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ORIGINAL ARTICLE

A comment on Baron and Ferejohn (1989): The open rule equilibrium and coalition formation

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Abstract I present a more general characterization of the symmetric stationary subgame perfect equilibrium to the Baron and Ferejohn (1989) open rule divide-the-dollar game. Specifically, I show that an amender can follow several different randomization strategies when deciding whom to make offers to, and each can be sustained as a distinct equilibrium with slightly different payoffs. The result demonstrates that, when building coalitions in bargaining settings where an offer is already on the table, those with the worst offers need not be the ones "bought up" first.

Keywords Legislative bargaining · Coalition formation

Introduction

The Baron and Ferejohn (1989) divide-the-dollar game has been utilized in dozens of applications.[1] A legislature must decide how to allocate a dollar among *n* identical districts. Each legislator wishes to maximize the share of the dollar his district receives. Two variants of the model are presented in the original paper: a closed rule model, in which offers made are not subject to amendment; and an open rule model, in which offers are subject to amendment. The focus here is on the open rule model.

In Baron and Ferejohn's Proposition 4, they characterize a symmetric subgame perfect stationary equilibrium to the open rule model. With one minor exception, Baron and Ferejohn are silent about whether the equilibrium is unique.[2] I show that, even within the class of equilibria that are symmetric, subgame perfect, and stationary, the one stated in Proposition

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[1] As of May 2006, it has nearly 200 cites in the Social Sciences Citation Index.

[2] The exception is the first sentence of the paragraph that precedes Proposition 4, where Baron and Ferejohn state, "The equilibrium for a legislature with a simple open rule is characterized in the following proposition." A more accurate sentence would replace "The" with "An."

4 is not unique. Specifically, an amender can follow several different randomization strategies when deciding whom to make offers to, and each can be sustained as a distinct equilibrium with slightly different payoffs.[3]

While the more general formulation here does not produce significantly different results than those reported by Baron and Ferejohn, it represents a step forward for an undertilized model. The closed rule model is widely used in applied models and substantive work, and it has been the subject of significant theoretical advances (e.g., Snyder, Ting, & Ansolabehere, 2005; Eraslan, 2002). By contrast, as Ansolabehere, Snyder, Strauss, and Ting put it, "We know of no general characterization of the open-rule proposal bargaining model" (2005, 559). I intend for this comment to represent a step toward additional basic research into the open rule model and its properties.

Model

The structure of the game is as follows. A legislature must decide how to divide a dollar; each legislator wants to maximize the share of the dollar awarded to his or her district. In period one, a legislator is selected at random to make an offer consisting of a proposed allocation of the dollar. Then, another legislator is selected at random and can either "move the previous question," at which point the initial proposal comes up for a vote, or propose an amendment to the proposal. If this legislator chooses to move the previous question, then the proposal is voted on by majority rule. If the proposal is successful, the dollar is divided and the game ends. If not, then a new agenda setter is chosen, and the process begins again (with the dollar discounted to account for delay in bargaining). If this legislator chooses not to move the previous question and instead opts to propose an amendment, then the legislature decides between the bill and the amendment, with the winner of that vote becoming the new bill on the table. Then, another player is selected at random to either move the previous question or propose an amendment. This process continues until the previous question is moved and is approved by a majority vote. Discounting occurs whenever the previous question is voted down or an amendment is proposed. The equilibrium concept is symmetric subgame perfect Nash restricted to the consideration of stationary strategies, in which players must take the same actions at every node in which the game is structurally identical. In addition, a legislator who is indifferent between voting for or against a proposal or amendment is assumed to vote for it. To eliminate equilibria that involve legislators voting against legislation that gives them higher utility than the alternative or voting for legislation that gives them lower utility than the alternative, weakly dominated strategies are also ruled out.

Let n denote the size of the legislature, where n is odd, and $m(\delta, n) \geq \frac{n-1}{2}$ denote the number of members offered projects in equilibrium, where $\delta \in (0, 1]$ is a discount factor. Let $\hat{y}^a$ denote the portion of the dollar retained by the proposer. Here is a partial statement of Baron and Ferejohn's equilibrium, correcting a small error.[4] The complete statement of the equilibrium, as well as a proof, can be found on pp. 1196 and 1202–1204 of the original paper. The open rule process is depicted in Baron and Ferejohn's Figure 2.

[3] The equilibria are similar enough that a researcher may select one to use for analysis or for future model-building without altering results dramatically, though the choice should be explicitly stated when solving the model.

[4] The original proposition uses the term $n - 1 - m(\delta, n)$ where $2m(\delta, n) - n + 2$ belongs.

Proposition 1. (Baron and Ferejohn (1989), Proposition 4). *In an n-member, majority-rule legislature governed by a simple open rule and equal probabilities of recognition, a stationary equilibrium is a strategy configuration in which (1) the member recognized first makes a proposal $\hat{y}$ that allocates $\frac{(1-\hat{y}^a)}{m(\delta,n)}$ to $m(\delta, n)$ other members, where $\frac{n-1}{2} \leq m(\delta, n) \leq (n-1)$, and allocates $\hat{y}^a$ to him- or herself; (2) if one of those $m(\delta, n)$ members is recognized next, this member moves the previous question, the proposal is approved by the proposer of $\hat{y}$ plus the $m(\delta, n)$ members, and the legislature adjourns; and (3) if one of the $(n-1-m(\delta, n))$ other members is recognized next, that member offers an amendment that allocates $\hat{y}^a$ to him- or herself and $\frac{(1-\hat{y}^a)}{m(\delta,n)}$ to $m(\delta, n)$ other members excluding the member that made the previous proposal. Those members include those offered zero in the proposal on the floor plus $(2m(\delta, n) - n + 2)$ others chosen randomly from those offered $\frac{(1-\hat{y}^a)}{m(\delta,n)}$ in that proposal. This amendment defeats the prior motion and becomes the motion on the floor. Then repeat strategies 2 and 3.*

The intuition for this result centers on the power of the amender and the agenda setter. The agenda setter wants to make an offer to members of his coalition such that, if selected to amend the offer, any of them will prefer to leave the proposal unchanged. The size of the coalition he builds will depend on the tradeoff between the probability that a legislator receiving no project is selected to make an amendment (and chooses to do so) and the share of the dollar the agenda setter can keep. Minimum winning coalitions typically result in this model for sufficiently high n and δ.

The aspect of the equilibrium that is offered without proof in Baron and Ferejohn (1989) is strategy (3). Specifically, no reason is given for why the amender who has been offered no project in the bill first selects players who received zero in the initial bill and then randomizes among all other legislators who received offers in the initial bill. The typical way of determining who is bought "first" is to ascertain who is the cheapest to buy (i.e., who would settle for less today rather than see the game continue) and then buy those legislators first. However, all legislators view the game identically going forward, and so they are all equally expensive (except the agenda setter). Because of this, any randomization among legislators can be sustained as an equilibrium; by stationarity, the same randomization must be used in each round.

In this paper I prove this more general case: that any randomization among the two types of players – those with projects on the table (except for the agenda setter), and those without (except for the amender) – can be sustained as a stationary subgame perfect Nash equilibrium, provided that the randomization results in a coalition of size m. This proof subsumes the procedure in Baron and Ferejohn, a simple randomization procedure among all legislators except the agenda setter, and any other randomization procedure (for instance, choosing x members of a size-m coalition from among players with offers on the table, and the remainder, $x - m$, from among those players with nothing on the table). Equilibrium parameter values are in the proof.

Let p equal the probability that a player with an offer on the table when an amender is selected is made an offer by the amender. Let q be the probability that a player with no offer on the table is made an offer. In the equilibrium presented by Baron and Ferejohn, $q = 1$ and $p = \frac{2m-n+2}{m}$, because players without offers must always be bought, and then the remainder of the coalition is bought from among those with offers on the table. For a simple randomization procedure, where all legislators (except the proposer) have an equal chance of receiving a project, $q = p = \frac{m}{n-2}$. For any strategy of randomizing between the two types of players, a p and q can be found that characterize the randomization. The values of p and q

are restricted to instances where they correspond to a randomization, given m and n.[5] While very similar, the equilibria implied by different randomizations are not payoff-equivalent; payoffs vary with p and q.

Proposition 2. *In an n-member, majority-rule legislature governed by a simple open rule and equal probabilities of recognition, the following characterizes any symmetric stationary equilibrium to the game: (1) the member recognized first makes a proposal $\hat{y}$ that allocates $\frac{(1-\hat{y}^a)}{m(\delta,n)}$ to $m(\delta, n)$ other members, where $\frac{n-1}{2} \leq m(\delta, n) \leq (n-1)$, and allocates $\hat{y}^a$ to him- or herself; (2) if one of those $m(\delta, n)$ members is recognized next, this member moves the previous question, the proposal is approved by the proposer of $\hat{y}$ plus the $m(\delta, n)$ members, and the legislature adjourns; and (3) if one of the $(n-1-m(\delta, n))$ other members is recognized next, that member offers an amendment that allocates $\hat{y}^a$ to him- or herself and $\frac{(1-\hat{y}^a)}{m(\delta,n)}$ to $m(\delta, n)$ other members selected randomly from those with projects and those without, such that p is the probability that a player with a project is offered a project by the amender, and q is the probability that a player without a project is offered a project. This excludes the member that made the previous proposal. This amendment defeats the prior motion and becomes the motion on the floor. Then repeat strategies 2 and 3.*

Proof: Denote 1 as the player who makes the first proposal; in equilibrium he offers $\frac{1-\hat{y}^a}{m}$ to $m(n, \delta)$ other legislators and keeps the balance for himself. The size of the coalition and the share of the dollar allocated to others are determined in equilibrium, with the share of the dollar and continuation values V being a function of coalition size. Denote $V_b^m(\hat{y}^c)$ as the continuation value of the game for player b given that an offer has just been made by player c for a coalition of size m. The m superscript will be omitted in the remainder of the proof for notational convenience.

A legislator receiving an offer votes for the bill instead of making an amendment if $\frac{1-\hat{y}^a}{m} \geq \delta V_1(\hat{y}^1)$, since by stationarity the individual who can make an offer is in the same position as the first player. If an amender is selected who did not receive an offer from the proposer – call this player j – he will not move the previous question and will instead offer an amendment. This will put the proposer in the same position as the player not receiving an offer. This occurs with probability $(1 - \frac{m}{n-1})$. With probability $\frac{m}{n-1}$, the amender is in the agenda setter's coalition and will accept the offer. Therefore, we can write $V_1(\hat{y}^1) = (\frac{m}{n-1})\hat{y}^a + (1 - \frac{m}{n-1})\delta V_1(\hat{y}^j)$. Next, by stationarity, $V_1(\hat{y}^j) = V_j(\hat{y}^1)$. This gives,

$$V_1(\hat{y}^1) = \left(\frac{m}{n-1}\right)\hat{y}^a + \left(1 - \frac{m}{n-1}\right)\delta V_j(\hat{y}^1). \tag{1}$$

To calculate $V_j(\hat{y}^1)$, note that there are three possible outcomes. First, with probability $\frac{m}{n-1}$, a player receiving an offer, denoted k, will be selected and move the previous question, giving j no share of the dollar. With probability $\frac{1}{n-1}$, player j will be selected to make an amendment, which will put him in the same position as the proposer, giving him $\delta V_1(\hat{y}^1)$, with δ accounting for the delay implied by an amendment being made. With probability $(1 - \frac{m+1}{n-1})$, another player receiving no offer, denoted l, is selected to make an offer. The

[5] For instance, suppose $m = 3$ and $n = 7$. Then $p = q = 1$ is not a legitimate randomization, since it implies that all legislators except the agenda setter and the amender are offered projects (i.e., $m = 5$).

continuation value in this case is $\delta V_j(\hat{y}^l)$. This implies

$$V_j(\hat{y}^1) = \left(\frac{m}{n-1}\right)0 + \left(\frac{1}{n-1}\right)\delta V_1(\hat{y}^1) + \left(1 - \frac{m+1}{n-1}\right)\delta V_j(\hat{y}^l). \tag{2}$$

Next, we need to calculate $V_j(\hat{y}^l)$. With probability $1-q$, the amender does not select j to be in the coalition, at which point j is in the same position he was in when 1 made no offer to him, which implies a continuation value of $V_j(\hat{y}^1)$. Next, with probability q, j will receive an offer. In this case, there is a $(\frac{m}{n-1})$ probability that one of the m members receiving an offer is selected next and moves the previous question, giving j the equilibrium allocation $\delta V_1(\hat{y}^1)$. There is a $(1-\frac{m}{n-1})$ probability that a legislator not receiving a project is selected, in which case this player makes a new proposal. This puts j in the same position as k was when l was the proposer, giving him $\delta V_k(\hat{y}^l)$. Putting all of this together gives

$$\begin{aligned} V_j(\hat{y}^l) = {} & q\left[\left(\frac{m}{n-1}\right)\delta V_1(\hat{y}^1) + \left(1 - \frac{m}{n-1}\right)\delta V_k(\hat{y}^l)\right] \\ & + (1-q)\left[\left(\frac{m}{n-1}\right)0 + \left(\frac{1}{n-1}\right)\delta V_1(\hat{y}^1) + \left(1 - \frac{m+1}{n-1}\right)\delta V_j(\hat{y}^l)\right]. \end{aligned} \tag{3}$$

Next, we need to calculate $V_k(\hat{y}^l)$. The logic is identical to the construction of $V_j(\hat{y}^l)$, except the q and $(1-q)$ terms are replaced by different probabilities, p and $(1-p)$. This gives

$$\begin{aligned} V_k(\hat{y}^l) = {} & p\left[\left(\frac{m}{n-1}\right)\delta V_1(\hat{y}^1) + \left(1 - \frac{m}{n-1}\right)\delta V_k(\hat{y}^l)\right] \\ & + (1-p)\left[\left(\frac{m}{n-1}\right)0 + \left(\frac{1}{n-1}\right)\delta V_1(\hat{y}^1) + \left(1 - \frac{m+1}{n-1}\right)\delta V_j(\hat{y}^l)\right]. \end{aligned} \tag{4}$$

Rearranging terms and simplifying in (4) gives

$$V_k(\hat{y}^l) = \frac{(pm + (1-p))\delta V_1(\hat{y}^1)}{(n-1) - \delta p(n-1-m)} + \frac{(n-2-m)(1-p)\delta V_j(\hat{y}^l)}{(n-1) - \delta p(n-1-m)}. \tag{5}$$

Substituting (5) into (3) gives

$$V_j(\hat{y}^l) = \alpha^{-1}\beta\delta V_1(\hat{y}^1), \text{ where} \tag{6}$$

$$\alpha = 1 - \frac{q(n-1-m)\delta^2(n-2-m)(1-p)}{(n-1-(1-q)\delta(n-2-m))(n-1-\delta p(n-1-m))}, \tag{7}$$

and

$$\begin{aligned} \beta = {} & \frac{1-q+qm}{(n-1-(1-q)\delta(n-2-m))} \\ & + \frac{q(n-1-m)\delta(pm+(1-p))}{(n-1-(1-q)\delta(n-2-m))(n-1-\delta p(n-1-m))}. \end{aligned} \tag{8}$$

Substituting (6) into Equation (2) gives

$$V_j(\hat{y}^1) = \left(\frac{1}{n-1}\right)\delta V_1(\hat{y}^1) + \left(1 - \frac{m+1}{n-1}\right)\delta\alpha^{-1}\beta V_1(\hat{y}^1). \tag{9}$$

Substituting (9) into (1) gives

$$V_1(\hat{y}^1) = \left(\frac{m}{n-1}\right)\hat{y}^a + \delta\left(1 - \frac{m}{n-1}\right)\left[\left(\frac{\delta}{n-1}\right) + \frac{\beta}{\alpha}\left(\frac{\delta^2(n-2-m)}{(n-1)}\right)\right]V_1(\hat{y}^1). \tag{10}$$

Next, recall that $\hat{y}^a \geq 1 - m(\delta, n)\delta V_1(\hat{y}^1)$. Because the proposer will never want to give away more of the dollar than necessary, $\hat{y}^a = 1 - m(\delta, n)\delta V_1(\hat{y}^1)$. Substituting this into Equation (10) and rearranging gives

$$V_1(\hat{y}^1) = \frac{m}{n-1}\left(1 + \frac{\delta m^2}{n-1} - \delta^2\left(\frac{n-1-m}{(n-1)^2}\right)\left(1 + \frac{\beta}{\alpha}\delta(n-2-m)\right)\right)^{-1}. \tag{11}$$

The size of the coalition $m(\delta, n)$ can be determined as follows: To check whether a coalition of size m_0 forms an equilibrium, solve $V_1(\hat{y}^1)$ for m_0 and note the continuation values. Then, check whether the agenda setter wants to deviate by maximizing the following with respect to m and $\hat{y}^a$ (restricting m to be an integer in $[\frac{n-1}{2}, n-1]$):

$$\max \left(\frac{m}{n-1}\right)\hat{y}^a + \left(1 - \frac{m}{n-1}\right)\delta V_1(\hat{y}^j)$$

$$\text{s.t.} \frac{1-\hat{y}^a}{m} \geq \delta V_1(\hat{y}^1).$$

If the maximization gives a different solution, then m_0 is not an equilibrium, since the continuation values are not consistent with utility-maximizing behavior on the part of the agenda setter. If the equilibrium values are unchanged, then $m_0 = m^*$. Do this for all possible m_0. □

Example

Consider a five person legislature with $\delta = 1$. In the equilibrium presented by Baron and Ferejohn, we can calculate the agenda setter's share of the dollar as 43.6 cents, with two other randomly-selected legislators receiving 28.2 cents each. If we move to the equilibrium where $p = q$, then the agenda setter's share of the dollar increases to 44 cents. It turns out that the equilibrium with the highest expected payoff for the agenda setter occurs when the amender keeps the coalition the same and merely replaces himself with the agenda setter (i.e., $p = 1, q = 0$). In this equilibrium, the agenda setter's share of the dollar goes to 45.5 cents. The intuition is as follows: When constructing the initial offer, the agenda setter knows that the coalition he builds is "privileged" in that those coalition members will continue to receive offers from amenders. Because of this, the downside to rejecting an offer outright is larger, meaning that a slightly smaller portion of the dollar can be offered. For the opposite reason, the Baron and Ferejohn equilibrium represents the worst equilibrium for the agenda setter.

Coalition members have to be offered a premium because their position in the coalition is not secure. This example demonstrates that the equilibrium results will not be payoff-equivalent as p and q change.

Conclusion

This comment established that the original open rule equilibrium presented by Baron and Ferejohn is not unique. In fact, at the amendment stage, any mixing of legislators with and without offers, such that the mixing produces the correct coalition size, can be established as an equilibrium. This result illustrates that building the "cheapest" coalition in a bargaining model need not mean selecting players who have the worst offers on the table first. In addition, these equilibria are not payoff-equivalent. The Baron and Ferejohn closed rule model continues to serve as a foundation for studying legislative bargaining. I hope that this comment begins a trend toward more basic research into the open rule model, as well.

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References

Ansolabehere, S., Snyder, J.M. Jr., Strauss, A.B., & Ting, M.M. (2005). Voting weights and formateur advantages in the formation of coalition governments. *American Journal of Political Science, 49*, 550–563.

Baron, D.P., & Ferejohn, J.A. (1989). Bargaining in legislatures. *American Political Science Review, 83*, 1181–1206.

Eraslan, H. (2002). Uniqueness of stationary equilibrium payoffs in the Baron-Ferejohn model. *Journal of Economic Theory, 103*, 11–30.

Snyder, Jr., James M., Ting, M.M., & Ansolabehere, S. (2005). Legislative bargaining under weighted voting. *American Economic Review, 95*, 981–1004.