Real Analysis

Problem 1. If $F : \mathbb{R} \to \mathbb{R}$ is a monotone function, show that

$$||F||_{TV([a,b])} = F(b) - F(a)$$

for any interval $[a, b]$, and that F has bounded variation on $\mathbb{R}$ if and only if it is bounded. Here

$$||F||_{TV([a,b])} = \sup_{a \le x_0 \le x_1 \le \cdots \le x_n \le b} \sum_i |F(x_i) - F(x_{i+1})|.$$

Problem 2. Compute the area of a regular 2^n-gon, $n = 2, 3, \dots$, inscribed in the unit circle. Rigorously prove that this quantity converges to π as $n \to \infty$.

For the purposes of this problem, please use the definition which states that π equals the ratio of a circle's circumference to its diameter. Do not use the definition that π is the ratio of the area of a circle to the square of its radius.

Problem 3. Suppose that $f_n, f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.. Prove that

$$\lim_{n\to\infty} ||f_n - f||_{L^p(\mathbb{R}^n)} = 0 \text{ if and only if } \lim_{n\to\infty} ||f_n||_{L^p(\mathbb{R}^n)} = ||f||_{L^p(\mathbb{R}^n)}.$$

Problem 4. Compute the three dimensional Lebesgue measure of the following subset of $\mathbb{R}^2 \times [0, \pi)$:

$$\left\{(x, y, \theta) \in \mathbb{R}^2 \times [0, \pi) : x^2 + y^2 \le 1; \theta \in [0, \pi); (x + \cos\theta)^2 + (y + \sin\theta)^2 \le 1\right\}.$$

For a bit of extra credit, give a simple geometric interpretation of the quantity you just computed.

Problem 5. i) State the Fubini theorem.

ii) Define $f : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Compute

$$\int_0^1 \left(\int_0^1 f(x, y)dx\right) dy \text{ and } \int_0^1 \left(\int_0^1 f(x, y)dy\right) dx.$$

Why doesn't the result contradict the Fubini theorem?

Complex Analysis

1. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z^3(z^2+9)}$$

in certain domains, and specify those domains.

2. Let $D \subset \mathbb{R}^2$ be a domain, such that ∂D (i.e., the frontier of D) is a positively oriented simple contour. Prove that the area of D is given by

$$\frac{1}{2i} \int_{\partial D} \overline{z}\, dz.$$

3. Let f be an analytic function inside and on a positively oriented simple contour γ, also having no zeros on γ. Prove that if f has n zeros z_k $(1 \le k \le n)$ inside C, where each z_k has multiplicity m_k, then

$$\int_\gamma \frac{z\, f'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

4. Compute, using the residue theorem and including complete justifications,

$$\int_0^\infty \frac{\ln x}{(1+x)^3}\, dx.$$

5. Let $D = \{z \in \mathbb{C} |\, Re\, z > 0\}$ and $f : D \to D$ a holomorphic function. Prove that

$$|f'(z)| \le \frac{Re\, f(z)}{Re\, z}, \qquad (\forall) z \in D,$$

where $Re\, z$ is the real part of the complex number z, i.e., $Re\,(x+iy) = x$.

Algebra I

Problems

1. Part a) Describe all finite groups with only two conjugacy classes.
   Part b) Describe all finite groups with only three conjugacy classes.

2. (a) Let $\mathbb{F}_3 := \mathbb{Z}/3\mathbb{Z}$. Find all values of $a \in \mathbb{F}_3$ such that the quotient ring

$$\mathbb{F}_3[x]/(x^3 + x^2 + ax + 1)$$

   is a field. Justify your answer.

   (b) Let F be a field and E an integral domain. Suppose F is a subring of E. Prove that if the dimension of E as a vector space over F is finite, then E is a field.

3. Prove that all groups of order 12 are solvable. (Note: you cannot simply state Burnside's theorem.)

4. All PID's are UFD's. Prove the first part of this assertion. That is, if R is a PID and $r \in R$, then there exists irreducible elements $p_1, \ldots, p_n \in R$ such that $r = p_1 \cdots p_n$.

5. Prove that normality of fields is not transitive. That is, give an explicit example of field extensions $F \le K \le E$ such that E/K and K/F are normal, but E/F is not. (Make sure to justify any statements you make about your example.)

Algebra II

1. (a) Let $F \le K \le E$ be field extensions. Suppose K/F is Galois, and E/K is Galois. Prove or give a counterexample that E/F is Galois.
   (b) Let $f \in \mathbb{Q}[x]$ be an odd degree polynomial with cyclic Galois group. Prove that all the roots of f are real.

2. Let ζ_n be a primitive n-th root of unity. Suppose n is odd and composite. Prove that the Galois group $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ of the cyclotomic extension over $\mathbb{Q}$ is not cyclic.

3. Let A be an integral domain with quotient field K, and let L be a finite separable extension of K. Let B be the set of elements of L that are integral over A. Prove that L is the fraction field of B.

4. If $E = \mathbb{Q}(\alpha)$ where α is a root of the cubic $x^3 - 3x + 1$, find the norm and trace of α^2 over E.

5. Suppose
$$0 \longrightarrow N_1 \longrightarrow M \longrightarrow N_2 \longrightarrow 0$$
is an exact sequence of R-modules. Prove that if N_1 and N_2 are finitely generated then M is finitely generated. Give a counterexample to the converse; explicitly describe the ring R and modules involved in your example.

Topology

Problem 1.

1. Let X be a topological space. Let $\Delta_X = \{(x, x) : x \in X\}$ be the diagonal in the product space $X \times X$. Then **prove** that X is Hausdorrf **iff** the diagonal Δ_X is closed in $X \times X$.

2. Prove that the topological space X is Hausdorrf **iff** every net $(s_n)_{n \in D}$ in X converges to at most one point.

3. Let C and D be disjoint compact subsets of a Hausdorrf space X. Then prove that there exist disjoint open sbsets U and V of X such that $C \subset U$ and $D \subset V$.

Problem 2. Recall that a topological space X is *connected* **iff** whenever U and V are disjoint open subsets of X such that $X = U \cup V$ then either $U = \emptyset$ or $V = \emptyset$. Recall also that a subset A of X is said to be *connected* **iff** it is connected for the relative topology from X – i.e., **iff** whenever U and V are open subsets of X such that $U \cap V \cap A = \emptyset$ and such that $A \subset U \cup V$ then either $A \cap U = \emptyset$ or $A \cap V = \emptyset$.

1. If A is a connected subset of the topological space X, then **prove** that the subset $\overline{A}$ of X is connected.

2. If X and Y are connected topological spaces, then **prove** that $X \times Y$ is connected. Use this to prove that the Cartesian product of finitely many connected spaces is connected.

3. Let $(X_i)_{i \in I}$ be an indexed family of non-empty connected topological spaces. Suppose that we choose an element $t_i \in X_i$ for every $i \in I$. If K is any finite subset of the index set I, then let $X_K = \{(x_i)_{i \in I} : x_i = t_i, \text{ for all } i \notin K\}$. Then **prove** that X_K is homeomorphic to $\prod_{i \in K} X_i$. Use this and what you proved above to **prove** that X_K is connected, for every finite subset K of I.

4. In class, we proved that, if X is a topological space, $x \in X$ and if $\mathfrak{A}$ is a collection of connected subsets of X such that $x \in A$, for all $A \in \mathfrak{A}$, then the subset $\cup_{A \in \mathfrak{A}} A$ of X is connected.

   If $(X_i)_{i \in I}$ and t_i, $i \in I$ are as above, then let $Y = \{(x_i)_{i \in I} : x_i = t_i, \text{ for all but finitely many } i \in I\}$. Then, using your results above, **prove** that the subset Y of $\prod_{i \in I} X_i$ is connected.

5. **Prove** that Y is dense in $\prod_{i \in I} X_i$.

6. Using your results above, **prove** that $\prod_{i \in I} X_i$ is connected.

Problem 3. Let X be a set and let $(s_n)_{n\in D}$ be a net in the set X. Then recall that a *subnet* of the net $(s_n)_{n\in D}$ is a net $(t_m)_{m\in E}$ together with an order-preserving function $T : E \longrightarrow D$ such that the subset $T(E)$ of D is *cofinal* (i.e, such that $n \in D$ implies $\exists\ m \in E$ such that $n \leq T(m)$), and such that $s_{T(m)} = t_m$, for all $m \in E$. Recall also that, if X is a topological space and $x \in X$, then we say that the net $(s_n)_{n\in D}$ *converges* to x **iff** the net $(s_n)_{n\in D}$ is eventually in every neighborhood of x in X (i.e., **iff** U a neighborhood of x in X implies there exists $n \in D$ such that $m \geq n$ in D implies $s_m \in U$.) We say that x *is a cluster point* of the net $(s_n)_{n\in D}$ **iff** the net $(s_n)_{n\in D}$ is frequently in every neighborhood of x (i.e., **iff** U a neighborhood of x in X and $n \in D$ implies there exists $m \geq n$ in D such that $s_m \in U$.)

1. Let $(s_n)_{n\in D}$ be a net in the topological space X, and let x be a cluster point of x. Then **prove** that there exists a subnet $(t_m)_{m\in E}$ of the net $(s_n)_{n\in D}$ that converges to x.

2. Let $(s_n)_{n\in D}$ be a net in the topological space X, and suppose that we have a subnet $(t_m)_{m\in E}$ of the net $(s_n)_{n\in D}$ that converges to x. Then **prove** that x is a cluster point of the net $(s_n)_{n\in D}$.

Problem 4. If $(X, \mathfrak{U})$ is a uniform space, then in class we proved that the set $\mathfrak{B}$ of all closed entourages in $\mathfrak{U}$ is a base for the uniformity.

1. $(X, \mathfrak{U})$ and $\mathfrak{B}$ as above, let $C = \cap_{U \in \mathfrak{B}} U$. Then **prove** that, for every neighborhood V of the diagonal Δ_X in $X \times X$, we have that $B \subset V$.

2. Suppose also that X is compact as topological space. Then so is $X \times X$, and, since a closed subset of a compact topological space is compact, it follows that all the entourages in $\mathfrak{B}$ are compact. If V is any neighborhood of Δ_X in X, then by the last part of this problem, you know also that $\cap_{U \in \mathfrak{B}} U \subset V$. Use this to **prove** that $V \in \mathfrak{U}$.

3. Using what you've shown above, **prove** that, if $(X, \mathfrak{U})$ is any *compact* uniform space, then the uniformity $\mathfrak{U}$ of X is necessarily equal to the set of all neighborhoods of the diagonal Δ_X in $X \times X$.

4. Using the above, **prove** that, if $f : (X, \mathfrak{U}) \longrightarrow (Y, \mathfrak{V})$ is a function of uniform spaces, and if f is continuous, and if X is compact as topological space, then the function $f : (X, \mathfrak{U}) \longrightarrow (Y, \mathfrak{V})$ is uniformly continuous.

Problem 5.

1. Let X be an arbitrary topological space. Let x be such that $x \notin X$. Let X^* be the set $X \cup \{x\}$. Then let τ be the set of all open subsets of X together with X^*. Then show that τ is a topology on X^* that is compact and such that X is an open subspace of X^*.

   NOTE: The topology that you've constructed above is almost *never* the same as the topology that makes X^* into what is usually called the *one-point* compactification of X.

2. For every closed compact subset C of X, let $U_C = (X \setminus C) \cup \{x\}$. Let μ be the topology of X. Then let $\rho = \mu \cup \{U_C$: such that C is a closed compact subset of $X\}$. Then *prove* that ρ is a topology on X^*.

   NOTE: The topology ρ that you've just constructed is the *one-point* compactification of X.

3. **Prove** that the one-point topology on X^* is the finest topology on X^* such that X^* is compact and such that X is an open subspace of X^*.

Geometry

1. Consider the subset of $\mathbb{R}^3$ which is the graph $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(\vec{x}) = |\vec{x}|$.
a) Can this graph be given a differentiable structure?
b) Can this graph be a differentiable submanifold of $\mathbb{R}^3$ with its standard differentiable structure?
c) Could this set be the image of a differentiable function?

2. Consider $M_k = \{(x, y, z) \in \mathbb{R}^3 | z^2 + xy = k\}$.
a) For which values of k is M_k a smooth manifold ?
b) For $k = 0$, is M_0 connected ? Is M_0 compact ?

3. An exact form α is a differential form that is the exterior derivative of another differential form β, i.e. $\alpha = d\beta$.
a) Determine whether the two-form $\alpha = zdx \wedge dy$ is exact in $\mathbb{R}^3$.
b) Let M represents an embedded submanifold of $\mathbb{R}^3$ given by
$M = \{(x, y, z) \in \mathbb{R}^3 | z - x^2 - y^2 = 1\}$. Determine whether the restriction of α to M is exact.

4. (15 pts) Let M be a differentiable manifold. A one-parameter group of transformations, ϕ, on M, is a differentiable map from $M \times \mathbb{R}$ onto M such that $\phi(x, 0) = x$ and $\phi(\phi(x, t), s) = \phi(x, t + s)$ for all $x \in M$, $t, s \in \mathbb{R}$. Show that the family of maps $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $\phi_t(x, y) = (e^{at}x, e^{bt}y)$, with $a, b \in \mathbb{R}$ form one parameter group of transformations.

5. **a)** Let M be a compact connected orientable n-manifold without boundary. Let $\beta \in \Omega^{n-1}(M)$ be a $(n-1)$-differential form. Show that there exists a point $p \in M$ such that $d\beta(p) = 0$.

b) Prove that there is no embedding $f : S^1 \to \mathbb{R}$ where S^1 is the unit circle.