UR Math Prelims, December 2012

Real Analysis

Instructions: Please provide complete proofs, and justifications for answers and calculations. You may cite a result from class or the text (fully stating what you are using) *unless it is, or is clearly equivalent to, the result you are being asked to prove.* Some notation, terminology and background: *Measurable* refers to Lebesgue measurable sets and functions. $\mathcal{M}(\mathbb{R}^n)$ denotes the measurable subsets of $\mathbb{R}^n$. You may assume that $\mathcal{M}(\mathbb{R}^n)$ is known to contain the closed sets, and also that the continuous functions of compact support form a dense subspace of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

1. (i) Define what it means for $f : [a, b] \to \mathbb{R}$ to be of *bounded variation.*

(ii) If f, g are of bounded variation on $[a, b]$, prove that fg is of bounded variation.

(iii) If also $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, prove that $g \circ f$ is of bounded variation.

2. (i) Define what it means for a *set* $E \subset \mathbb{R}^n$ to be measurable.

(ii) Prove that $\mathcal{M}(\mathbb{R}^n)$ is a σ-algebra.

3. (i) Define what it means for a *function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ to be measurable.

(ii) Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable.

(iii) Find a nontrivial condition on g so that if $f : \mathbb{R}^n \to \mathbb{R}$, $f \in L^1(\mathbb{R}^n)$, then $g \circ f \in L^1(\mathbb{R}^n)$. [Here, *nontrivial* will mean that it is satisfied by all g in some infinite-dimensional function space.]

4. Suppose $f : [0,1] \to \overline{\mathbb{R}}$ is a measurable function such that $f \in L^p[0,1]$, $\forall\, 1 \leq p < \infty$.

(i) Show that $||f||_{L^p}$ is an increasing function of p.

(ii) If, for some $1 \leq p < q < \infty$, it happens that $||f||_{L^p} = ||f||_{L^q}$, what can you conclude?

(iii) Find $\lim_{p \to +\infty} ||f||_{L^p}$.

5. (i) If $f, g \in L^p(\mathbb{R}^n)$, some $1 \leq p < \infty$, show that $||f(\,\cdot - t\,) - f(\,\cdot\,)||_{L^p} \to 0$ as $t \to 0$ in $\mathbb{R}^n$.

(ii) Prove that the convolution, defined by $f * g(x) = \int f(x-y)g(y)dy$, is defined everywhere if $f, g \in L^2(\mathbb{R}^n)$ and that $f * g$ is a bounded, continuous function $\to 0$ as $|x| \to \infty$ in $\mathbb{R}^n$.

Topology

1. Prove that the product of two connected spaces is connected.

2. Prove that the product of two path-connected spaces is path connected.

3. Let A be an open cover of a compact metric space (X, d). Prove there is a number λ such that for each subset $B \subset X$ with diameter $B < \lambda$, there is an element $\mathcal{U}$ in $\mathcal{A}$ such that $B \subset \mathcal{U}$.

4. Let X be a topological space, not necessarily Hausdorff. X is said to be *locally compact* if each point in X is contained in an open set with compact closure. Assume that X is locally compact and Hausforff. Prove X has a basis of open sets with compact closures. If you use the one-point compactification of X, state explicitly which of its properties you use in your proof.

5. Let (X, d) be a metric space. Prove that the set of points where X is locally connected is the countable intersection of open subsets of Z.

Algebra

1a) Show that A_4 has no subgroup of order 6, and explain why this implies that the "converse" of Lagrange's theorem is false.

b) Find the commutator subgroups of S_4 and A_4 and of S_n and A_n if $n \geq 5$, and completely justify your answers. *(Hint: you can do this without computing any commutators.)*

2) Assume G acts on a finite set S in such a way so that there is only one orbit. Let $s \in S$, and let G_s be the isotropy subgroup of s (sometimes called the stabilizer of s).

a) Give a formula for $|S|$, the cardinality of S, and prove that your answer is correct.

b) If $s' = gs$ for some $g \in G$ what is the relation between G_s and $G_{s'}$?

c) Suppose $\{g \in G | gx = x \, \forall x \in S\} = \{e\}$. Show if N is any normal subgroup of G that is contained in G_s, then $N = \{e\}$.

3) Let p, q, r be distinct primes such that$p > q > r$, and G be a group of order pqr.

a) Prove that either a Sylow p- subgroup or a Sylow $q-$subgroup must be normal.

b) Use your answer to a to prove the Sylow $p-$subgroup must be normal.

c) Are there any non-abelian groups of order pqr? Justify your answer.

4) Let A be a commutative ring with $1 \neq 0$, and S a multiplicatively closed subset of A containing 1 and not containing 0. For any ideal I in A, define the ideal $S^{-1}(I)$ in $S^{-1}(A)$ to equal $\left\{\frac{x}{s} \mid x \in I, s \in S\right\}$, and note $S^{-1}(I)$ equals the smallest ideal in $S^{-1}(A)$ containing I.

a) Prove that any ideal in $S^{-1}(A)$ is of the form $S^{-1}(I)$ for some ideal I in A.

b) Let $f : A \longrightarrow S^{-1}(A)$ be the canonical map, and let p be a prime ideal in A not meeting S. Prove that the contraction of $S^{-1}(p)$ (or equivalently the inverse image of$S^{-1}(p)$ under$f$) equals p.

c) Must the conclusion of part b still be true if one drops the hypothesis that p is a prime ideal (in other words if one lets p be an arbitrary ideal not meeting S)? Prove it or give a counterexample.

5) Let A be a commutative ring with $1 \neq 0$. Recall that the Jacobson radical $\mathcal{J}_A$ of A is defined to equal the intersection of all maximal ideals in A, and the nilradical $\mathcal{N}_A$ of A (whch is defined to be the set of all nilpotent elements of A) can be shown to equal the intersection of all prime ideals in A. For each of the following, tell whether the statement $\mathcal{J}_A = \mathcal{N}_A$ is always, sometimes, or never true for rings of that type. Briefly justify your answers. If the answer is "sometimes", give an example to illustrate each possibility.

For purposes of the prelims, you must have at least 4 of these completely correct to count as having the problem correct.

a) An Artin ring

b) A Noetherian ring that is not Artin

c) A local integral domain that is not a field

d) An integral domain with only a finite number of maximal ideals that is not a field

e) A ring that has only a finite number of prime ideals in which it is not the case that all primes are maximal.