PRELIMS, DEC. 2011

Real Analysis

Problem #1: Suppose that $f \in L^1(\mathbb{R}^n)$.

i) Prove that

$$\lim_{h \to 0} \int_{\mathbb{R}^n} |f(x) - f_h(x)| \, dx = 0,$$

where

$$f_h(x) = f(x - h).$$

ii) Suppose that $f \in L^1(\mathbb{R}^n)$. Is it true that given $\epsilon > 0$ there exists $h > 0$ such that

$$|\{(x : |f(x) - f_h(x)| > \epsilon\}| < \epsilon?$$

Problem #2: i) Let $E_1, E_2, \ldots, E_N \subset [0, 1]$, $N \geq 2$, be measurable sets such that every $x \in [0, 1]$

is contained in at least n of these sets. Prove that

$$|E_1| + |E_2| + \cdots + |E_N| \geq n.$$

ii) Under the assumptions of part i), prove that

$$\sum_{j=1}^{N} \sum_{j'=1}^{N} |E_j \cap E_{j'}| \geq n^2.$$

Problem #3: i) State Jensen's inequality and use it to prove that

$$a_1 a_2 \ldots a_N \leq \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j}$$

where $a_j \geq 0$, $p_j \geq 1$ and $\sum_{j=1}^{N} \frac{1}{p_j} = 1$.

ii) Use part i) to conclude that a cube in $\mathbb{R}^3$ has the smallest sum of lengths of edges connected

to each vertex among all three-dimensional boxes with the same volume.

Problem #4: Let $f(x)$ be a measurable and bounded function on $[0,1]$ satisfying

$$f(x+y) = f(x) + f(y) \ \forall \ x, y \text{ and } f(1) = 1.$$

Prove that $f(x) = x$.

Hint: First prove that f is continuous by using integrals and the assumption above in a mildly
clever way. Then show that $f(x) = x$ on rational numbers and extend by continuity.

Problem #5: Let $\epsilon > 0$ and define $A = \cup_{j=1}^{\infty}(x_j - \epsilon, x_j + \epsilon)$, $x_j \in \mathbb{R}$. Suppose that $A \cap [0,1]$ is
dense in $[0,1]$. Then $|A \cap [0,1]| = 1$.

Hint: Use the Lebesgue differentiation theorem.

Algebra I

(1) (a) Please derive the class equation:

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)].$$

(b) Prove that the center of a (nontrivial) p-group is nontrivial.

(2) Let M be a $\mathbb{Z}$-module generated by m_1, m_2, and m_3. Suppose the generators satisfy the relations

$$\begin{aligned} 8m_1 + 4m_2 + 8m_3 &= 0 \\ 4m_1 + 8m_2 + 4m_3 &= 0. \end{aligned}$$

Please decompose M into cyclic $\mathbb{Z}$-modules. (Hint: one approach is to use Smith Normal Form.)

(3) Let D be a UFD and F its quotient field. Assume D does not equal F. Prove that for every positive integer n there is a field extension E of F with $[E : F] = n$.

(4) Let $N \trianglelefteq G$ and suppose that N has order 5, and that $G/N \cong S_3$. Prove that G has order 30. Further, prove G has a normal subgroup of order 15, and that G has three subgroups of order 10 that are not normal.

(5) (a) Construct a field with 8 elements.

(b) Demonstrate the multiplication table of any three elements not including 0 or 1.

Topology

Problem 1. Recall that a topological space is *connected* **iff** whenever A and B are disjoint open subsets of X such that $X = A \cup B$ then either $A = \emptyset$ or $B = \emptyset$.

(1) If X is topological space and $x \in X$, and if A_i is a connected subset containing x for all $i \in I$, then prove that $\cup_{i \in I} A_i$ is connected.
(2) Let X be a topological space and let $x \in X$. Then prove that there exists a maximum connected subset of X containing x (i.e, a connected subset containing x that contains every other connected subset that contains x). **NOTE:** The maximum connected subset of X that contains x is called the *connected component* of x in X.
(3) Show that a topological space has the property that $[x \in X \implies$ $\exists$ a neighborhood of x that is connected$]$ **iff** the connected components of X are all open.
(4) Using your results in the last problem, show that if X is a topological space, then the following two conditions are equivalent:
   (a) For every $x \in X$, $\{$ connected neighborhoods of $x\}$ is a base for the neighborhood system at x.
   (b) For every open subset U of X, we have that all of the connected components of U are open.

   NOTE: Such a topological space X is called *locally connected*.

Problem 2.

(1) Let X_i be a topological space, for all $i \in I$, and let C_i be a closed subset of X_i, all $i \in I$. Then prove that the subset $\prod_{i \in I} C_i$ of $\prod_{i \in I} X_i$ is a closed subset.
(2) Recall that a topological space X is *regular* **iff** whenever $x \in X$ and C is a closed subset of X such that $x \notin C$, we have that there exist disjoint open subsets U and V of X such that $x \in U$ and $C \subset V$. In class, we proved that X is regular *iff* $\{$closed neighborhoods of $x\}$ is a base for the neighborhood system at x, for all $x \in X$. Use this latter fact to prove that, if X_i is a regular topological space for all $i \in I$, then $\prod_{i \in I} X_i$ is regular.

Problem 3. Recall that if $(s_n)_{n \in D}$is a net in a topological space X, then a point x in X is a *cluster point* of the the net $(s_n)_{n \in D}$ **iff** the net $(s_n)_{n \in D}$ is frequently in every neighborhood of x.

Let $(s_n)_{n\in D}$ be a net in the topological space X. For each $n \in D$, let $A_n = \{s_i : i \geq n\}$. Then **prove** that {cluster points of the net $(s_n)_{n\in D}\} = \cap_{n\in D} A_n$.

Problem 4. Recall that a topological space X is *compact* **iff** every open covering has a finite subcovering; equivalently **iff** every family of closed subsets with the finite intersection property has a non-empty intersection.

Prove that a topological space X is compact **iff** every net in X has a cluster point.

HINT: In proving $\Longleftarrow$, note that if $\mathfrak{C}$ is collection of closed subsets of X with the finite intersection property, and if $\mathfrak{D}$ is the set of all finite intersections of elements of $\mathfrak{C}$, then $\mathfrak{D}$ is a directed set ordered by reverse inclusion. Build an appropriate net $(s_n)_{n\in\mathfrak{D}}$ indexed by $\mathfrak{D}$, and use the fact that it must have a cluster point. [You can use the Axiom of Choice.]

Problem 5. Let $(X, \mathfrak{U})$ be a uniform space. Let $(s_n)_{n\in D}$ be a net in X.

(1) What does it mean for the the net $(s_n)_{n\in D}$ to be a *Cauchy* net?
(2) If $(s_n)_{n\in D}$ is a Cauchy net in the uniform space $(X, \mathfrak{U})$ and x is a cluster point of the net $(s_n)_{n\in D}$, then **prove** that the net $(s_n)_{n\in D}$ converges to x.