Election Forensics: Latent Dimensions of Election Frauds and Strategic Voting*

Walter R. Mebane, Jr.†

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†Professor, Department of Political Science and Department of Statistics, University of Michigan, Haven Hall, Ann Arbor, MI 48109-1045 (E-mail: wmebane@umich.edu).

Abstract

Many statistical methods that use low-level election vote count data to detect election frauds have the limitation that they have a hard time distinguishing distortions in vote counts that stem from voters' strategic behavior from distortions that originate from election frauds. Identifying latent components that underlie various election forensics statistics (e.g., digit tests, Klimek-related estimates) can help show the extent to which the statistics measure fraudulent as opposed to strategic behavior. We use polling station voting data and postelection complaint data from a recent German election to illustrate the latent variable methods. Geographic ambiguity about the locations at which some complaints occur motivates embedding a geographic mixture structure in the latent variable model. "Fraud" probabilities that stem from an idea that elections without fraud are unimodal in fact respond strongly to strategic behavior.

1 Introduction

A key challenge for election forensics—the field devoted to using statistical methods to try to determine whether the results of an election are accurate—is to be able to tell whether patterns in election results that may appear anomalous in statistical estimates and tests are the results of election frauds or of strategic behavior. Many methods for trying to detect election frauds have been proposed (e.g. Myagkov, Ordeshook and Shaikin 2009; Levin, Cohn, Ordeshook and Alvarez 2009; Shikano and Mack 2009; Mebane 2010; Breunig and Goerres 2011; Pericchi and Torres 2011; Cantu and Saiegh 2011; Deckert, Myagkov and Ordeshook 2011; Beber and Scacco 2012). Both strategic behavior and frauds can cause the patterns such methods look for (Mebane 2013, 2014). Here we examine whether a finite mixture model implementation of the key ideas in Klimek, Yegorov, Hanel and Thurner (2012) can help avoid possible confusion and clearly identify circumstances in which frauds occur. The finite mixture model we use corrects the one introduced by Mebane, Egami, Klaver and Wall (2014).

We use data from the 2005 federal election in Germany to help assess the efficacy of the modified Klimek et al. (2012) model. Mebane et al. (2014) find in data from German and Mexican elections that the probability of each of the two kinds of frauds that the Klimek et al. (2012) model uses—Incremental Fraud and Extreme Fraud—is associated with measures of strategic voting. But Mebane et al. (2014) also show that the fraud probability estimates relate to the occurrence of complaints about the elections filed after the elections. While the complaints do not necessarily concern what might be considered genuine frauds, they have face validity as imperfect measures of potentially serious irregularities. Ziblatt (2009) uses such complaints to measure the occurrence of election frauds in Germany during the years 1871–1912, and Breunig and Goerres (2011) make a similar usage with regard to more recent elections.

We use a latent variable model to assess how both measures of strategic voting and the distribution of different types of complaints across districts relate to the distribution of the fraud probability parameter estimates. The latent variable model features generalized linear associations between manifest variables and a set of latent variables that various manifest variables have in common. For some manifest variables the relationship to common latent variables is simply linear, while for the binary variables that measure the incidence of complaints the relationship goes through a probit model and for the fraud probabilities the relationship goes through a Dirichlet link. We use a corrected version of the finite mixture variant of the Klimek et al. (2012) model developed by Mebane et al. (2014) to estimate the fraud probabilities used in the latent variable model.

If fraud-detecting variables are genuinely ambiguous in the sense that they are triggered both by frauds and by strategic behavior, then common latent variables should connect the supposed fraud measures to the complaint and strategic variables. The generalized linear functional forms are restrictive, in that the variables may be associated in essentially nonlinear ways. It is reasonable, however, to begin with such models.

2 Model

The current analysis relies on two kinds of models: a finite mixture likelihood implementation of the key ideas in the simulation model introduced by Klimek et al. (2012); and a latent variable model that uses Monte Carlo Markov Chain (MCMC) Bayesian methods to relate a diverse set of manifest variables to a smaller set of common latent variables using an exploratory factor analysis model specification. Estimation proceeds in two stages. Fraud probabilities estimated using the finite mixture model are treated as manifest variable data when estimating the latent variable model. We describe the two models in turn.

2.1 Klimek-like model

In the Klimek et al. (2012) model the baseline assumption is that votes in an election with no fraud are produced through the interaction of processes whose effects can be summarized by two Normal distributions: there is one distribution for turnout proportions and another, independent distribution for the proportion of votes going to the "winner" (that is, the party with the most votes). Klimek et al. (2012) condition on the number of eligible voters. Klimek et al. (2012) assume that election fraud means that votes are added to the votes for the winner. Some votes are transferred to the winner from the opposition, and some are transferred from nonvoters. The two kinds of election fraud refer to how many of the opposition and nonvoters votes are shifted: with "incremental fraud" moderate proportions of the votes are shifted; with "extreme fraud" almost all of the votes are shifted. Klimek et al. (2012) have parameters that specify the probability that each unit experiences each type of election fraud: f_{i} is the probability of incremental fraud and f_{e} is the probability of extreme fraud. Other parameters fully describe bimodal and trimodal distributions that the model characterizes as being consequences of election frauds.

Core idea of the Klimek et al. (2012) model: Klimek et al. (2012) describe a simulation protocol that includes three kinds of votes: votes without fraud; votes with "incremental fraud"; and votes with "extreme fraud." With no fraud the distribution of votes, given the number of eligible voters, is a product Normal distribution. The fraud conditions correspond to differing proportions of votes going to the winning party that should have gone to other parties or should not have been counted as votes at all. With incremental fraud a small proportion x of what should have been nonvotes are counted for the winner while a proportion x^{α}, $\alpha > 0$, of votes that should have gone to opposition instead go to the winner. With extreme fraud a large proportion $1 - y$ of the nonvotes are counted for the winner and a proportion $(1 - y)^{\alpha}$ of genuine opposition votes instead go to the winner.

A formal description of the key steps in the simulation protocol that relate to model specification follows. Features of the simulation protocal that relate to parameter selection are not described.[1] Nor do we describe the features of the protocol used to assign each observation to one of the three categories no fraud, incremental fraud or extreme fraud. We describe the parts of the protocol that are the point of departure for the finite mixture likelihood.

Observed data come from n electoral units (e.g., polling stations), and the number of eligible voters in each unit is N_i. Votes for parties are observed as the count of votes for the winning party, denoted W_i, and the sum of votes cast for all other parties (the "opposition"), denoted O_i. The number of observed nonvotes ("abstentions") is $A_i = N_i - W_i - O_i$. The observed number of valid votes is $V_i = N_i - A_i$.

Using $\mathcal{N}(\mu, \sigma)$ to denote a normally distributed simulated random variable with mean μ and standard deviation σ, the protocol involves two kinds of fraud and is applied to each observation i. For each electoral unit $i = 1, \ldots, n$, for some $\alpha \geq 0$ do:

1. Sample turnout: $\tau_i \sim \mathcal{N}(\tau, \sigma_\tau)$.

2. Sample the winner's vote proportion: $\nu_i \sim \mathcal{N}(\nu, \sigma_\nu)$.

3. (Incremental fraud) With probability f_{i} sample the proportion of nonvotes that are turned into votes: $x_i \sim |\mathcal{N}(0, \theta)|$ subject to $0 < x_i < 1$. Set the number of votes for the winning party as $W_i = N_i (\tau_i \nu_i + x_i (1 - \tau_i) + x_i^\alpha (1 - \nu_i) \tau_i)$, the number of votes for the opposition as $O_i = N_i (1 - x_i^\alpha)(1 - \nu_i) \tau_i$ and the number of nonvoters as $A_i = N_i (1 - x_i)(1 - \tau_i)$.

4. (Extreme fraud) With probability f_{e} sample the proportion of nonvotes that are not turned into votes: $y_i \sim |\mathcal{N}(0, \sigma_x)|$, $\sigma_x = 0.075$, subject to $0 < y_i < 1$. Set the number of votes for the winning party as

[1] See Klimek et al. (2012) for a description and Mebane et al. (2014) for a critique of the simulation method.

$W_i = N_i \left(\tau_i \nu_i + (1 - y_i)(1 - \tau_i) + (1 - y_i)^\alpha (1 - \nu_i) \tau_i\right)$, the number of votes for the opposition as $O_i = N_i \left(1 - (1 - y_i)^\alpha\right)(1 - \nu_i) \tau_i$ and the number of nonvoters as $A_i = N_i y_i (1 - \tau_i)$.

5. (No fraud) With probability $f_0 = 1 - f_{\mathrm{i}} - f_{\mathrm{e}}$, the number of votes for the winning party is $W_i = N_i \tau_i \nu_i$, the number of votes for the opposition is $O_i = N_i \tau_i (1 - \nu_i)$, and the number of nonvoters is $A_i = N_i (1 - \tau_i)$.

The intuition is that, depending on the value of α, incremental fraud involves shifting to the winner some of the votes from the opposition and from nonvoters, while extreme fraud involves shifting to the winner almost all of those votes. Smaller values of α mean that larger fractions of votes are shifted from opposition to winner.

A finite mixture likelihood version of the model of Klimek et al. (2012): As in Mebane et al. (2014), we treat the model of Klimek et al. (2012) from a likelihood point of view. The "no fraud," incremental fraud and extreme fraud cases define three distinct components that can fit together in a finite mixture model. The following corrects the finite mixture model of Mebane et al. (2014). Let $\mathbf{W}$, $\mathbf{O}$, $\mathbf{A}$ and $\mathbf{N}$ be vectors containing, repectively, the n observations of W_i, O_i, A_i and N_i. The finite mixture likelihood is

$$\mathcal{F}(\mathbf{W}, \mathbf{O}, \mathbf{A} \mid \mathbf{N}; \mathbf{\Psi}) = \sum_{j \in \{0,\mathrm{i},\mathrm{e}\}} f_j \prod_{i=1}^{n} g_{jW}(W_i \mid N_i; \mathbf{\Psi}) g_{jA}(A_i \mid N_i; \mathbf{\Psi}), \tag{1}$$

where f_0, f_{i} and f_{e} are probabilities with $f_0 + f_{\mathrm{i}} + f_{\mathrm{e}} = 1$. To adapt the language of Klimek et al. (2012), f_0 is the probability of "no fraud." $g_{jW}(W_i \mid N_i; \mathbf{\Psi})$ and $g_{jA}(A_i \mid N_i; \mathbf{\Psi})$ are conditional densities and scalar parameters are in a vector $\mathbf{\Psi} = (\alpha, \nu, \tau, \sigma_\nu, \sigma_\tau, \theta)'$.

Because $O_i = N_i - A_i - W_i$, the variable O_i given N_i and A_i is just W_i. So the joint density of (W_i, O_i, A_i), given (N_i), is just the conditional density of (W_i, A_i) given N_i. Such is exactly the situation in (1), where no additional term for a conditionl density of O_i appears.

Let $\phi(x, \mu, \sigma)$ denote the Normal density with mean μ and standard deviation σ evaluated at quantile x. Let $\upsilon(x, \sigma) = 2\exp(-x^2/2\sigma^2)/(\sigma\sqrt{2\pi})$ denote the density matching $x \sim |\mathcal{N}(0, \sigma)|$ (a folded Normal density); we use the error function, $\operatorname{erf}\left(1/\sqrt{2\theta}\right) = \int_0^1 \upsilon(x, \theta)dx = 2\int_{-\infty}^{\sqrt{1/\theta}} d\phi(x, 0, 1) - 1$, to rescale the density for the censoring implied by $0 < x < 1$.

Densities for A_i: With no fraud the density of A_i is

$$g_{0A}(A_i \mid N_i; \mathbf{\Psi}) = \frac{\phi(A_i, N_i(1-\tau), N_i\sigma_\tau)}{\int_0^1 \phi(\tau_i, \tau, \sigma_\tau)\, d\tau_i} \tag{2}$$

Because incremental fraud implies $A_i = N_i(1-x_i)(1-\tau_i)$, with incremental fraud the density of A_i is

$$g_{\mathrm{i}A}(A_i \mid N_i; \mathbf{\Psi}) = \frac{\displaystyle\int_0^1 \phi\left(\frac{A_i}{1-x_i}, N_i(1-\tau), N_i\sigma_\tau\right) \frac{\upsilon(x_i, \theta)}{1-x_i}\, dx_i}{\operatorname{erf}\left(1/\sqrt{2\theta}\right) \int_0^1 \phi(\tau_i, \tau, \sigma_\tau)} . \tag{3}$$

Because extreme fraud implies $A_i = N_i y_i (1-\tau_i)$, with extreme fraud the density of A_i is

$$g_{\mathrm{e}A}(A_i \mid N_i; \mathbf{\Psi}) = \frac{\displaystyle\int_0^1 \phi\left(\frac{A_i}{y_i}, N_i(1-\tau), N_i\sigma_\tau\right) \frac{\upsilon(y_i, \sigma_x)}{y_i}\, dy_i}{\operatorname{erf}\left(1/\sqrt{2\sigma_x}\right) \int_0^1 \phi(\tau_i, \tau, \sigma_\tau)} . \tag{4}$$

Densities for W_i: The density of W_i with no fraud is

$$g_{0W}(W_i \mid N_i; \mathbf{\Psi}) = \frac{\phi\left(\frac{W_i}{(1-A_i/N_i)}, N_i\nu, N_i\sigma_\nu\right) \frac{\phi((1-A_i/N_i), \tau, \sigma_\tau)}{(1-A_i/N_i)}}{\int_0^1 \phi(\nu_i, \nu, \sigma_\nu)\, d\nu_i \int_0^1 \phi(\tau_i, \tau, \sigma_\tau)\, d\tau_i} \tag{5}$$

With incremental fraud the density of W_i is

$$g_{\mathrm{i}W}(W_i \mid N_i; \mathbf{\Psi}) =$$
$$\frac{\int_0^1 \int_0^1 \phi\left(\frac{W_i}{\tau_i\,(1-x_i^\alpha)}, \mu_{\mathrm{i}}, N_i\sigma_\nu\right) \frac{(1-x_i)^{\frac{1}{\alpha}-1}\upsilon(x_i,\theta)\phi(\tau_i,\tau,\sigma_\tau)}{\alpha\tau_i\,(1-x_i^\alpha)}\, dx_i\, d\tau_i}{\mathrm{erf}\left(1/\sqrt{2\theta}\right)\left(\int_0^1 \phi(\nu_i,\nu,\sigma_\nu)\,d\nu_i\right)\left(\int_0^1 \phi(\tau_i,\tau,\sigma_\tau)\,d\tau_i\right)} \tag{6a}$$

$$\mu_{\mathrm{i}} = N_i\left(\nu + \frac{x_i(1-\tau_i)}{\tau_i(1-x_i^\alpha)} + \frac{x_i^\alpha}{1-x_i^\alpha}\right) \tag{6b}$$

With extreme fraud the density of W_i is

$$g_{\mathrm{e}W}(W_i \mid N_i; \mathbf{\Psi}) =$$
$$\frac{\int_0^1 \int_0^1 \phi\left(\frac{W_i}{\tau_i\,(1-(1-y_i)^\alpha)}, \mu_{\mathrm{e}}, N_i\sigma_\nu\right) \frac{(1-y_i)^{\frac{1}{\alpha}-1}\upsilon(y_i,\sigma_x)\phi(\tau_i,\tau,\sigma_\tau)}{\alpha\tau_i\,(1-(1-y_i)^\alpha)}\, dy_i\, d\tau_i}{\mathrm{erf}\left(1/\sqrt{2\sigma_x}\right)\left(\int_0^1 \phi(\nu_i,\nu,\sigma_\nu)\,d\nu_i\right)\left(\int_0^1 \phi(\tau_i,\tau,\sigma_\tau)\,d\tau_i\right)} \tag{7a}$$

$$\mu_{\mathrm{e}} = N_i\left(\nu + \frac{(1-y_i)(1-\tau_i)}{\tau_i(1-(1-y_i)^\alpha)} + \frac{(1-y_i)^\alpha}{1-(1-y_i)^\alpha}\right) \tag{7b}$$

In (6a), $f(x_i) = z = (1-x_i^\alpha)$ implies $f^{-1}(z) = (1-z)^{1/\alpha}$, so $|\partial f^{-1}(z)/\partial z| = (1-x_i)^{\frac{1}{\alpha}-1}/\alpha$ gives the Jacobian to use to express the density of $(1-x_i^\alpha)$ in terms of $\upsilon(x_i,\theta)$. In (7a), $(1-y_i)^{\frac{1}{\alpha}-1}/\alpha$ arises similarly. Note that we fix $\sigma_x = 0.075$.

Restrictions on parameters: As in Mebane et al. (2014), to match an aspect of the Klimek et al. (2012) specification,[2] we specify upper bounds for parameters ν, τ, σ_ν and σ_τ: $\nu \le \mathrm{median}(W_i/V_i)$, $\tau \le \mathrm{median}(V_i/N_i)$, and using the sets

[2]Klimek et al. (2012) restrict τ and ν to be less than the first local maxima of the empirical distributions respectively of V_i/N_i and W_i/V_i. They similarly constrain σ_τ and σ_ν.

$\mathfrak{W}_u = \{W_i/V_i : W_i/V_i \leq \text{median}(W_i/V_i)\}$ and $\mathfrak{V}_u = \{V_i/N_i : V_i/N_i \leq \text{median}(V_i/N_i)\}$,

$$\sigma_\nu \leq 2\left(\frac{1}{|\mathfrak{W}_u|}\sum_{W_i/V_i\in\mathfrak{W}_u}\left(W_i/V_i - \frac{\sum_{W_i/V_i\in\mathfrak{W}_u} W_i/V_i}{|\mathfrak{W}_u|}\right)^2\right)^{1/2},$$

$$\sigma_\tau \leq 2\left(\frac{1}{|\mathfrak{V}_u|}\sum_{V_i/N_i\in\mathfrak{V}_u}\left(V_i/N_i - \frac{\sum_{V_i/N_i\in\mathfrak{V}_u} V_i/N_i}{|\mathfrak{V}_u|}\right)^2\right)^{1/2},$$

where $|\mathfrak{W}_u|$ is the cardinality of $\mathfrak{W}_u$.

Estimation: As in Mebane et al. (2014), to estimate the parameters of the finite mixture model we use an EM algorithm (Dempster, Laird and Rubin 1977; Wu 1983; McLachlan and Peel 2000; McLachlan and Krishnan 2008) with random starting values, using GENOUD (Mebane and Sekhon 2011) to execute the maximization steps in the EM algorithm. The algorithm includes a thresholding technique to handle instances in which frauds appear not to occur: if estimates $\hat{f}_{\text{i}}$ or $\hat{f}_{\text{e}}$ fall below 10^{-9} at any point in the algorithm, then the corresponding component is dropped from the likelihood and the referent probability is set to zero. When using GENOUD we suppress all use of BFGS. As described in Mebane et al. (2014), the functional form of the model that includes a folded-Normal variable x raised to a real exponent (x^α), when both the variance of x and the exponent need to be estimated, creates a situation in which Newton-Raphson and similar hill-climbing optimization algorithms fail (Hager and Bain 1970)).

2.2 Latent Variable model

We use a latent variable model to study how variation in district-specific fraud probability parameter estimates $\hat{f}_{\text{i}i}$ and $\hat{f}_{\text{e}i}$ (with $i = 1, \ldots, 299$ indexing districts)—estimates based on the plurality rule votes (*Erststimmen*) at the polling stations in each single-member district—relates to variation across districts in the occurrence of postelection complaints and to the measures of voters' strategic behavior.

There are K binary manifest variables that measure complaints, y_{ki}, $k = 1, \ldots, K$, $i = 1, \ldots, 299$. Each of the complaint variables relates to an unobserved continuous variable x_k that is itself related to one or more common latent variables ξ_ℓ through equations of the following form,

$$x_{ki} = c_k + \sum_{\ell=1}^{J} \lambda_{k\ell} \xi_{\ell i}\,, \quad k = 1, \ldots, K+2\,, \tag{8}$$

where the values of intercept c_k and of factor loading $\lambda_{k\ell}$ that are not constant[3] have Normal distributions and the ξ_ℓ are multivariate Normal with mean $\boldsymbol{\gamma} = (\gamma_i, \ldots, \gamma_J)'$ and precision matrix $\boldsymbol{\Upsilon}$.[4] Usually the district associated with a complaint is known with certainty, in which case the manifest variables y_k take the (probit) index variable form

$$\text{Prob}(y_{ki} = 0) = \int_{-\infty}^{0} \phi(x_{ki}, \psi_k)\, df \tag{9a}$$

$$\text{Prob}(y_{ki} = 1) = \int_{0}^{\infty} \phi(x_{ki}, \psi_k)\, df \tag{9b}$$

where $\phi(x, \psi)$ is the Normal density with mean x and precision ψ.[5]

But sometimes there is uncertainty about the assignment of a complaint to a district which leads to uncertainty about whether the observation y_{ki} should be $y_{ki} = 0$ or $y_{ki} = 1$. In this case we mix over the two possible values. Let v_1 and v_2 be the number of registered voters in the portion of each of the two districts for which the district assignment is

[3]See the discussion on page 11.

[4]Adapting specifications given by Lee (2007), the prior for each mean γ_ℓ is Normal, and the prior for $\boldsymbol{\Upsilon}$ is Wishart. Further details about the prior specifications are in the Model Appendix.

[5]The precisions ψ_k have gamma priors.

uncertain.[6] Generate $r_i \in \{0,1\}$ using probabilities that have Dirichlet priors $\mathcal{D}([v_1, v_2])$ by[7]

$$\pi_i \sim \mathcal{D}([v_1, v_2])$$
$$L_i \sim \text{dcat}(\pi_i)$$
$$r_i = L_i - 1$$

With this prior π_i has mean $\left(\frac{v_1}{v_1+v_2}, \frac{v_2}{v_1+v_2}\right)$. Assuming the probabilities in π_i refer to the events in the order $(y_{ki} = 1,\ y_{ki} = 0)$, let

$$\text{Prob}(y_{ki}) = r_i \int_{-\infty}^{0} \phi(x_{ki}, \psi_k)\, df + (1 - r_i) \int_{0}^{\infty} \phi(x_{ki}, \psi_k)\, df\ . \tag{10}$$

We generate π_i—separately for each pair of districts—because we are uncertain about the chances that any ambiguously locatable complaint should be associated with one of the two districts to which it could relate. The components of each π_i range over the unit interval and each r_i switches between the values zero and one as the MCMC algorithm proceeds.

The S manifest variables that relate to strategic behavior connect to the common latent variables straightforwardly. Denote these variables by x_{ki}, $k = K+4+1, \ldots, K+4+S$, and define

$$x_{ki} = c_k + \sum_{\ell=1}^{J} \lambda_{k\ell}\xi_{\ell i} + \epsilon_{ki}\ , \quad k = K+4+1, \ldots, K+4+S\ , \tag{11}$$

where ϵ_{ki} is Normal with mean zero.[8]

For the manifest variables $\hat{f}_{\text{i}i}$, $\hat{f}_{\text{e}i}$ and $\hat{f}_{0i} = 1 - \hat{f}_{\text{i}i} - \hat{f}_{\text{e}i}$ we use M_i, the number of polling stations in district i, to specify a Dirichlet likelihood for the probability vector

[6] Details about how complaints are assigned to districts and about how v_1 and v_2 are measured are in the District Association paragraph of the Data Appendix in Mebane and Klaver (2015).

[7] The dcat() function in the following specification returns positive integer values for L_i in $\{1,2\}$ (Lunn, Jackson, Best, Thomas and Spiegelhalter 2013, 352).

[8] The precisions of these variables' distributions have gamma priors.

$(\hat{f}_{\mathrm{i}i}, \hat{f}_{\mathrm{e}i}, \hat{f}_{0i})$ that depends on the common latent variables.[9] The loglikelihood is

$$\mathfrak{D}(\hat{f}_{\mathrm{i}i}, \hat{f}_{\mathrm{e}i}, \hat{f}_{0i}) = \log\Gamma\left(\sum_{j\in\{0,\mathrm{i},\mathrm{e}\}} \zeta_{ji}\right) - \sum_{j\in\{0,\mathrm{i},\mathrm{e}\}} \log\Gamma(\zeta_{ji}) + \sum_{j\in\{0,\mathrm{i},\mathrm{e}\}} (\zeta_{ji}-1)\log(f_j)$$

$$\zeta_{0i} = M_i \frac{1}{1+\exp(x_{\mathrm{i}i})+\exp(x_{\mathrm{e}i})}$$

$$\zeta_{ji} = M_i \frac{\exp(x_{ji})}{1+\exp(x_{\mathrm{i}i})+\exp(x_{\mathrm{e}i})}\,, \quad j \in \{\mathrm{i},\mathrm{e}\}$$

$$x_{ji} = c_j + \sum_{\ell=1}^{J} \lambda_{j\ell}\xi_{\ell i}\,, \quad j \in \{\mathrm{i},\mathrm{e}\}\,.$$

Evidently $x_{\mathrm{i}i}$ and $x_{\mathrm{e}i}$ have the same form as x_{ki} in (8), summarizing generalized linear connections to the common latent variables.[10] If the loadings $\lambda_{\mathrm{i}\ell}$ and $\lambda_{\mathrm{e}\ell}$ for a latent variable are positive, then an increase in the latent variable tends to go with higher values of f_{i} and f_{e} and lower values of f_0. If $\lambda_{\mathrm{i}\ell}$ is positive and greater than $\lambda_{\mathrm{e}\ell}$, then an increase in the latent variable tends to go with values of $f_{\mathrm{i}i}$ that are high relative to $f_{\mathrm{e}i}$: $f_{\mathrm{i}i}/f_{\mathrm{e}i}$ tends to be larger. Including M_i in the likelihood ensures that potentially important information about varying district sizes—measured by the numbers of polling stations—is not ignored.

The scales and means of the latent variables are set by matching them to particular manifest variables, as follows. The factor loadings $\lambda_{k\ell}$ are fixed equal to zero when the manifest variable is not a measure of latent variable ξ_ℓ. For each latent variable there is one $\lambda_{k\ell}$ that is fixed equal to 1.0 (using a different k for each ℓ), thus establishing a unit of measurement (scale) for the latent variables. Each latent variable has $\lambda_{k\ell}$ fixed equal to 1.0 for one distinct value k; and for that value k, $\lambda_{k\ell'} = 0$ for all $\ell' > \ell$ while the $\lambda_{k\ell}$ are free to take on any value for all other combinations of manifest and latent variables. To set the mean of each latent variable we fix $c_k = 0$ for the values k that have $\lambda_{k\ell}$ fixed equal to 1.0. These lower triangular restrictions on $\lambda_{k\ell}$ and zero restrictions on c_k are sufficient to identify the parameters of an "exploratory" factor analysis model (Anderson and Amemiya

[9]Because in the algorithm to estimate $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ any value of $\hat{f}_{\mathrm{i}i}$ or $\hat{f}_{\mathrm{e}i}$ less than 10^{-9} is truncated to zero, we set 10^{-9} as the smallest possible value of $\hat{f}_{\mathrm{i}i}$ or $\hat{f}_{\mathrm{e}i}$. Therefore $2(10^{-9}) \le \hat{f}_{0i} \le 1-2(10^{-9})$.

[10]In Tables 3–5 loadings $\lambda_{j\ell}$, $j \in \{\mathrm{i},\mathrm{e}\}$, are denoted $\lambda_{K+3\ell}$ and $\lambda_{K+4\ell}$.

1988, 760).

3 Data

The manifest variables we use in our latent variable models are based on three kinds of measures. These are (1) several measures of strategic behavior, (2) fraud probabilities estimated using an improved version of Mebane et al. (2014)'s finite mixture variant of the Klimek et al. (2012) model (see section 2.1) and (3) codes representing postelection complaints filed with a committee of the *Bundestag*. To motivate the latent variable analysis we first show how the "fraud" probability estimates are strongly related to measures of strategic behavior, relationships that pose sharply the measurement challenge for election forensics.

3.1 Measures of Strategic Behavior

Ample evidence exists to demonstrate that strategic voting occurs in the mixed system used in German federal elections (Bawn 1999; Pappi and Thurner 2002; Gschwend 2007). The *Erststimmen*, being plurality votes for a single winner, are affected by "wasted vote" reasoning such as Cox (1994) analyzes. The proportional representation tier votes (*Zweitstimmen*) exhibit "threshold insurance" strategic behavior intended to insure that key smaller parties gain seats in the *Bundestag* (Herrmann and Pappi 2008; Shikano, Herrmann and Thurner 2009).

As measures of strategic behavior we use variables that have been argued to measure effects of strategic voting. Germany's mixed system gives opportunity to observe different distributions of votes being cast in the same district at the same time under both plurality (*Erststimme*) and proportional representation (*Zweitstimme*) rules. The difference between those votes is often used as a measure of strategic behavior (e.g. Cox 1997, 83; Bawn 1999). For each of the five most prominent parties, we use variables defined as the proportion of

Zweitstimmen for a party in a district minus the proportion of *Erststimmen* in the same district. The variables are denoted by `ze-SPD`, `ze-CDUCSU`, `ze-FDP`, `ze-Green` and `ze-Left`.

With the plurality election results alone, measures such as the difference between each of the top two finisher's and the third-place candidate's votes connect to strategic behavior (Cox 1994). We use two margin difference variables: the difference between the first-place candidate's proportion of *Erststimmen* and the third-place proportion ($\mathfrak{M}_{13}$); and the difference between the second- and third-place proportions ($\mathfrak{M}_{23}$).

Pericchi and Torres (2011) argue that deviations in the distribution of the second significant digits of votes signal frauds, but Mebane (2013, 2014) finds that the conditional mean of the second significant digits of votes both for the winning and second-place candidates in a plurality election varies in relation both to strategic behavior and to district imbalances. We use the means of those two candidates' polling station vote counts as measures of strategic behavior. The second-digit mean for the winning candidate is denoted $\hat{j}_{\underline{1}}$ and the second-digit mean for the second-place candidate is denoted $\hat{j}_{\underline{2}}$. Mebane (2013, 2014) finds that patterns in which such means relate to strategic behavior and to district imbalances exhibit complicated nonlinearities, so the second-digit means may not fit well in our setting that imposes generalized linear functional forms. To the extent that the means measure frauds, essential nonlinearities in their relationships to frauds may reasonably be expected as well.

Some of the measures of strategic behavior seem to be related to one another, others not. Scatterplots of the *Zweitstimmen* minus *Erststimmen* variables, in Figures 1, show no apparent relationship across districts between `ze-SPD` and `ze-CDUCSU`, but as `ze-SPD` increases `ze-Green` decreases and as `ze-CDUCSU` increases `ze-FDP` decreases. Whether these patterns reflect the consequences of wasted-vote actions or of threshold-insurance actions is of course not clear from the scatterplots. `ze-Left` shows structure in its relationships to the *Zweitstimmen* minus *Erststimmen* differences for other parties that is not easy to summarize.

*** Figure 1 about here ***

The margin and second-digit mean variables do not appear to relate to one another in any simple fashion. Scatterplots in Figure 2 show that $\mathfrak{M}_{13}$ relates somewhat positively to $\mathfrak{M}_{23}$. In addition the joint distribution of $\mathfrak{M}_{13}$ and $\mathfrak{M}_{23}$ appears to be roughly bimodal, as the equilibrium analysis of Cox (1994) suggests it should be. The second-digit mean variables $\hat{j}_{\underline{1}}$ and $\hat{j}_{\underline{2}}$ appear unrelated both to one another and to the margin variables.

*** Figure 2 about here ***

Relating the *Zweitstimmen* minus *Erststimmen*, margin and second-digit mean variables to one another shows signs of some linear relationship between `ze-SPD` and `ze-CDUCSU`, one the one hand, and $\mathfrak{M}_{13}$ and $\mathfrak{M}_{23}$, on the other (Figure 3). Some relationship between these two kinds of variables is to be expected if wasted-vote actions are part of why the *Zweitstimmen* proportions differ from the *Erststimmen* proportions. But the relationships do not appear to be very strong, and the marginal distribution of `ze-SPD` and `ze-CDUCSU` lacks the bimodality that is apparent in the distribution of $\mathfrak{M}_{13}$ and $\mathfrak{M}_{23}$. This may suggest that two very different kinds of strategies are at work in these elections: one, within each district, that is tied to wasted vote logic; and one, spanning the whole election system, that connects to motives to ensure smaller parties' gain seats in the *Bundestag* and hence to threshold insurance. The second-digit mean measures appear unrelated to the other variables.

*** Figure 3 about here ***

3.2 "Fraud" Probabilities

We use polling station vote count data from the 2005 election (Bundeswahlleiter 2010) to estimate parameters f_{i} and f_{e} for the *Erststimmen* (single-member district plurality rule votes) in each district.[11] For each district i, $i = 1, \ldots, 299$, we obtain estimates $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$.

[11]Polling stations include both in-person (*Urnenwahlbezirke*) and mail (*Briefwahlbezirke*) vote districts.

Figure 4 shows the distribution of the fraud probability estimates across the German districts.[12] Colors correspond to rescaled values $\hat{f}_{\mathrm{i}i}/\max(\hat{f}_{\mathrm{i}i})$ and $\hat{f}_{\mathrm{e}i}/\max(\hat{f}_{\mathrm{e}i})$. The color is red for values equal to 1.0 and blue for 0.0, and intermediate colors correspond to intermediate values. The maximum values of $\hat{f}_{\mathrm{i}i}$ are large: $\max(\hat{f}_{\mathrm{i}i}) = .74$. The maximum values of $\hat{f}_{\mathrm{e}i}$ are very small: $\max(\hat{f}_{\mathrm{e}i}) = ..0047$. A few districts have $\hat{f}_{\mathrm{i}i}$ or $\hat{f}_{\mathrm{e}i}$ values near the maximum values, but most values are zero or very near zero.

*** Figure 4 about here ***

Strategic Entanglements: The estimates of fraud probabilities from the Klimek et al. (2012) model are related to measures of strategic voting. Estimates in particular of $\hat{f}_{\mathrm{i}i}$ relate to measures of strategic voting in 2005. We use two-dimensional nonparametric regressions[13] in which the fraud probability estimates $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ are the outcome variable to illustrate the marginal relationships between fraud probability estimates and strategic voting measures.

Figure 5 illustrates the strong relationships between fraud probability estimates and strategic voting measures in 2005. In all four regressions displayed in the parts of Figure 5, the rug plot along the x-axis shows the distribution of the proportional difference between the *Zweitstimmen* and *Erststimmen* cast for CDU-CSU in each district. That difference is one of the covariates in all of the regressions. The other covariates, with district distributions shown along the y-axes, are (a) the district *Erststimme* winner's proportion of the votes, (b) the margin between the winner and the third-place candidate, (c) the mean of the second significant digits in the winner's vote in each polling station, and (d) the mean of the second significant digits in the second-place candidate's vote in each polling station. The titles above the graphics in each part of the figure report the p-value for a test of the nonparametric regression model compared to the model of no effects.

[12] District (*Wahlkreis*) shapefiles are `Geometrie_Wahlkreise_16DBT_VG1000.shp`, found via `http://www.bundeswahlleiter.de/de/bundestagswahlen/`.

[13] We use the `sm` (Bowman and Azzalini 1997) package of **R** to compute the nonparametric regressions.

*** Figure 5 about here ***

In two of the four regressions in Figure 5, the nonparametric regression relationships are significant. In Figure 5(a) the relationship concerns the tendency for $\hat{f}_{\mathrm{i}i}$ to increase as negative values of the *Zweitstimmen−Erststimmen* difference become even more negative and the winner's margin over the third-place finisher increases.[14] In Figure 5(d) $\hat{f}_{\mathrm{i}i}$ tends to increase as the mean of the second digits of the second-place finisher's polling station vote counts decreases and as the *Zweitstimmen−Erststimmen* difference becomes more negative.

Figure 6 shows the same kinds of regressions except using the *Zweitstimmen−Erststimmen* difference for SPD instead of CDU-CSU. The relationships are significant in all four regressions. In all four regressions the principal variable that is associated with $\hat{f}_{\mathrm{i}i}$ is `ze-SPD`: the vertical contours in the nonparametric regressions indicate that primarily the variation in $\hat{f}_{\mathrm{i}i}$ depends on variation in `ze-SPD`.

*** Figure 6 about here ***

Similar patterns occur in regressions estimated for the FDP, Green and Left parties. In 2005 $\hat{f}_{\mathrm{i}i}$ has significant relationships with several measures of strategic voting. None of the relationships between $\hat{f}_{ei}$ and measures of strategic behavior are significant.

3.3 Postelection Complaints in Germany

The *Bundestag* committee that primarily receives and adjudicates postelection complaints is the *Ausschuss für Wahlprüfung, Immunität und Geschäftsordnung* (Committee for Election Verification, Immunity and Rules of Procedure). In contrast to other election complaint systems—such as that used in Mexico—political parties in Germany do not play a central role in the complaint process. Complainants tend to be individuals who either

[14] In 2005 there are two districts in which the number of *Zweitstimmen* for CDU-CSU is greater than the number of *Erststimmen*: district 176 (Main-Kinzig-Kreis (part), Hesse) and district 236 (Stadt Weiden in der OberPfalz, Bavaria). These districts appear prominently on the right in Figure 5(a), but neither of them is responsible for the significant association between $\hat{f}_{\mathrm{i}i}$ and the various covariates.

directly experienced a failure of election administration or who are otherwise dissatisfied with the prevailing electoral system or political order more generally in Germany.

Two unusual events relating to election administration dominated public perceptions of the federal election in 2005: mismatched *Briefwahl* (mail ballots) in Dortmund and a *Nachwahl* (late election) in Dresden. For more details about these incidents see Mebane and Klaver (2015). Note that the situation in Dresden was controversial due to the strategic advantage held by those voters who would cast their ballots already knowing the outcome in the rest of the country. That knowledge encouraged conservative voters to behave strategically and cast a "coalition vote," i.e., to cast their *Erststimmen* for the CDU and their *Zweitstimmen* for the FDP (Behnke 2008).

Mebane and Klaver (2015) code the complaint documents using a scheme that as much as possible follows the Election Incident Reporting System (EIRS) coding scheme developed for elections in the United States (Verified Voting Foundation 2005; Hall 2005; Johnson 2005). We use these so-called EIRS+ coded data. Nineteen EIRS+ types of complaints occur in 2005. Two additional complaint types refer to either *Briefwahl in Dortmund* or *Nachwahl in Dresden*.

The manifest variable we use in the latent variable model for the complaints of type k in district i, y_{ki}, is a binary indicator for whether at least one complaint of type k occurs for district i. As described in Mebane and Klaver (2015), sometimes ambiguity about the district to which a complaint refers produces an ambiguous count of the number of complaint instances. When the ambiguity is between a count of zero and a positive value, ambiguity is induced in y_{ki}.

The ambiguity also produces variety in the totals of the binary indicators across districts. Table 1 shows two total counts for each type of complaint, the least that can occur and the most. See Mebane and Klaver (2015) for detailed descriptions of the codes' meanings. For most types the two counts are the same, and in a few instances the counts differ by one. Despite the variations, the types of EIRS+ complaints that are the most frequent are the same: Electoral System; Absentee-ballot Related Problem; and Polling Place Problem.

*** Table 1 about here ***

4 Latent Variable Model Estimation Results

The model specification we use features $K = 6$ common latent dimensions. Using more or fewer common latent dimensions produces posterior distributions for the mean (c_k and $\boldsymbol{\gamma}$) and loading ($\lambda_{k\ell}$) parameters that are severely multimodal, featuring parameters with both positive and negative modes. Posteriors in the specifications we use are all unimodal and for the most part symmetric.

We use complaints variables to set the scales for three of the common latent variables (i.e., $\lambda_{k\ell} = 1$ and $c_k = 0$) and we use variables that measure strategic behavior to set the scales for the other three common latent variables. The variables we use to set the scales of each common latent variables are as follows: ξ_1, `AbsenteeB`; ξ_2, `Electoral`; ξ_3, `PollingPl`; ξ_4, `ze-SPD`; ξ_5, `ze-CDUCSU`; ξ_6, $\mathfrak{M}_{13}$. The exploratory factor analysis loading pattern means that while we use the named variables to set the scales of the common latent variables, we are not doing anything to make sure that "complaint" manifest variables and "strategic" manifest variables remain separated. A manifest variable we label as "strategic" may well have a significantly nonzero loading for a common factor that otherwise is associated primarily with "complaint" manifest variables, and vice versa. The interpretability of the common latent variables is not guaranteed by a prespecified factor loading pattern.

The common latent variables are for the most part not correlated with one another, but the nonzero covariances that do appear already raise questions about whether the complaints variables can be sharply distinguished from the strategic variables. None of the posterior means of the covariances between latent variables are exactly zero (see $\boldsymbol{\Phi} = \boldsymbol{\Upsilon}^{-1}$ in Table 2), but most of the covariances have 95% credible intervals that include zero. The three exceptions are that the 95% credible intervals for $\mathbf{\Phi}_{13}$ and $\mathbf{\Phi}_{16}$ contain all positive values and the 95% credible interval for $\mathbf{\Phi}_{14}$ contains all negative values. Expressed as correlations the posterior means of the covariances that are significantly different from zero are $\mathbf{r}_{13} = 0.36$, $\mathbf{r}_{16} = 0.39$ and $\mathbf{r}_{14} = -0.46$. One of the nominal "complaints" common latent variables (ξ_1, whose scale is set by `AbsenteeB`) is correlated with two of the nominally "strategic" common latent variables: positively with ξ_6, whose scale is set by $\mathfrak{M}_{13}$, and negatively with ξ_4, whose scale is set by `ze-SPD`.

*** Table 2 about here ***

The factor loadings show that most of the complaints manifest variables positively depend on the first common latent variable. In Table 3, which shows 95% credible intervals and posterior medians for the loading parameters $\lambda_{k\ell}$ of the first two common latent variables, loadings whose 95% credible intervals contain all positive values are highlighted in green, and loadings whose 95% credible intervals contain all negative values are highlighted in red. The loadings that are fixed to set the scales of the common latent variables are highlighted in gray. Most of the complaints manifest variables have significantly positive loadings on the first common latent variable, ξ_1, including `Dortmund` and `Dresden`. Also loading positively on ξ_1 are `ze-SPD` and $\hat{f}_{\mathrm{i}i}$. $\mathfrak{M}_{13}$ loads negatively on ξ_1. Despite the two nominally strategic manifest variables that load on ξ_1, it seems reasonable to interpret ξ_1 as a variable that connects many of the problems people complained about that disrupted the election. In that light it is notable that $\hat{f}_{\mathrm{i}i}$ relates positively to this common latent variable.

*** Table 3 about here ***

The second common latent variable, ξ_2, whose scale is set by `Electoral`, is measured by a few of the other complaints manifest variables (loadings are in Table 3). Notably `Dresden` loads positively on ξ_2 but `Dortmund` does not. Complaints about the general electoral system connect specifically to complaints about the kinds of unfairness that the *Nachwahl in Dresden* aggravated. None of the strategic manifest variables have a loading on ξ_2 that differs credibly from zero, nor do either of the fraud probabilities load on ξ_2.

The third common latent variable, ξ_3, whose scale is set by `PollingPl`, is measured by a few of the other complaints manifest variables but with a mix of positive and negative loadings (see Table 4). The fact that `Dortmund` loads negatively on ξ_3 perhaps emphasizes that this common latent variable has little directly to do with the kinds on absentee ballot problems that arose in Dortmund, but keep in mind that ξ_1 and ξ_3 are positively correlated. In any case $\hat{f}_{\mathrm{i}i}$ also loads negatively on ξ_3.

*** Table 4 about here ***

The fourth common latent variable, ξ_4, whose scale is set by `ze-SPD`, is not measured by any other strategic manifest variables, as one might expect, but instead has positive loadings for two manifest complaints variables and a negative loading for `Dresden` (see Table 4). Perhaps the positive loading of `Countingo` on ξ_4 is a matter of collocation: perhaps counting controversies happened to arise in places where strategic voting involving the SPD was especially strong. The negative loading for `Dresden` is understandable as a matter of collocation, because the *Nachwahl in Dresden* presented a strategic opportunity for conservative voters who favored the CDU-CSU and the FDP but not for those with preferences more to the left. $\hat{f}_{\mathrm{i}i}$ loads positively on ξ_4. It is striking that both $\hat{f}_{\mathrm{i}i}$ and `ze-SPD` have positive loadings on both ξ_4 and ξ_1, but the correlation between ξ_4 and ξ_1 is negative.

The fifth common latent variable, ξ_5, whose scale is set by `ze-CDUCSU`, is also not measured by any other strategic manifest variables, but instead has nonzero loadings for three manifest complaints variables and a positive loading for `Dresden` (see Table 5). The positive loading for `Dresden` is understandable given that the *Nachwahl in Dresden* offered precisely the kind of strategic opportunity for the CDU-CSU that `ze-CDUCSU` relates to. In that light it's a bit puzzling that `ze-FDP` does not have a nonzero loading on ξ_5. Neither $\hat{f}_{\text{i}i}$ nor $\hat{f}_{\text{e}i}$ has a nonzero loading on ξ_5.

*** Table 5 about here ***

The sixth common latent variable, ξ_6, whose scale is set by $\mathfrak{M}_{13}$, also is devoid of nonzero loadings for other strategic manifest variables, although it does have nonzero loadings of mixed signs for two manifest complaints variables (see Table 5). Why $\mathfrak{M}_{23}$ does not also have a positive loading on ξ_6 is mildly puzzling. $\hat{f}_{\text{i}i}$ has a negative loading on ξ_6.

Overall, strategic dimensions of the data are related to dimensions that connect the various complaints. If the complaints manifest variables are plausibly interpreted as connected to frauds, then frauds are related to strategies in ways that the essentially linear relationships in the model can capture. Covariances occur between complaint-related and strategic common latent variables. Manifest variables of one broad type—complaint-related or strategic—load on common latent variables whose scale is set by a variable of the other broad type. The relations across broad types are not enough to make it impossible to characterize common latent variables as being essentially of one broad type or the other, but boundaries are not sharp.

The fraud probabilies relate to both complaint-related and strategic common latent variables. At least, $\hat{f}_{\text{i}i}$ does. $\hat{f}_{\text{i}i}$ has nonzero loadings on four of the six common latent variables. Two of those loadings are positive and two are negative. $\hat{f}_{\text{i}i}$ has a positive loading on ξ_1, which is the common latent variable on which most of the manifest complaint variables load positively. It also has a positive loading on ξ_4, the common latent variable that relates to the one of the most substantial strategic movements of votes in the election, the movement that relates to shifts in votes for SPD between *Erststimmen* and *Zweitstimmen*. $\hat{f}_{\text{i}i}$ also loads on ξ_6, the latent variable that relates to strategic shifts in votes inside each district. In the 2005 election, $\hat{f}_{\text{i}i}$ is a valid but ambiguous measure. $\hat{f}_{\text{i}i}$ measures both frauds—to the extent that the complaints can be considered as referring to frauds—and strategic behavior.

5 Discussion

If the complaints variables and the latent variables they have in common reflect real irregularities in the administration of the election, do the relationships between those latent variables and $\hat{f}_{\text{i}i}$ and $\hat{f}_{\text{e}i}$ suggest that $\hat{f}_{\text{i}i}$ and $\hat{f}_{\text{e}i}$ merit being described as "fraud" probabilities? In the 2005 German federal election, the loadings for $\hat{f}_{\text{i}i}$ suggest that the probabilities do relate in a meaningful way to irregularities. Whether these irregularities should be called *frauds* is an interpretive matter we will not try to resolve. But $\hat{f}_{\text{i}i}$ also relates to measures of strategic voting. The "fraud" probabilities in the 2005 election are ambiguous. When $\hat{f}_{\text{i}i}$ is large, it is not clear whether the reason is that something went wrong with the voting or that voters themselves moved the votes around by acting strategically.

The parameters of a model inspired by Klimek et al. (2012) that purport to measure the probability of election frauds sometimes also respond to strategic voting. The Klimek et al. (2012) parameters describe particular bimodal and trimodal distributions that are viewed as "unusual." But such distributions might arise as a matter of course, because of voters' strategic behavior. Strategic behavior being essential in politics, perhaps multimodal distributions should not be viewed as being generically odd.

Before reaching any final judgment regarding the general ambiguity of $\hat{f}_{\text{i}i}$, $\hat{f}_{\text{e}i}$ or any other putative measure of election frauds, it is necessary to study how the measures perform in additional election contexts. Findings in Mebane and Klaver (2015) suggest that the patterns in Germany are different even in the next federal election, in 2009. We plan to apply a latent variable model to that election, for which very similar data are available. We also have immediate plans to use a similar latent variable approach with similar data from Mexican elections. Certainly Germany and probably Mexico cannot be considered situations in which elections are affected by notorious and massive frauds. It would be good to find situations where frauds are rampant but auxiliary measures such as the postelection complaints used here are available to estimate latent variable models in which there are multiple low-level indicators of frauds—and of strategic behavior. That frauds are massive does not imply that voters do not act strategically.

To be able to distinguish strategic behavior from election frauds is a key challenge for election forensics. If some voters change how they vote based on strategic considerations, then the distribution of votes differs from what it would have been had the voters not done that. Votes can also change due to fraudulent manipulations. Statistical methods for detecting frauds that focus on identifying "unusual" patterns in votes need to be insensitive to patterns induced by strategic behavior, else users of the methods need to be aware of the potential for confused inferences. Mebane (2013, 2014) argues especially that methods based on vote counts' second significant digits are highly sensitive to strategic behavior. The concern is that all election forensic methods may be sensitive to strategic behavior. Perhaps putative fraud measures are inherently and always ambiguous. The task then is to tease out when they mean one thing and when another.

Our latent variable analysis suggests that estimates of the "fraud" probability parameters $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ do relate meaningfully to the irregularities that provoke postelection complaints to the *Bundestag* in German elections. Whether the incidents that provoke the complaints should be described as "frauds" is a matter of interpretation, but $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ appear to be valid but not perfect measures of those incidents. That is, to be a bit more precise, the bimodal and trimodal distributions that the Klimek et al. (2012) model highlights appear to be valid but not perfect measures of the "frauds" that occur in German federal elections.

5.1 Model Appendix

5.1.1 BUGS Code

The code we use with `OpenBUGS` (Lunn, Spiegelhalter, Thomas and Best 2009; OpenBUGS 2013; Lunn et al. 2013) to run the MCMC algorithms is as follows.

2005:

```
model{
  for(i in 1:N){
    for (j in 1:20) {
      # geo location indicator
      L[i,j] ~ dcat(pi[i,j,1:2])
      # prior for mixture probability vector
      alpha[i,j,1] <- w1[i,j]
      alpha[i,j,2] <- w2[i,j]
      pi[i,j,1:2] ~ ddirch(alpha[i,j,1:2])
    }
    for (j in 21:29) {
      z1[i,j+3] ~ dnorm(mu[i,j], psi[j])
    }

    #measurement equation model
    for(j in 1:20){
      r[i,j] <- L[i,j]-1
      y1[i,j]~dnorm(mu[i,j],psi[j])I(thd[1,z1[i,j]],thd[1,z1[i,j]+1])
      y2[i,j]~dnorm(mu[i,j],psi[j])I(thd[1,z2[i,j]],thd[1,z2[i,j]+1])
      y[i,j] <- r[i,j]*y1[i,j] + (1-r[i,j])*y2[i,j]
      ephat[i,j]<-y[i,j]-mu[i,j]
    }
# "zero trick" Dirichlet likelihoods for fraud probabilities
    for (j in 1:2) {
      p[i,j]<-exp(d[j]+mu[i,j+29])/(1+exp(d[1]+mu[i,30])+exp(d[2]+mu[i,31]))
      theta[i,j] <- p[i,j]*z1[i,23]
      lGtheta[i,j] <- loggam(theta[i,j])
      thp[i,j] <- (theta[i,j]-1)*log(z1[i,j+20])
      ephat[i,j+29]<-z1[i,j+20]-theta[i,j]/(theta[i,1]+theta[i,2]+theta[i,3])
    }
    theta[i,3] <- (1-p[i,1]-p[i,2])*z1[i,23]
    lGtheta[i,3] <- loggam(theta[i,3])
    thp[i,3] <- (theta[i,3]-1)*log(1-z1[i,21]-z1[i,22])
    logL[i] <- loggam(theta[i,1]+theta[i,2]+theta[i,3])-
      (lGtheta[i,1]+lGtheta[i,2]+lGtheta[i,3])+(thp[i,1]+thp[i,2]+thp[i,3])
    Zero[i] <- 0
    Zero[i] ~ dpois(Dphi[i])
    Dphi[i]  <- -logL[i] + 100

    # four factors
    mu[i,1]<- xi[i,1]                             # Absenteeb
    mu[i,2]<- lam[1]*xi[i,1]  + lam[29]*xi[i,2] + lam[56]*xi[i,3] + lam[82]*xi[i,4] +
      lam[107]*xi[i,5] + lam[131]*xi[i,6] +c[1]  # Allegatio
    mu[i,3]<- lam[2]*xi[i,1]  + lam[30]*xi[i,2] + lam[57]*xi[i,3] + lam[83]*xi[i,4] +
      lam[108]*xi[i,5] + lam[132]*xi[i,6] +c[2]  # Ballotrel
    mu[i,4]<- lam[3]*xi[i,1]  + lam[31]*xi[i,2] + lam[58]*xi[i,3] + lam[84]*xi[i,4] +
```

```
  lam[109]*xi[i,5] + lam[133]*xi[i,6] +c[3]  # Countingo
mu[i,5]<- lam[4]*xi[i,1]  + lam[32]*xi[i,2] + lam[59]*xi[i,3] + lam[85]*xi[i,4] +
  lam[110]*xi[i,5] + lam[134]*xi[i,6] +c[4]  # Criminals
mu[i,6]<- lam[5]*xi[i,1]  + lam[33]*xi[i,2] + lam[60]*xi[i,3] + lam[86]*xi[i,4] +
  lam[111]*xi[i,5] + lam[135]*xi[i,6] +c[5]  # Disabilit
mu[i,7]<- lam[6]*xi[i,1]  + xi[i,2]          # Electoral
mu[i,8]<- lam[7]*xi[i,1]  + lam[34]*xi[i,2] + lam[61]*xi[i,3] + lam[87]*xi[i,4] +
  lam[112]*xi[i,5] + lam[136]*xi[i,6] +c[6]  # IDrelated
mu[i,9]<- lam[8]*xi[i,1]  + lam[35]*xi[i,2] + lam[62]*xi[i,3] + lam[88]*xi[i,4] +
  lam[113]*xi[i,5] + lam[137]*xi[i,6] +c[7]  # ImproperC
mu[i,10]<-lam[9]*xi[i,1]  + lam[36]*xi[i,2] + lam[63]*xi[i,3] + lam[89]*xi[i,4] +
  lam[114]*xi[i,5] + lam[138]*xi[i,6] +c[8]  # ImproperD
mu[i,11]<-lam[10]*xi[i,1] + lam[37]*xi[i,2] + lam[64]*xi[i,3] + lam[90]*xi[i,4] +
  lam[115]*xi[i,5] + lam[139]*xi[i,6] +c[9]  # ImproperS
mu[i,12]<-lam[11]*xi[i,1] + lam[38]*xi[i,2] + lam[65]*xi[i,3] + lam[91]*xi[i,4] +
  lam[116]*xi[i,5] + lam[140]*xi[i,6] +c[10] # PartyList
mu[i,13]<-lam[12]*xi[i,1] + lam[39]*xi[i,2] + lam[66]*xi[i,3] + lam[92]*xi[i,4] +
  lam[117]*xi[i,5] + lam[141]*xi[i,6] +c[11] # PoliceHar
mu[i,14]<-lam[13]*xi[i,1] + lam[40]*xi[i,2] + xi[i,3]   # Pollingpl
mu[i,15]<-lam[14]*xi[i,1] + lam[41]*xi[i,2] + lam[67]*xi[i,3] + lam[93]*xi[i,4] +
  lam[118]*xi[i,5] + lam[142]*xi[i,6] +c[12] # Problemwi
mu[i,16]<-lam[15]*xi[i,1] + lam[42]*xi[i,2] + lam[68]*xi[i,3] + lam[94]*xi[i,4] +
  lam[119]*xi[i,5] + lam[143]*xi[i,6] +c[13] # Registrat
mu[i,17]<-lam[16]*xi[i,1] + lam[43]*xi[i,2] + lam[69]*xi[i,3] + lam[95]*xi[i,4] +
  lam[120]*xi[i,5] + lam[144]*xi[i,6] +c[14] # Unspecifi
mu[i,18]<-lam[17]*xi[i,1] + lam[44]*xi[i,2] + lam[70]*xi[i,3] + lam[96]*xi[i,4] +
  lam[121]*xi[i,5] + lam[145]*xi[i,6] +c[15] # Voterinti
mu[i,19]<-lam[18]*xi[i,1] + lam[45]*xi[i,2] + lam[71]*xi[i,3] + lam[97]*xi[i,4] +
  lam[122]*xi[i,5] + lam[146]*xi[i,6] +c[16] # Dortmund
mu[i,20]<-lam[19]*xi[i,1] + lam[46]*xi[i,2] + lam[72]*xi[i,3] + lam[98]*xi[i,4] +
  lam[123]*xi[i,5] + lam[147]*xi[i,6] +c[17] # Dresden
mu[i,21]<-lam[20]*xi[i,1] + lam[47]*xi[i,2] + lam[73]*xi[i,3] + xi[i,4]  # SPD
mu[i,22]<-lam[21]*xi[i,1] + lam[48]*xi[i,2] + lam[74]*xi[i,3] + lam[99]*xi[i,4] +
  xi[i,5]                                    # CDUCSU
mu[i,23]<-lam[22]*xi[i,1] + lam[49]*xi[i,2] + lam[75]*xi[i,3] + lam[100]*xi[i,4] +
 lam[124]*xi[i,5] + lam[148]*xi[i,6] +c[18] # FDP
mu[i,24]<-lam[23]*xi[i,1] + lam[50]*xi[i,2] + lam[76]*xi[i,3] + lam[101]*xi[i,4] +
 lam[125]*xi[i,5] + lam[149]*xi[i,6] +c[19] # GR.NE
mu[i,25]<-lam[24]*xi[i,1] + lam[51]*xi[i,2] + lam[77]*xi[i,3] + lam[102]*xi[i,4] +
 lam[126]*xi[i,5] + lam[150]*xi[i,6] +c[20] # DIE.LINKE
mu[i,26]<-lam[25]*xi[i,1] + lam[52]*xi[i,2] + lam[78]*xi[i,3] + lam[103]*xi[i,4] +
 lam[127]*xi[i,5] + xi[i,6]                 # fmt
mu[i,27]<-lam[26]*xi[i,1] + lam[53]*xi[i,2] + lam[79]*xi[i,3] + lam[104]*xi[i,4] +
 lam[128]*xi[i,5] + lam[151]*xi[i,6] +c[21] # smt
mu[i,28]<-lam[27]*xi[i,1] + lam[54]*xi[i,2] + lam[80]*xi[i,3] + lam[105]*xi[i,4] +
 lam[129]*xi[i,5] + lam[152]*xi[i,6] +c[22] # dwinner
mu[i,29]<-lam[28]*xi[i,1] + lam[55]*xi[i,2] + lam[81]*xi[i,3] + lam[106]*xi[i,4] +
```

```
   lam[130]*xi[i,5] + lam[153]*xi[i,6] +c[23] # dsecond

  mu[i,30]<-lam[154]*xi[i,1] + lam[155]*xi[i,2] + lam[156]*xi[i,3] + lam[157]*xi[i,4] +
   lam[158]*xi[i,5] + lam[159]*xi[i,6] +c[24] # fi
  mu[i,31]<-lam[160]*xi[i,1] + lam[161]*xi[i,2] + lam[162]*xi[i,3] + lam[163]*xi[i,4] +
   lam[164]*xi[i,5] + lam[165]*xi[i,6] +c[25] # fe

  #structural equation model
  xi[i,1:6]~dmnorm(u[1:6],phi[1:6,1:6])
}# end of i

#thresholds
for(j in 1:29){
  thd[j,1]<-alpbot
  thd[j,2]<-alpmid
  thd[j,3]<-alptop
}

for(i in 1:6){u[i]<-gam[i]}

#priors on loadings and coefficients
var.lam[1]<-4.0*psi[3]     var.lam[2]<-4.0*psi[3]     var.lam[3]<-4.0*psi[3]
var.lam[4]<-4.0*psi[3]     var.lam[5]<-4.0*psi[3]     var.lam[6]<-4.0*psi[6]
var.lam[7]<-4.0*psi[7]     var.lam[8]<-4.0*psi[8]     var.lam[9]<-4.0*psi[9]
var.lam[10]<-4.0*psi[10]   var.lam[11]<-4.0*psi[11]     var.lam[12]<-4.0*psi[11]
var.lam[13]<-4.0*psi[10]   var.lam[14]<-4.0*psi[11]     var.lam[15]<-4.0*psi[11]
var.lam[16]<-4.0*psi[10]   var.lam[17]<-4.0*psi[11]     var.lam[18]<-4.0*psi[11]
var.lam[19]<-4.0*psi[10]   var.lam[20]<-4.0*psi[11]     var.lam[21]<-4.0*psi[11]
var.lam[22]<-4.0*psi[10]   var.lam[23]<-4.0*psi[11]     var.lam[24]<-4.0*psi[11]
var.lam[25]<-4.0*psi[10]   var.lam[26]<-4.0*psi[11]     var.lam[27]<-4.0*psi[11]
var.lam[28]<-4.0*psi[10]   var.lam[29]<-4.0*psi[11]     var.lam[30]<-4.0*psi[11]
var.lam[31]<-4.0*psi[10]   var.lam[32]<-4.0*psi[11]     var.lam[33]<-4.0*psi[11]
var.lam[34]<-4.0*psi[10]   var.lam[35]<-4.0*psi[11]     var.lam[36]<-4.0*psi[11]
var.lam[37]<-4.0*psi[10]   var.lam[38]<-4.0*psi[11]     var.lam[39]<-4.0*psi[11]
var.lam[40]<-4.0*psi[10]   var.lam[41]<-4.0*psi[11]     var.lam[42]<-4.0*psi[11]
var.lam[43]<-4.0*psi[10]   var.lam[44]<-4.0*psi[11]     var.lam[45]<-4.0*psi[11]
var.lam[46]<-4.0*psi[10]   var.lam[47]<-4.0*psi[11]     var.lam[48]<-4.0*psi[11]
var.lam[49]<-4.0*psi[10]   var.lam[50]<-4.0*psi[11]     var.lam[51]<-4.0*psi[11]
var.lam[52]<-4.0*psi[10]   var.lam[53]<-4.0*psi[11]     var.lam[54]<-4.0*psi[11]
var.lam[55]<-4.0*psi[10]   var.lam[56]<-4.0*psi[11]     var.lam[57]<-4.0*psi[11]
var.lam[58]<-4.0*psi[10]   var.lam[59]<-4.0*psi[11]     var.lam[60]<-4.0*psi[11]
var.lam[61]<-4.0*psi[10]   var.lam[62]<-4.0*psi[11]     var.lam[63]<-4.0*psi[11]
var.lam[64]<-4.0*psi[10]   var.lam[65]<-4.0*psi[11]     var.lam[66]<-4.0*psi[11]
var.lam[67]<-4.0*psi[10]   var.lam[68]<-4.0*psi[11]     var.lam[69]<-4.0*psi[11]
var.lam[70]<-4.0*psi[10]   var.lam[71]<-4.0*psi[11]     var.lam[72]<-4.0*psi[11]
var.lam[73]<-4.0*psi[10]   var.lam[74]<-4.0*psi[11]     var.lam[75]<-4.0*psi[11]
var.lam[76]<-4.0*psi[10]   var.lam[77]<-4.0*psi[11]     var.lam[78]<-4.0*psi[11]
```

```
  var.lam[79]<-4.0*psi[10]   var.lam[80]<-4.0*psi[11]     var.lam[81]<-4.0*psi[11]
  var.lam[82]<-4.0*psi[10]   var.lam[83]<-4.0*psi[11]     var.lam[84]<-4.0*psi[11]
  var.lam[85]<-4.0*psi[10]   var.lam[86]<-4.0*psi[11]     var.lam[87]<-4.0*psi[11]
  var.lam[88]<-4.0*psi[10]   var.lam[89]<-4.0*psi[11]     var.lam[90]<-4.0*psi[11]
  var.lam[91]<-4.0*psi[10]   var.lam[92]<-4.0*psi[11]     var.lam[93]<-4.0*psi[11]
  var.lam[94]<-4.0*psi[10]   var.lam[95]<-4.0*psi[11]     var.lam[96]<-4.0*psi[11]
  var.lam[97]<-4.0*psi[10]   var.lam[98]<-4.0*psi[11]     var.lam[99]<-4.0*psi[11]
  var.lam[100]<-4.0*psi[10]   var.lam[101]<-4.0*psi[11]     var.lam[102]<-4.0*psi[11]
  var.lam[103]<-4.0*psi[10]   var.lam[104]<-4.0*psi[11]     var.lam[105]<-4.0*psi[11]
  var.lam[106]<-4.0*psi[10]   var.lam[107]<-4.0*psi[11]     var.lam[108]<-4.0*psi[11]
  var.lam[109]<-4.0*psi[10]   var.lam[110]<-4.0*psi[11]     var.lam[111]<-4.0*psi[11]
  var.lam[112]<-4.0*psi[10]   var.lam[113]<-4.0*psi[11]     var.lam[114]<-4.0*psi[11]

  for (k in 115:165){var.lam[k]<-4.0*psi[11]}

  for(i in 1:165){lam[i]~dnorm(0.8,var.lam[i])}

  var.b<-4.0*psi[1]
  for(j in 1:6){gam[j]~dnorm(0.1,var.b)}

  var.c<-4.0*psi[2]
    for(j in 1:25){c[j]~dnorm(0.1,var.c)}

  var.d<-4.0*psi[2]
    for(j in 1:2){d[j]~dnorm(0.1,var.d)}

  #priors on precisions
  for(j in 1:P){
    psi[j]~dgamma(10,8)
    sgm[j]<-1/psi[j]
  }

  phi[1:6,1:6]~dwish(R[1:6,1:6], 30)
  phx[1:6,1:6]<-inverse(phi[1:6,1:6])
} #end of model
```

5.1.2 Credible Intervals for Additional Model Parameters

*** Tables 6, 7 about here ***

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Table 1: Frequency of Postelection Complaint Types, Germany 2005

Type	Description	2005	
		I[a]	II[b]
AbsenteeB	Absentee-ballot Related Problem	29	29
Electoral	Electoral System	67	68
PollingPl	Polling Place Problem	24	24
Allegatio	Allegations of Official Corruption	8	8
BallotRel	Ballot Related Problem	6	6
Countingo	Counting of the Votes	6	6
CriminalS	Criminal Status Related Problem	5	5
Disabilit	Disability Access Problem	2	2
IDrelated	Identification Related Problem	6	6
ImproperC	Improper Campaigning Influence	11	11
ImproperD	Improper District Boundaries	1	1
ImproperS	Improper Statistics	4	4
PartyList	Party List Not on Ballot	19	20
Problemwi	Problems with the Creation of Party Lists	2	2
Registrat	Registration Related Problem	20	20
Unspecifi	Unspecified Other	10	10
PoliceHar	Police Harassment	1	1
VoterInti	Voter Intimidation	1	1
Dortmund	*Briefwahl in Dortmund*	12	12
Dresden	*Nachwahl in Dresden*	35	36

Note: Number of districts that have each type of complaint. [a] minimum number of districts with at least one complaint. [b] maximum number of districts with at least one complaint. Source: Mebane and Klaver (2015).

Table 2: Common Latent Variable Covariance Matrix ($\boldsymbol{\Phi}$), Germany 2005

lower[a]

0.4264	−0.08666	0.05193	−0.2329	−0.03624	0.03918
−0.08666	0.3325	−0.229	−0.2212	−0.02396	−0.1885
0.05193	−0.229	0.255	−0.07159	−0.05083	−0.00956
−0.2329	−0.2212	−0.07159	0.05848	−0.05254	−0.03799
−0.03624	−0.02396	−0.05083	−0.05254	0.05334	−0.02023
0.03918	−0.1885	−0.00956	−0.03799	−0.02023	0.06688

mean[b]

0.8012	0.1371	0.2045	−0.1205	0.04134	0.1059
0.1371	0.8842	0.01478	−0.06094	0.08407	−0.05535
0.2045	0.01478	0.3949	−0.008612	0.00327	0.03574
−0.1205	−0.06094	−0.008612	0.08402	−0.01174	−0.01547
0.04134	0.08407	0.00327	−0.01174	0.07496	0.0014
0.1059	−0.05535	0.03574	−0.01547	0.0014	0.08981

upper[c]

1.159	0.4321	0.3759	−0.03859	0.1441	0.1753
0.4321	1.57	0.2646	0.04877	0.2596	0.05272
0.3759	0.2646	0.6009	0.04471	0.05257	0.08343
−0.03859	0.04877	0.04471	0.1251	0.01017	0.01158
0.1441	0.2596	0.05257	0.01017	0.1217	0.02404
0.1753	0.05272	0.08343	0.01158	0.02404	0.124

Note: $\boldsymbol{\Phi} = \boldsymbol{\Upsilon}^{-1}$. $n = 299$. [a] 95% credible interval elementwise lower bounds, [b] elementwise posterior means, [c] 95% credible interval elementwise upper bounds.

Table 3: Six-LV Model Factor Loadings, Latent Variables 1 and 2, Germany 2005

Manifest Variable	Latent Variable 1 load.	lower[a]	mean[b]	upper[c]	Latent Variable 2 load.	lower[a]	mean[b]	upper[c]
AbsenteeB	λ_{101}		1.0[d]					
Electoral	λ_{102}	0.598	0.9559	1.457	λ_{202}		1.0[d]	
PollingPl	λ_{103}	0.7246	1.246	1.975	λ_{203}	−1.187	−0.4298	0.09714
Allegatio	λ_{104}	0.3718	1.18	2.075	λ_{204}	0.06383	3.703	7.631
BallotRel	λ_{105}	0.3007	1.027	1.832	λ_{205}	−3.1	−0.4353	1.433
Countingo	λ_{106}	0.2165	0.928	1.672	λ_{206}	−0.3376	0.6782	1.805
CriminalS	λ_{107}	0.348	1.095	1.925	λ_{207}	−4.211	−1.273	0.9556
Disabilit	λ_{108}	0.04326	0.8029	1.602	λ_{208}	−2.154	0.3434	2.552
IDrelated	λ_{109}	0.1833	0.7131	1.237	λ_{209}	−0.7514	0.2713	1.24
ImproperC	λ_{110}	−0.5569	0.3553	1.122	λ_{210}	−0.692	1.796	5.113
ImproperD	λ_{111}	0.2905	1.064	1.853	λ_{211}	−2.722	0.05223	2.703
ImproperS	λ_{112}	0.07697	1.3	2.629	λ_{212}	−0.2371	1.004	2.266
PartyList	λ_{113}	0.02539	1.848	3.893	λ_{213}	1.448	3.576	6.787
PoliceHar	λ_{114}	−2.031	0.3774	3.14	λ_{214}	0.1941	2.613	6.074
Problemwi	λ_{115}	−3.841	−1.418	1.666	λ_{215}	−6.339	−2.543	1.401
Registrat	λ_{116}	2.057	4.132	8.024	λ_{216}	−3.016	−0.9099	0.7234
Unspecifi	λ_{117}	−1.007	0.1907	1.38	λ_{217}	0.02247	1.126	2.355
VoterInti	λ_{118}	−2.768	−0.3234	1.991	λ_{218}	−1.941	0.7259	3.815
Dortmund	λ_{119}	4.303	7.583	11.72	λ_{219}	−1.352	0.859	3.227
Dresden	λ_{120}	0.4483	1.818	3.288	λ_{220}	0.5166	1.598	2.769
f_i	λ_{121}	1.593	2.144	3.196	λ_{221}	−2.016	−0.3041	1.231
f_e	λ_{122}	−1.038	1.302	4.309	λ_{222}	−2.92	−0.2948	2.122
ze-SPD	λ_{123}	0.07599	0.1625	0.2416	λ_{223}	−0.06374	0.03429	0.1552
ze-CDUCSU	λ_{124}	−0.1351	−0.04015	0.06194	λ_{224}	−0.2291	−0.1005	0.03706
ze-FDP	λ_{125}	−0.04025	0.004711	0.05034	λ_{225}	−0.03698	0.003198	0.04415
ze-Green	λ_{126}	−0.04525	6.062e-4	0.04652	λ_{226}	−0.03519	0.004346	0.04583
ze-Left	λ_{127}	−0.04629	−6.761e-4	0.04536	λ_{227}	−0.03942	5.928e-4	0.04101
$\mathfrak{M}_{13}$	λ_{128}	−0.2069	−0.1225	−0.02321	λ_{228}	−0.02063	0.1175	0.2835
$\mathfrak{M}_{23}$	λ_{129}	−0.06014	−0.00652	0.04668	λ_{229}	−0.03007	0.01598	0.07002
$\hat{j}_{\underline{1}}$	λ_{130}	−0.06681	−3.012e-4	0.06579	λ_{230}	−0.05008	0.01074	0.08024
$\hat{j}_{\underline{2}}$	λ_{131}	−0.07532	0.006582	0.08689	λ_{231}	−0.09299	−0.01062	0.07125

Note: $n = 299$. [a] 95% credible interval lower bound, [b] posterior mean, [c] 95% credible interval upper bound.
[d] fixed parameter. Loading parameters not shown with a value are fixed at zero.

Table 4: Six-LV Model Factor Loadings, Latent Variables 3 and 4, Germany 2005

Manifest	Latent Variable 3				Latent Variable 4			
Variable	load.	lower[a]	mean[b]	upper[c]	load.	lower[a]	mean[b]	upper[c]
PollingPl	λ_{303}		1.0[d]					
Allegatio	λ_{304}	2.868	8.043	13.99	λ_{404}	−0.7065	0.6654	2.012
BallotRel	λ_{305}	−5.325	−2.397	−0.08382	λ_{405}	−10.8	−3.895	5.074
Countingo	λ_{306}	0.4512	1.58	3.016	λ_{406}	0.4244	5.183	9.822
CriminalS	λ_{307}	0.4307	4.19	7.947	λ_{407}	−0.6123	0.7333	2.063
Disabilit	λ_{308}	−6.527	−3.865	−1.602	λ_{408}	−6.367	−0.746	5.264
IDrelated	λ_{309}	−0.1522	1.059	2.44	λ_{409}	−4.528	1.574	6.534
ImproperC	λ_{310}	0.9972	3.928	7.938	λ_{410}	−0.4275	0.9391	2.313
ImproperD	λ_{311}	0.07429	3.996	8.634	λ_{411}	−5.515	−0.08476	4.943
ImproperS	λ_{312}	−0.3766	0.9476	2.315	λ_{412}	−8.705	−2.744	2.822
PartyList	λ_{313}	−2.35	0.04622	2.252	λ_{413}	−0.3104	1.002	2.375
PoliceHar	λ_{314}	0.4368	3.808	7.451	λ_{414}	−3.118	2.168	7.497
Problemwi	λ_{315}	−0.4331	0.8274	2.096	λ_{415}	−5.388	0.9103	7.772
Registrat	λ_{316}	−0.9062	1.215	4.421	λ_{416}	−0.5683	0.775	2.128
Unspecifi	λ_{317}	−1.04	1.324	4.053	λ_{417}	2.96	7.634	12.83
VoterInti	λ_{318}	−0.6583	0.6253	1.886	λ_{418}	−3.149	2.518	7.695
Dortmund	λ_{319}	−6.149	−3.065	−0.4648	λ_{419}	−0.6265	0.7428	2.113
Dresden	λ_{320}	−1.186	0.782	3.091	λ_{420}	−10.41	−6.264	−2.366
f_i	λ_{321}	−4.267	−2.779	−1.505	λ_{421}	2.883	5.814	10.48
f_e	λ_{322}	−1.971	1.014	4.762	λ_{422}	−4.345	0.7347	5.879
ze-SPD	λ_{323}	−0.2081	−0.07658	0.06147	λ_{423}		1.0[d]	
ze-CDUCSU	λ_{324}	−0.1353	0.02143	0.1838	λ_{424}	−0.2527	0.01503	0.3011
ze-FDP	λ_{325}	−0.05264	−7.217e-4	0.05055	λ_{425}	−0.1304	0.01011	0.152
ze-Green	λ_{326}	−0.04759	0.003921	0.05636	λ_{426}	−0.1546	−0.01241	0.1299
ze-Left	λ_{327}	−0.0509	0.001084	0.05271	λ_{427}	−0.1432	−0.00182	0.1408
$\mathfrak{M}_{13}$	λ_{328}	−0.2139	−0.06358	0.06848	λ_{428}	−0.125	0.1768	0.4895
$\mathfrak{M}_{23}$	λ_{329}	−0.0639	−0.002302	0.05861	λ_{429}	−0.1683	0.002372	0.1716
$\hat{\jmath}_{\underline{1}}$	λ_{330}	−0.07327	0.003899	0.08246	λ_{430}	−0.2759	−0.05434	0.1674
$\hat{\jmath}_{\underline{2}}$	λ_{331}	−0.0564	0.0429	0.1496	λ_{431}	−0.4101	−0.1458	0.1209

Note: $n = 299$. [a] 95% credible interval lower bound, [b] posterior mean, [c] 95% credible interval upper bound.
[d] fixed parameter. Loading parameters not shown with a value are fixed at zero.

Table 5: Six-LV Model Factor Loadings, Latent Variables 5 and 6, Germany 2005

Manifest	Latent Variable 5				Latent Variable 6			
Variable	load.	lower[a]	mean[b]	upper[c]	load.	lower[a]	mean[b]	upper[c]
Allegatio	λ_{504}	−7.201	−1.207	4.665	λ_{604}	−4.671	1.057	6.798
BallotRel	λ_{505}	−7.746	−1.924	4.772	λ_{605}	−4.327	3.368	9.746
Countingo	λ_{506}	−0.562	0.7631	2.126	λ_{606}	−6.162	−1.002	3.182
CriminalS	λ_{507}	−1.7	2.981	7.854	λ_{607}	−0.1995	4.217	9.145
Disabilit	λ_{508}	−5.405	0.3238	6.195	λ_{608}	−4.141	1.582	6.595
IDrelated	λ_{509}	−0.7008	0.7192	2.185	λ_{609}	2.215	7.681	12.57
ImproperC	λ_{510}	−6.561	−0.6386	5.526	λ_{610}	−13.03	−8.06	−2.414
ImproperD	λ_{511}	−6.759	−1.627	3.45	λ_{611}	−4.004	1.336	6.312
ImproperS	λ_{512}	−1.764	3.979	9.983	λ_{612}	−4.775	1.042	6.923
PartyList	λ_{513}	−15.15	−9.882	−5.246	λ_{613}	−5.197	0.01934	6.922
PoliceHar	λ_{514}	−6.133	−0.8547	4.277	λ_{614}	−5.804	−0.4058	4.873
Problemwi	λ_{515}	−5.911	0.3278	6.553	λ_{615}	−5.778	−0.1962	5.135
Registrat	λ_{516}	−12.55	−7.398	−3.376	λ_{616}	−6.773	−2.631	2.207
Unspecifi	λ_{517}	1.091	6.174	11.43	λ_{617}	−2.571	2.11	6.595
VoterInti	λ_{518}	−4.726	0.6755	5.972	λ_{618}	−9.162	−4.223	0.4128
Dortmund	λ_{519}	−3.244	2.193	7.499	λ_{619}	−2.222	2.548	7.791
Dresden	λ_{520}	0.03598	5.247	9.778	λ_{620}	−6.688	−2.537	1.172
f_i	λ_{521}	−2.754	1.063	3.821	λ_{621}	−7.825	−5.678	−1.968
f_e	λ_{522}	−3.687	1.137	6.171	λ_{622}	−6.475	−1.272	3.724
ze-CDUCSU	λ_{524}		1.0^d					
ze-FDP	λ_{525}	−0.1491	−0.009162	0.1301	λ_{625}	−0.1269	0.002395	0.1313
ze-Green	λ_{526}	−0.154	−0.01486	0.1258	λ_{626}	−0.1225	0.008014	0.1375
ze-Left	λ_{527}	−0.1376	0.001223	0.1392	λ_{627}	−0.1295	0.001604	0.132
$\mathfrak{M}_{13}$	λ_{528}	−0.3961	−0.08959	0.2124	λ_{628}		1.0^d	
$\mathfrak{M}_{23}$	λ_{529}	−0.21	−0.04147	0.1269	λ_{629}	−0.09666	0.05916	0.2156
$\hat{j}_{\underline{1}}$	λ_{530}	−0.2487	−0.02984	0.1892	λ_{630}	−0.1054	0.09352	0.288
$\hat{j}_{\underline{2}}$	λ_{531}	−0.4654	−0.1929	0.07892	λ_{631}	−0.2793	−0.01607	0.239

Note: $n = 299$. [a] 95% credible interval lower bound, [b] posterior mean, [c] 95% credible interval upper bound.
[d] fixed parameter. Loading parameters not shown with a value are fixed at zero.

Table 6: Six-LV Model Means, Germany 2005

variable	mean	lower[a]	mean[b]	upper[c]
latent variable 1	γ_1	−1.617	−1.348	−1.008
latent variable 2	γ_2	−0.2493	0.1278	0.5145
latent variable 3	γ_3	−0.9074	−0.6781	−0.4399
latent variable 4	γ_4	−0.001968	0.1239	0.2612
latent variable 5	γ_5	−0.2018	−0.08302	0.04869
latent variable 6	γ_6	0.01167	0.11	0.2063
Allegatio	c_4	−24.76	−16.6	−10.12
BallotRel	c_5	−10.89	−6.063	−2.505
Countingo	c_6	−4.723	−2.499	−0.8138
CriminalS	c_7	−7.083	−4.193	−1.911
Disabilit	c_8	−13.72	−8.791	−4.568
IDrelated	c_9	−7.785	−4.923	−2.222
ImproperC	c_{10}	−10.17	−3.446	0.1425
ImproperD	c_{11}	−9.679	−5.976	−3.008
ImproperS	c_{12}	−16.76	−12.25	−8.589
PartyList	c_{13}	−7.168	−4.375	−1.898
PoliceHar	c_{14}	−12.59	−7.935	−4.115
Problemwi	c_{15}	−20.59	−9.984	−4.624
Registrat	c_{16}	−1.921	0.2114	2.206
Unspecifi	c_{17}	−7.483	−4.709	−2.121
VoterInti	c_{18}	−13.47	−7.062	−2.254
Dortmund	c_{19}	−7.577	−4.25	−1.527
Dresden	c_{20}	−2.649	−0.9169	0.7148
f_i	c_{21}	−5.959	−4.521	−3.624
f_e	c_{22}	−28.35	−23.61	−20.73
ze-FDP	c_{25}	−0.0146	0.05417	0.1229
ze-Green	c_{26}	−0.04018	0.02961	0.09805
ze-Left	c_{27}	−0.06165	0.007385	0.07627
$\mathfrak{M}_{23}$	c_{29}	0.1309	0.212	0.2929
$\hat{\jmath}_{\underline{1}}$	c_{30}	4.271	4.373	4.475
$\hat{\jmath}_{\underline{2}}$	c_{31}	4.151	4.276	4.401

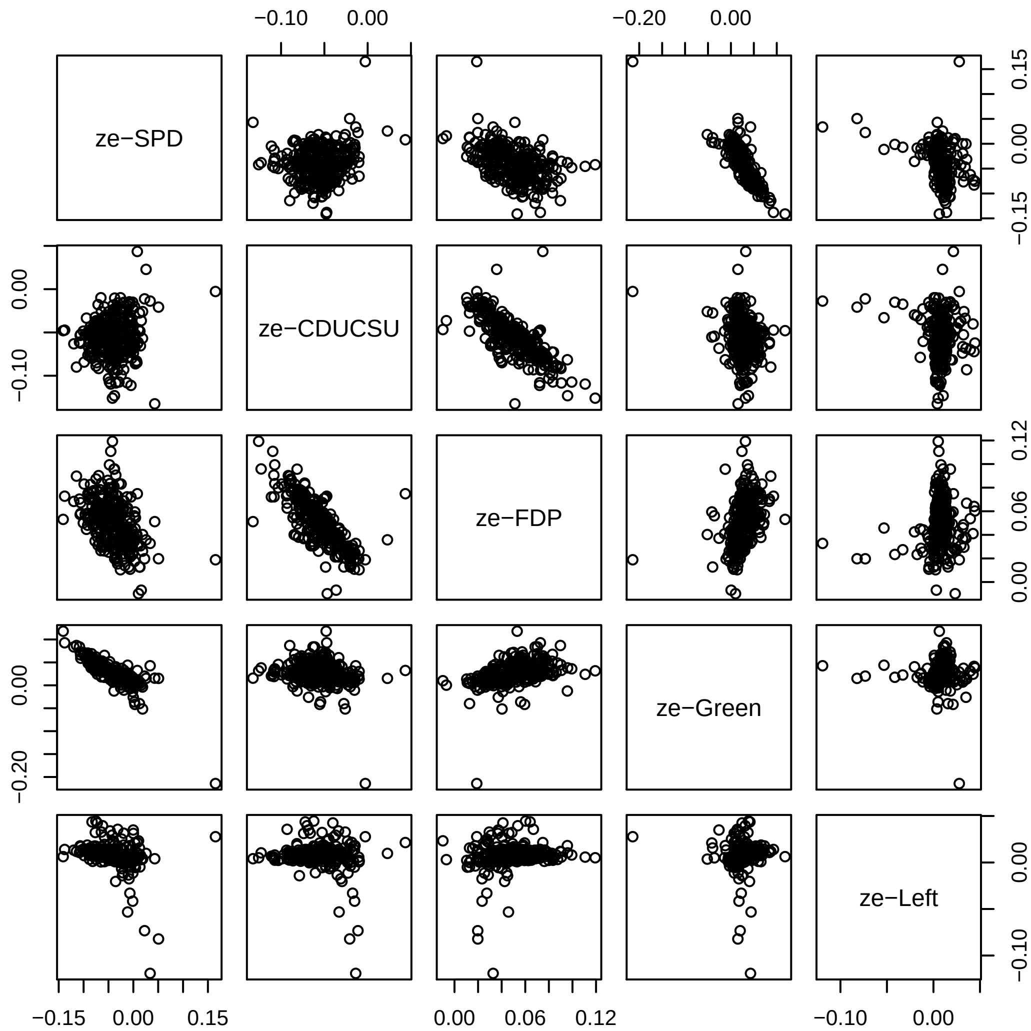
Note: $n = 299$. [a] 95% credible interval lower bound, [b] posterior mean, [c] 95% credible interval upper bound.
Mean parameters not shown with a value are fixed at zero.

Table 7: Six-LV Model Uniqueness Variances, Germany 2005

variable	var.	lower[a]	mean[b]	upper[c]
AbsenteeB	ψ_1^{-1}	0.3023	0.4518	0.6324
Electoral	ψ_2^{-1}	0.217	0.3123	0.447
PollingPl	ψ_3^{-1}	0.3612	0.6306	1.072
Allegatio	ψ_4^{-1}	61.72	106.1	184.4
BallotRel	ψ_5^{-1}	0.4027	0.7143	1.263
Countingo	ψ_6^{-1}	0.4121	0.7656	1.405
CriminalS	ψ_7^{-1}	0.4145	0.7348	1.283
Disabilit	ψ_8^{-1}	0.4205	0.7508	1.331
IDrelated	ψ_9^{-1}	0.412	0.7634	1.418
ImproperC	ψ_{10}^{-1}	0.3811	0.6551	1.125
ImproperD	ψ_{11}^{-1}	1.085	1.908	3.297
ImproperS	ψ_{12}^{-1}	23.12	38.38	61.78
PartyList	ψ_{13}^{-1}	0.3492	0.5778	0.9483
PoliceHar	ψ_{14}^{-1}	0.4486	0.8283	1.495
Problemwi	ψ_{15}^{-1}	0.4337	0.7919	1.431
Registrat	ψ_{16}^{-1}	0.4155	0.7238	1.253
Unspecifi	ψ_{17}^{-1}	0.37	0.6248	1.043
VoterInti	ψ_{18}^{-1}	0.4422	0.8097	1.454
Dortmund	ψ_{19}^{-1}	0.408	0.7177	1.244
Dresden	ψ_{20}^{-1}	0.3357	0.5392	0.8611
ze-SPD	ψ_{21}^{-1}	0.07049	0.08414	0.1003
ze-CDUCSU	ψ_{22}^{-1}	0.07123	0.0853	0.1019
ze-FDP	ψ_{23}^{-1}	0.04448	0.05196	0.0608
ze-Green	ψ_{24}^{-1}	0.04449	0.0522	0.06112
ze-Left	ψ_{25}^{-1}	0.04424	0.05179	0.06056
$\mathfrak{M}_{13}$	ψ_{26}^{-1}	0.08379	0.1009	0.1214
$\mathfrak{M}_{23}$	ψ_{27}^{-1}	0.0546	0.06383	0.07467
$\hat{j}_{\underline{1}}$	ψ_{28}^{-1}	0.07568	0.08858	0.1038
$\hat{j}_{\underline{2}}$	ψ_{29}^{-1}	0.09553	0.1122	0.1315

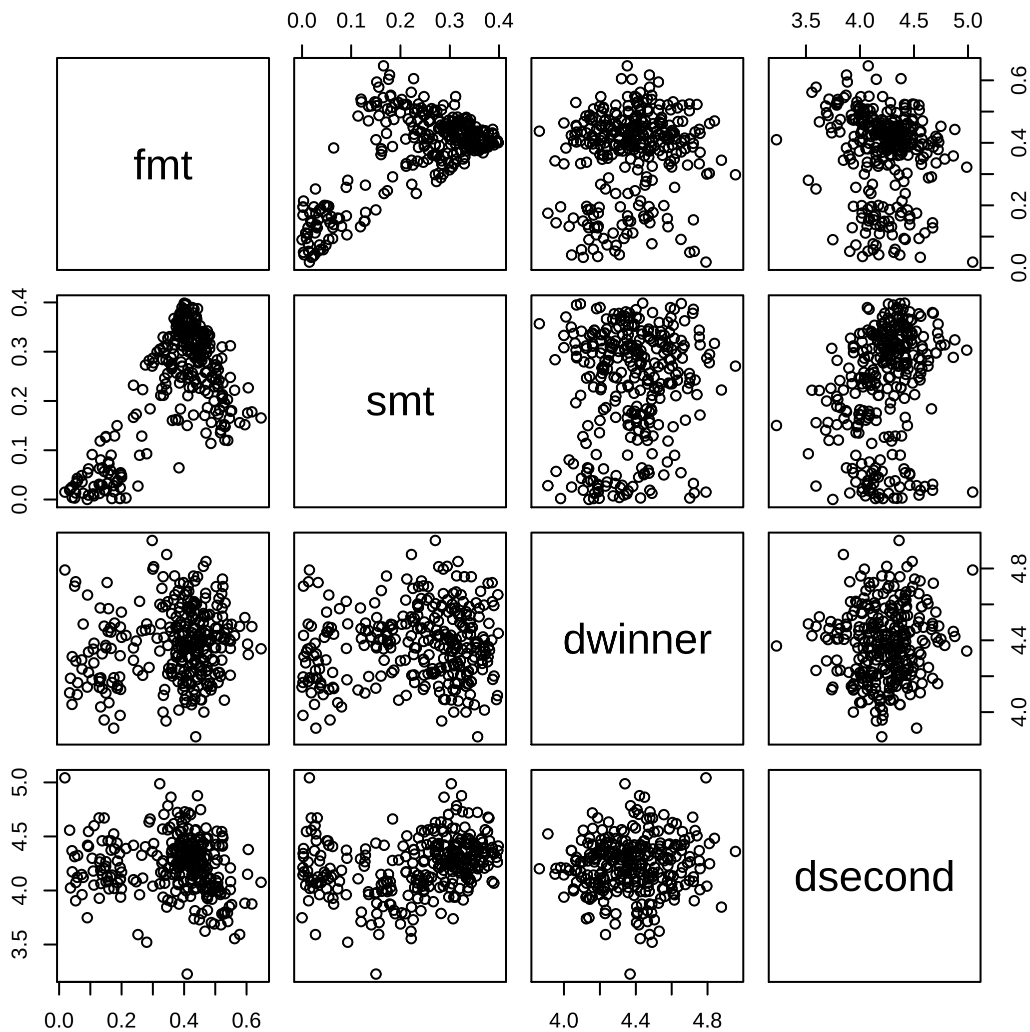
Note: $n = 299$. [a] 95% credible interval lower bound, [b] posterior mean, [c] 95% credible interval upper bound.

Figure 1: Strategic Voting Measures, Germany 2005: *Zweitstimmen* Minus *Erststimmen*



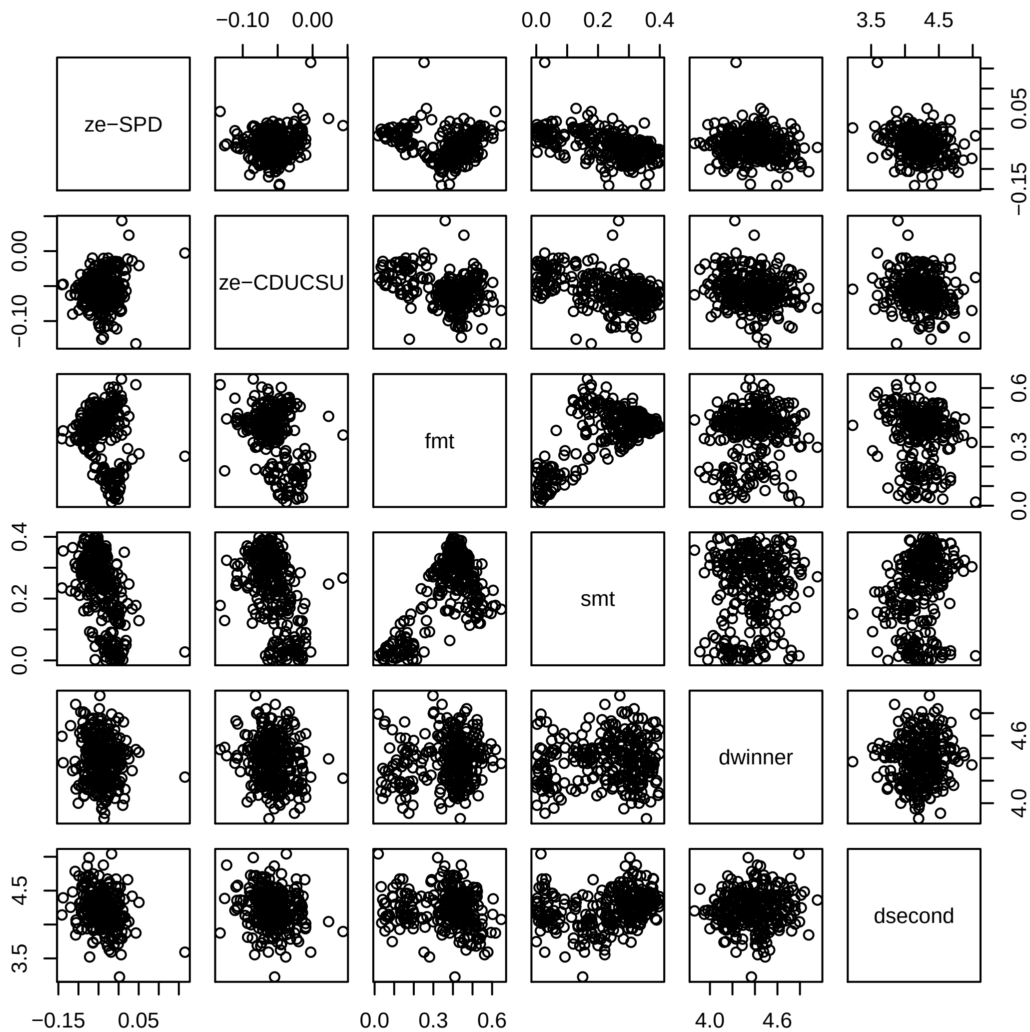
Note: ze-SPD, ze-CDUCSU, ze-FDP, ze-Green and ze-Left refer to the differences between the proportion of *Zweitstimmen* and of *Erststimmen* received by the referent party in each district.

Figure 2: Strategic Voting Measures, Germany 2005: Margins and Second Digit Means



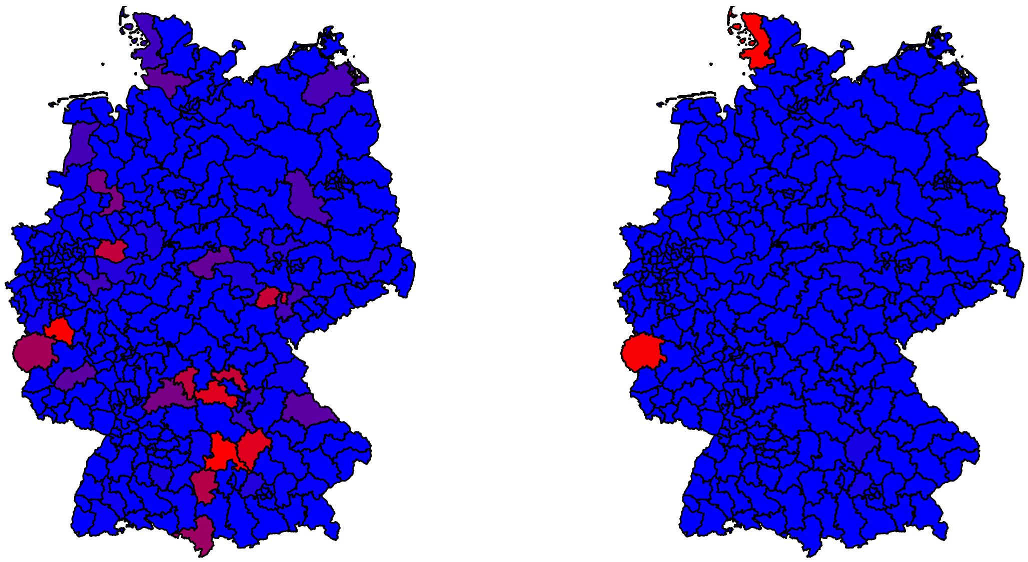
Note: fmt refers to the difference between the proportions of *Erststimmen* received by the first- and third-place parties in each district (denoted $\mathfrak{M}_{13}$ in Tables 3–6). smt refers to the difference between the proportions of *Erststimmen* received by the second- and third-place parties (denoted $\mathfrak{M}_{23}$ in Tables 3–6). dwinner is the mean of the second significant digits of the first-place party's polling place vote counts in each district (denoted $\hat{j}_{\underline{1}}$ in Tables 3–6). dwinner is the mean of the second significant digits of the first-place party's polling place vote counts in each district (denoted $\hat{j}_{\underline{2}}$ in Tables 3–6).

Figure 3: Strategic Voting Measures, Germany 2005



Note: see the notes in Figures 1 and 2 for descriptions of the variables.

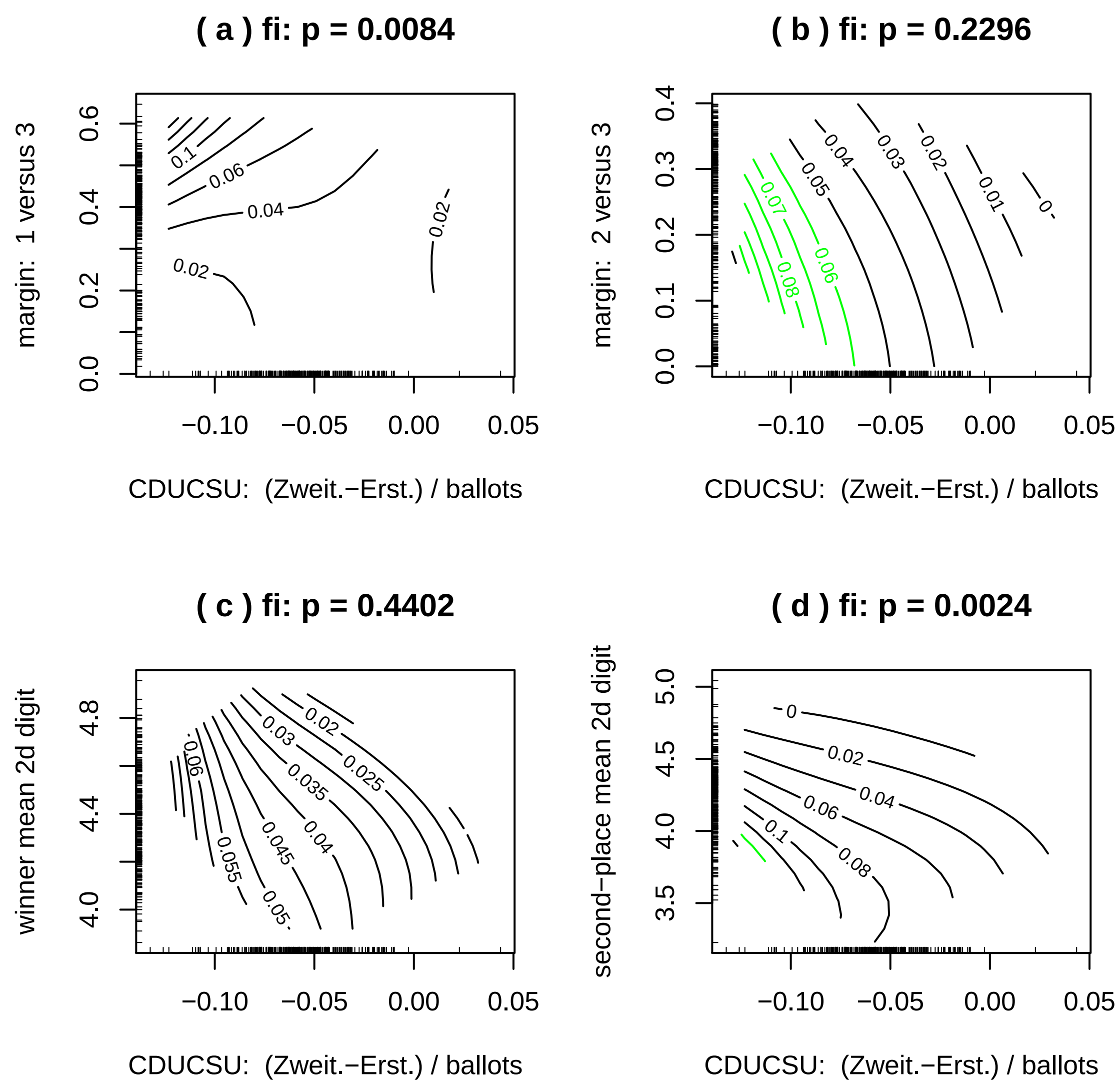
Figure 4: "Fraud" Probabilities, by District, Germany 2005 Bundestag Erststimmen



(a) Incremental (f_{i}) (b) Extreme (f_{e})

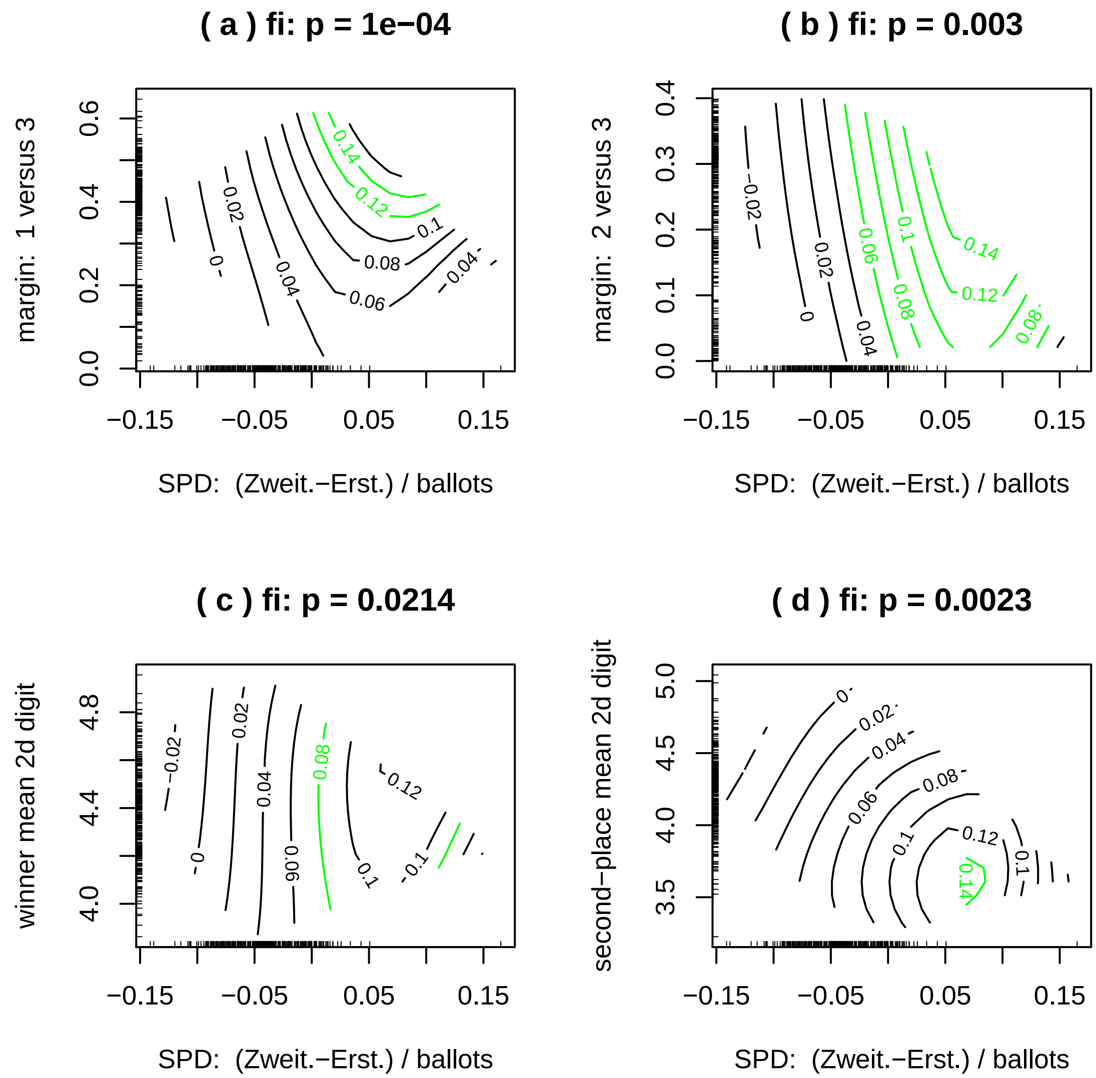
Note: $f_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ values estimated using a finite mixture variant of the Klimek et al. (2012) model separately for each for each district, $i = 1, \ldots, 299$, using polling center data. Color red means $\hat{f}_{\mathrm{i}i}/\max(\hat{f}_{\mathrm{i}i}) = 1$ or $\hat{f}_{\mathrm{e}i}/\max(\hat{f}_{\mathrm{e}i}) = 1$, color blue means $\hat{f}_{\mathrm{i}i} = 0$ or $\hat{f}_{\mathrm{e}i} = 0$, and intermediate values have colors that are weighted mixtures of red and blue.
(a) $\max(\hat{f}_{\mathrm{i}i}) = 0.74$. (b) $\max(\hat{f}_{\mathrm{e}i}) = 0.004659684$.

Figure 5: Incremental Fraud Probabilities by Strategic Measures, Germany 2005 Bundestag Erststimmen, CDU-CSU



Note: nonparametric regression contours for $\hat{f}_{\text{i}i}$. "[party]: (Zweit.-Erst.)/ballots" is the total of *Zweitstimmen* cast for [party] minus the number of *Erststimmen* cast for [party] divided by the total number of ballots used in the district. "margin: 1 versus 3" is the number of *Erststimmen* for the winning party in each district minus the number of votes for the third-place party divided by the total of *Erststimmen* cast in the district. Rug plots show locations of district values. p in each subfigure heading reports the p-value for a significance test versus the model of no effects.

Figure 6: Incremental Fraud Probabilities by Strategic Measures, Germany 2005 Bundestag Erststimmen, SPD



Note: nonparametric regression contours for $\hat{f}_{\text{i}i}$. "[party]: (Zweit.-Erst.)/ballots" is the total of *Zweitstimmen* cast for [party] minus the number of *Erststimmen* cast for [party] divided by the total number of ballots used in the district. "margin: 1 versus 3" is the number of *Erststimmen* for the winning party in each district minus the number of votes for the third-place party divided by the total of *Erststimmen* cast in the district. Rug plots show locations of district values. p in each subfigure heading reports the p-value for a significance test versus the model of no effects.