Real Analysis Prelim Questions

Day 1—August 27, 2013

Pass at the PhD level requires at least 3 problems worked correctly. There is no partial credit.

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are 5 questions. TIME LIMIT: 3 hours

Instructions: Measure and measurable refer to Lebesgue measure μ_n on $\mathbb{R}^n$, and $\mathcal{M}(\mathbb{R}^n)$ is the collection of measurable subsets. Please provide complete proofs and justifications for answers.

1. Let $1 \leq p_i < \infty$, $1 \leq i \leq k$, and $1 \leq q < \infty$ be such that $\sum_{i=1}^k \frac{1}{p_i} = \frac{1}{q}$. Prove that if $f_i \in L^{p_i}(\mathbb{R}^n)$, $1 \leq i \leq k$, then $\prod_{i=1}^k f_i \in L^q(\mathbb{R}^n)$ and

$$||\prod_{i=1}^k f_i||_{L^q} \leq \prod_{i=1}^k ||f_i||_{L^{p_i}}.$$

2. For $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}$, define $A_x := \{y \in \mathbb{R} | (x,y) \in A\}$.

(i) Show that if $A \in \mathcal{M}(\mathbb{R}^2)$, then $A_x \in \mathcal{M}(\mathbb{R})$ for almost every $x \in \mathbb{R}$.

(ii) Show that if $|A|_{\mu_2} = 0$, then $|A_x|_{\mu_1} = 0$ for almost every $x \in \mathbb{R}$.

3. Prove the $L^1(\mathbb{R}^n)$ norm triangle inequality: $||f+g||_{L^1} \leq ||f||_{L^1} + ||g||_{L^1}$. (Hint: first prove it for $\mathbb{R}$-valued functions, and then consider the case of $\mathbb{C}$-valued functions.)

4. Prove the continuity of the integral: If $f \in L^1(\mathbb{R})$, then

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = 0.$$

5. (i) Define what it means for a function $f : \mathbb{R}^n \to [0,\infty] = [0,\infty) \cup \{+\infty\}$ to be measurable.

(ii) Prove or find a counterexample of: If an f as in (i) also satisfies $||f||_{L^1(\mathbb{R}^n)} < \infty$, then f takes values in $[0, \infty)$ almost everywhere.

(iii) Prove or find a counterexample of: If an f as in (i) takes values in $[0, \infty)$ almost everywhere, then $||f||_{L^1(\mathbb{R}^n)} < \infty$.

Complex Analysis Questions

Day 1—August 27, 2013

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6. Suppose that $f(z)$ is holomorphic on $\mathbb{D} = \{z : |z| < 1\}$. Suppose also that for all $z \in \mathbb{D}$, $\Re f(z) > 0$ and $f(0) = 1$. Prove that for all $z \in \mathbb{D}$ we have

$$|f(z)| \leq \frac{1+|z|}{1-|z|}.$$

7. Suppose that Ω is an open connected set containing 0 and that $f(z)$ is holomorphic on Ω. Show that if $|f(1/n)| \leq e^{-n}$ for $n = 1, 2, \ldots$, then $f(z)$ is identically zero on Ω.

8. Let $\{f_k(z)\}$ be a sequence of functions holomorphic in the complex plane $\mathbb{C}$, which converges uniformly on compact subsets of $\mathbb{C}$ to a polynomial $P(z)$ of positive degree n. Prove that if k is sufficiently large, then $f_k(z)$ has at least n zeros (counting multiplicities).

9. Compute, using the residue theorem **and including complete justifications**,

$$\int_0^\infty \frac{\cos x}{9 + 10x^2 + x^4}\, dx.$$

10. Let $P(z)$ be a non-constant complex polynomial, all of whose zeros lie in a half-plane $\{z : \Re z < a\}$ where a is a real number. Show that all the zeros of $P'(z)$ lie in this same half-plane. (Hint: compute the logarithmic derivative of $P(z)$.

Algebra I Questions
Day 2—August 28, 2013

Pass at the PhD level requires at least 3 problems worked correctly. There is no partial credit.

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are five questions. TIME LIMIT: 3 hours

11. a) Show that A_4 has no subgroup of order 6, and explain why this implies that the "converse" of Lagrange's theorem is false.

b) Find the commutator subgroups of S_4 and A_4 and of S_n and A_n if $n \geq 5$, and justify your answers. (Hint: you can do this without actually computing any commutators.)

12. Let H be a proper subgroup of a finite group G. Show that G can not equal the union of conjugates of H.

13. a) Assume without proof that all groups of orders dividing 36 but strictly smaller than 36 are solvable, and use this to conclude that all groups of order 36 are solvable.

b) Show that if G is any group of order p^2q for distinct primes p, q then at least one of its Sylow subgroups must be normal. (Keep in mind that your argument should also work in the case $p = 2$ and $q = 3$.)

14. Recall that an element x of a ring A is called nilpotent if $x^n = 0$ for some positive integer n.

a) Suppose A is a commutative ring with $1 \neq 0$ satisfying the property that A_m (the localization of A outside of m) has no nonzero nilpotent elements for any maximal ideal m. Prove that A has no nonzero nilpotent elements.

b) Would the analogous result still be true if the phrase "no nonzero nilpotent elements" was replaced by the phrase "no zero-divisors", (where by definition a zero divisor can not equal 0)? Prove it or give a counterexample.

15. a) Prove that any prime ideal in an Artin ring is a maximal ideal. If you use any properties about Artin rings other than the definition of Artin rings in your answer, also include an explanation of why those properties are true.

b) Prove that an Artin ring has only a finite number of prime ideals.

Algebra II Questions
Day 2—August 28, 2013

Pass at the PhD level requires at least 3 problems worked correctly. There is no partial credit.

Do as many problems as you can in whatever order you wish. **Use a separate blue book for each problem.** Clearly indicate the exam date, the problem number and your name on the front of each book you use. There are six questions. TIME LIMIT: 3 hours

16. a) Let k be a finite field with q elements. Prove that

$$x^{q^n} - x = \prod_{d|n} \prod_{f_d} f_d(x) \tag{1}$$

where the inside product is over all monic irreducible polynomials of degree d over k, and the outside product is over all positive integers d dividing k.

b) Give an example of a field k and a polynomial $f(x) \in k[x]$ of degree bigger than 1 that is irreducible over k, but is neither separable nor purely inseparable over k if it is possible to do so. If it is not possible to do so, briefly explain why not. Be sure to justify all your claims.

17. Let k be a field, $f(x)$ an irreducible polynomial in $k[x]$, and K a finite normal extension of k. If $g(x)$, $h(x)$ are monic irreducible polynomials in $K[x]$ that divide $f(x)$, show there exists an automorphism σ of K over k such that $g(x) = h^{\sigma}(x)$.

b) Give an example that shows the conclusion need not be valid if we drop the hypothesis that K is normal over k.

18. a) Let $\Phi_n(x)$ denote the n^{th} cyclotomic polynomial. Prove that $\Phi_n(x)$ is irreducible over $\mathbf{Q}$.

b) Prove that the Galois group of $\mathbf{Q}(\zeta_n)$ over $\mathbf{Q}$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^*$ (where ζ_n represents a primitive n^{th} root of 1 in characteristic 0.

c) Is it also true that the Galois group of $\mathbf{F}_p(\zeta_n)$ over $\mathbf{F}_p$ must be isomorphic to $(\mathbf{Z}/n\mathbf{Z})^*$ (where ζ_n now represents a primitive n^{th} root of 1 in characteristic p in a case where $p \not| n|$)? Prove it or give a counterexample.

19. To answer this question you can use the additive form of Hilbert's Theorem 90 without proof if you want to.

Let k be a field of characteristic $p \neq 0$. Prove each of the following.

a) Let K be a cyclic extension of k of degree p. Then $K = k(\alpha)$ for some $\alpha \in K$ that is a root of a polynomial $x^p - x - c$ for some $c \in k$.

b) Conversely, for any $c \in k$, the polynomial $x^p - x - c$ either has one root in k, in which case, all its roots are in k, or it is irreducible. Moreover, in the latter case, $k(\alpha)$ is Galois and cyclic of degree p over k.

20. Let k be a field of characteristic $\neq$2, 3. Let $f(x)$ be an irreducible cubic polynomial over k and $g(x)$ be an irreducible quadratic polynomial over k of the form $g(x) = x^2 - c$ for $c \in k$. Assume that $[k(\Delta^{1/2}) : k] = 2$ and $k(\Delta^{1/2}) \neq k(c^{1/2})$ where Δ denotes the discriminant of f . Let α be a root of $f(x)$ and let β be a root of $g(x)$ in an algebraic closure. Prove

a) The splitting field of $f(x)g(x)$ over k has degree 12 over k.

b) Let $\gamma = \alpha + \beta$. Then $[k(\gamma) : k] = 6$.

Topology Questions
Day 3—August 29, 2013

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21. Let $A \subset X$. Let C be a connected subspace of X that intersects both A and $X - A$. Prove that C intersects the boundary of A, BdA.

22. Let $Y \subset X$ and assume that both X and Y are connected. Suppose that $X - Y = A \cup B$ with $\bar{A} \cap B = \phi = A \cap \bar{B}$. Prove $Y \cup A$ is connected.

23. Prove that if Y is compact, then the projection $\bar{v}_1 : X \times Y \to X$ maps closed sets to closed sets.

24. Let $\{X_\alpha\}$ be an indexed family of non-empty spaces. Suppose $\prod_\alpha X_\alpha$ is locally compact. Prove that each X_α is locally compact and X_α is compact for all but finitely many values of α.

25. Prove that a closed countable subset of a complete metric space has an isolated point.

Geometry Questions
Day 3—August 29, 2013

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26. A form β is closed if $d\beta = 0$. The form β is exact if there exists a form γ such that $\beta = d\gamma$.

In R^3 let $\alpha = xdy \wedge dz$ be a two form.

(a) Determine whether α is exact. Either exhibit a one form γ such that $\alpha = d\gamma$ or show that no such form exists.

(b) Given the submanifold $M = \{(x, y, z) \mid x - y^2 - z^2 = 1\}$ let β be the form α restricted to the submanifold M. Determine whether β is exact by either exhibiting a one form whose exterior derivative is β or proving that no such form exists.

27. A derivation δ_p at the point p on a manifold M is a linear functional on $C^\infty(M)$ which also satisfies $\delta_p(fg) = \delta_p(f)g(p) + f(p)\delta_p(g)$.

(a) Prove that δ_p is identically 0 on all constant functions.

(b) List the properties of a C^∞ bump function.

(c) Assuming the existence of $C^\infty(R^n)$ bump functions prove the following: If $f(x) = g(x)$ for all x such that $|x - p| < \epsilon$ for some positive ϵ then $\delta_p f = \delta_p g$.

(d) Show that the space of derivations δ_p on R^n is isomorphic to the space of directional derivatives D_{v_p} at p where v_p is a vector in the tangent space at p.

28. Consider the set G defined as the set of invertible matrices which satisfy

$$G = \{A \mid A^T J A = J\} \text{ where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It is easily shown that G forms a group called the group of symplectic matrices.

(a) Prove that G is a smooth submanifold of R^4

b) Describe the elements in the tangent space to G at the identity element e and determine the dimension of the submanifold.

(c) If Y is the left invariant vector field on G whose value at $g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is

$$Y_g = \begin{pmatrix} 2 & 1 - 2a \\ 0 & -2 \end{pmatrix}$$

what is the value of Y at e? $Y_e =$.

(d) What is the dimension of the space of left invariant vector fields on G?

29. For which values of a is $M_a = \{(x, y) \mid y^2 = x(x-1)(x-a)\}$ a submanifold? What is the dimension of the submanifold?

30. Let M, N and X be smooth manifolds. Let $\pi\colon M \to N$ be a surjective submersion and let $F\colon M \to X$ be a smooth function.

If $G\colon N \to X$ satisfies $G(y) = F(x)$ for all x such that $\pi(x) = y$ and $F = G \circ \pi$ then G is a smooth function from $N \to X$.

Prove this statement. (Use the properties of a surjection or the "constant rank" theorem.)