e consider curves $A(t)$ with $A(0) = I$ and $A^T(t) J A(t) = J$. Differentiating and evaluating at $t = 0$ we have $A'(0) = -JA'(0)^T J$. If we let b_{ij} be the components of the tangential matrix $A'(0)$ we see that the following identities are satisfied: $b_{ii} = 0$, $b_{12} = b_{21}, b_{13} = b_{31}$ and $b_{23} = -b_{32}$.

The tangent space is 3 dimensional. The lie algebra of G is the space of left invariant vector fields on G. These fields can be completely determined by their value at e and therefore the lie algebra is also a 3 dimensional vector space.

(b)(7) If Y is a left invariant vector field on G and

$$Y_e = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

what is the value of the vector field Y at g (that is Y_g) if

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$Y_g =$.

Answers:

Because the left action of g on G is given as multiplication by a (constant) matrix (also called g) we have

$$g_*(Y_e) = g_*(A'(0)) = \frac{d}{dt} gA(t) \mid_{t=0} = g \frac{d}{dt} A(t) \mid_{t=0} = gY_e$$

Multiplying the matrix g against Y_e gives $Y_g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

(c)(6) Compare the vector space of all smooth vector fields on G and the space of left invariant vector fields on G including their dimensions.

Answers:

There are many vector fields on G. There is no finite basis for this space and the dimension is infinite.

There are relatively few left invariant vector fields Y on G and each such field is determined by its value Y_e at a single point, say the identity. Then Y at g is given by the matrix formula $Y_g = gY_e$. Y_e is in the tangent space at the identity which is 3 dimensional by the first result above and therefore the space of left invariant vector fields on G is a 3 dimensional vector space.

3.

(a) (8) If $F\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $F(x, y) = x^3 + xy + y^3 = a$ for which values of a are the level sets of F guaranteed to be embedded submanifolds of $\mathbb{R}^2$? Explain your reasoning – you may quote theorems.

Answers:

$F(x, y) = x^3 + xy + y^3$. $\frac{\partial F}{\partial x} = 3x^2 + y$, $\frac{F}{y} = x + 3y^2$. The inverse function theorem guarantees that the level set is a submanifold unless it contains a point where DF has rank 0, i.e. both partial derivatives are 0. Some algebra shows that this can happen only if $x = y = -\frac{1}{3}$ or $x = y = 0$. $F(0, 0) = 0$ and $F(-\frac{1}{3}, -\frac{1}{3}) = \frac{1}{27}$ so unless a equals one of those two values the level set is a submanifold. If a is equal to one of those two values then more analysis is needed to determine whether the level set is a submanifold at the critical points.

(b)(2) Give an example of an immersed submanifold which is not embedded.

Answers:

There are two canonical examples. One is a figure eight which is the image of an open unit interval. At the "crossing point" the image is not a submanifold (it is not locally homeomorphic to an interval). The second is the result of taking S to be a line in $\mathbb{R}^2$ with irrational slope and then taking the quotient of the plane with respect to the integer lattice to obtain a torus. The line S never closes and is dense in the torus, hence it's image does not have any charts exhibiting S as a submanifold.

(c)(10) Show that if $i\colon S^k \to M^n$ is an injective immersion of a k-manifold into an n-manifold then for any $p \in S$ there is an open set $p \in U \subseteq S$ for which $i(U)$ is an embedded submanifold and $i\colon U \to M$ is a diffeomorphism onto $i(U)$. (Hint: Use constant rank theorem).

Answers:

Given $i\colon S^k \to M^n$ and a point p we have that a i_* has rank k at every point because i is an immersion. By the constant rank theorem there are open sets U and V and chart functions ϕ and ψ such that the map $\psi \circ i \circ \phi^{-1}\mathbb{R}^n \to \mathbb{R}^n$ is given by $x^1, \ldots x^k \to (x^1, \ldots, x^k, 0, \ldots)$. This shows that $i(U)$ is the zero set of $x^{k+1} = \ldots x^n = 0$ (restricted to V) and therefore $i(U)$ is an embedded submanifold.

Comments: The map I doesn't need to be injective. In the example, the effect of restricting to U is to remove the parts of $i(S^k$ which are too close to $i(p)$ and are preventing $i(S^k)$ from being an embedded submanifold.

4.

(a)(5) Explain how one determines whether a function between manifolds $F\colon M^n \to N^k$ is C^∞ at a point $p \in M$. Explain how the definition of a manifold insures that this definition is well defined.

Answers:

$F\colon M \to N$ is smooth at p off for some charts $\phi\colon p \in U \to \mathbb{R}^n$, $\psi\colon F(p) \in V \to \mathbb{R}^k$ we have $\tilde{F} = \psi \circ F \circ \phi^{-1}\colon \mathbb{R}^n \to \mathbb{R}^k$ is C^∞. We only need to check one chart because if alternate charts are used the smooth transition functions τ, ρ insure that $\tilde{F}$ is smooth if and only if the alternative representation $\tau \circ \tilde{F} \circ \rho$ is smooth.

(b)(15) We consider three types of manifold structures related to the manifold M^n: the n dimensional manifold M itself, a k dimensional embedded submanifold S contained in M and the $2n$ dimensional tangent bundle TM. Explain the difference between the charts and the transition functions used to describe each of these structures. (Don't forget to explain the difference between the tangent bundle transition functions and those for an arbitrary vector bundle over M).

Answers:

(a) For M the transition functions between charts must be smooth diffeomorphisms from $\mathbb{R}^n \to \mathbb{R}^n$. Among other things this allows you to unambiguously specify the smooth functions on M as above.

(b) For $S^k \subseteq M^n$ the charts must identify points of S with the natural embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$ along the first k coordinates.. The transition functions, in addition to being smooth, must map this $\mathbb{R}^k$ subspace to the $\mathbb{R}^k$ subspace on the next chart.

For the manifold TM the $\mathbb{R}^{2n}$chart is a product $\mathbb{R}^n \times \mathbb{R}^n$ and the transition map is the product of a diffeomorphism ϕ and a linear map L. $(x^1, \ldots, x^n, a^1, \ldots a^n) \mapsto (y^1(\vec{x}), \ldots, y^n(\vec{x}), L(\vec{a})$ where the coefficients of the matrix L can depend on $\vec{x}$. In order to be the tangent bundle we must have that $L = D\phi$, that is the linear map is the linearization of the diffeomorphism ϕ.

5.

Assume the existence of C^∞ bump functions and partitions of unity on M.

(a)(5) Let A and B be two disjoint closed sets in a manifold M. Find a C^∞ function F on M such that F is identically 1 on A and is identically 0 on B.

Answers:

Consider the partition of unity subordinate to $(M - A)$ and $(M - B)$. $supp \phi_{M-B} \subseteq M - B$ hence ϕ_{M-B} restricted to B is identically 0. On the other hand $\phi_{M-B} + \phi_{M-A} = 1$ and ϕ_{M-A} is identically 0 on A. This means ϕ_{M-B} is identically 1 on A.

(b) (5) Prove or give counter-example: If S is a smooth submanifold of M then every C^∞ function $g \colon M \to \mathbb{R}$ restricts to a C^∞ function on S.

Answers:

If S is a smooth submanifold then $i \colon S \to M$ is C^∞ and the restriction of g to S is simply $g \circ i$ which is smooth by the chain rule. S does not need to be an embedded submanifold for this is true for immersed sub manifolds as well.

(c) (10) Prove or give counter-example: If S is a smooth embedded submanifold of M then every C^∞ function $f \colon S \to \mathbb{R}$ is the restriction of some globally defined C^∞ function $g \colon M \to \mathbb{R}$. $f = g|_S$.

Answers:

This is more complicated. One uses the structure of the submanifold charts and a partition of unity on M.

Let U_α be an atlas for $S \subset M$ with partition of unity ρ_α. For each $\psi_\alpha(p) \in \psi_\alpha(S)$ extend the function f to be a constant in the $x^{k+1}, \ldots, x^n$ variables and then multiply by ρ_α.

$$f_\alpha \circ \psi^{-1}(x^1, \ldots, x^n) = \rho_\alpha f \circ \psi^{-1}(x^1, \ldots, x^k) = \rho_\alpha f(p)$$

f_α can be defined as 0 outside U_α so it is defined and smooth on all of M. $\sum f_\alpha = \tilde{f}$ is defined and smooth on all of M since the sums are locally finite. For each $p \in S$ we have $\sum f_\alpha(p) = \sum \rho_\alpha f(p) = (\sum \rho_\alpha) f(p) = f(p)$ so $\tilde{f}$ is an extension of f to the entire manifold M. f is the restriction of a smooth function on M to S.