Mixed Strategy Equilibrium and Covering in Multidimensional Electoral Competition

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Outline of talk

- Downsian Elections
- Background Literature
- Results
  - Existence
  - Deep Covering
  - Equilibrium Bounds
- Extensions
  - Existence with Mixed Motives
  - Multiple Candidates
  - General Voting Voting Behavior
  - Generalized Covering
  - Back to Bounds
Downsian Elections

• Candidates $A$ and $B$ simultaneously commit to $x_A$ and $x_B$ in $X \subseteq \mathbb{R}^d$.

• An odd number $n$ of voters cast ballots (no abstention).

• Majority rule election (coin flip in case of tie). An outcome is $(C, x_C)$.

• Office-motivated candidates and policy-oriented voters:

\[
u_A(C, x_C) = \begin{cases} 
1 & \text{if } C = A \\
-1 & \text{else}
\end{cases}
\]

\[
u_i(C, x_C) = u_i(x_C).
\]
Downsian Elections (cont.)

- Focus on locational aspects of elections, ignoring
  - information aspects
  - temporal aspects
  - financial aspects

- But positions $x_A$ and $x_B$ can be interpreted more generally as “campaign strategies.” Analysis is still static.

- Unidimensional $X$ and single-peakedness $\Rightarrow$ Median Voter Theorem

- Multidimensional $X$
  $\Rightarrow$ generic non-existence of pure strategy equilibrium.
Downsian Elections (cont.)

• Responses (within commitment paradigm):
  – probabilistic voting
  – policy-motivated candidates
  – repeated elections
  – mixed strategies

• Interpretations of mixed strategies:
  – candidate beliefs
  – incomplete information among candidates (?)
  – ambiguous speeches
Downsian Elections (cont.)

- Issues for mixed strategy equilibrium:
  - Existence: Not a problem for finite $X$, but discontinuities present difficulties in general case.
  - Characterization: Can we find bounds on supports of equilibrium strategies?
  - Robustness: If a pure strategy equilibrium exists, and we perturb preferences, will equilibrium outcomes move continuously?
Background: Existence

- Finite case (tournaments)
  - Laffond, Laslier, and Le Breton (1993)

- Distributive case
  - Borel (1921)
  - Gross and Wagner (1950)
  - Laslier and Picard (2002)

- Multidimensional spatial model
  - Kramer (1978), continuum of voters
  - Duggan (2003), three voters
Background (cont.)

• General games
  – Dasgupta and Maskin (1986)
  – Simon (1987)
  – Reny (1999)

• Reny’s (1999) diagonal better reply security (2-person, symmetric, z-sum game):
  – for all $x$ and all $\xi$ s.t. $u(x, \xi) > 0$, there exists $\xi'$ and open $G$ containing $\xi$ such that
    \[
    \inf_{\hat{\xi} \in G} u(\xi', \hat{x}) > 0.
    \]

• These results all reduce to finite approximation arguments.
Background: Characterization

• **Strict** and **weak majority preference**:
  \[ xPy \iff \#\{i \mid u_i(x) > u_i(y)\} > \frac{n}{2} \]
  \[ xRy \iff \#\{i \mid u_i(x) \geq u_i(y)\} \geq \frac{n}{2} \]

• The **core**:
  \[ K = \{x \mid \text{there is no } y \text{ s.t. } yPx\} \]

• **Facts**: Assume \( u_i \) strictly quasi-concave.
  
  – if there is a core point, it is unique
  
  – in every pure strategy equilibrium, candidates locate at the core point
  
  – in multiple dimensions, the core is generically empty
Background (cont.)

- The alternatives that strictly beat $x$:
  \[ P(x) = \{ y \mid yPx \}, \]
  etc.

- Say $x$ “covers” $y$ iff...
  
  \[ xPy \text{ and } P(x) \subseteq P(y) \]
  
  \[ xPy \text{ and } R(x) \subseteq R(y) \]
  
  \[ xPy \text{ and } P(x) \subseteq P(y) \text{ and } R(x) \subseteq R(y) \]
  ("McKelvey covering," $MC$)

- The “McKelvey uncovered set”:
  \[ UMC = \{ y \mid \text{there is no } x \text{ s.t. } xMCyx \} \]
Background (cont.)

- Facts: Assume $u_i$ is strictly quasi-concave.
  
  - $UMC$ is non-empty
  
  - if the core is non-empty, then $UMC = K. . .$
  
  - and $UMC$ expands continuously if preferences are perturbed
Background (cont.)

- Finite case
  - Laffond, Laslier, and Le Breton (1993)
  - Dutta and Laslier (1999)

- Multidimensional spatial model
  - McKelvey (1986)

- General case
  - Banks, Duggan, and Le Breton (2002)
Results: Equilibrium Existence

- $X$ is compact, $u_i$ is continuous

- Pure strategies for candidates: $x_A, x_B \in X$

- Mixed strategies: $\xi_A, \xi_B$

- Pure strategies for voters:

$$v_i : X \times X \rightarrow \{A, B\}$$

- Mixed strategies: $\mu_i$
Existence (cont.)

- We consider subgame perfect equilibria where voting strategies are
  
  - **undominated:**
    \[ u_i(x_A) > u_i(x_B) \Rightarrow \mu_i(A|x_A, x_B) = 1 \]
  
  - **symmetric:** \[ \mu_i(C|x, y) = 1 - \mu_i(C|y, x) \]

  Such equilibria are SPEUV.

- Note:
  
  - If \( x_A P x_B \), there is a unique outcome, i.e., \( A \) wins.
  
  - The candidate location stage is symmetric, zero-sum.
Existence (cont.)

- **Theorem 1**: There exists a SPESUV.

- **Sketch**: Reduce to a “game with endogenous sharing rule” for the candidates, with payoff correspondence

  \[ \Psi(x_A, x_B) = \begin{cases} 
  (\alpha, -\alpha) & \text{if } x_APx_B \\
  \alpha = 1 & \text{if } x_BPx_A \\
  \alpha = -1 & \text{if } x_AIx_B \\
  \alpha \in [-1, 1] & \text{if } x_AIx_B 
  \end{cases}. \]

  From JSSZ (2002), there is a symmetric payoff selection \( \psi \) that admits a m.s.e. Now put the voters back in and specify voting equilibrium strategies that generate \( \psi \).
• The classical approach considers only the selection $\psi$ such that

$$
\psi(x_A, x_B) = \begin{cases} 
1 & \text{if } x_A P x_B \\
-1 & \text{if } x_B P x_A \\
0 & \text{else.}
\end{cases}
$$

• This pins down the behavior of indifferent voters in advance.

• We endogenize the behavior of indifferent agents, as usual when searching for mixed strategy equilibria.

• Earlier results bounding equilibrium strategies by UMC do not apply.
Results: Deep Covering

• Say $x$ deeply covers $y$, $xDCy$, iff

  $- R(x) \subseteq P(y)$.

• Note: If $xDCy$, then $xPy$.

• $DC$ is asymmetric, transitive, and open.

• Proposition 1: $xDCy$ if and only if $x$ weakly dominates $y$ in the stage game (assuming symmetric, undominated voting strategies).
Deep Covering (cont.)

- The *deep uncovered set*:
  \[ UDC = \{ x \mid \text{there is no } y \text{ s.t. } yDCx \} \]

- Facts:
  - *UDC* is non-empty, compact, and externally stable, characterized by 2-step principle
  - if \( u_i \) is strictly quasi-concave and majority core is non-empty, then \( UDC = K \).
  - *UDC* is u.h.c. as a function of voter preferences, i.e., it cannot expand discontinuously as preferences are varied.
Results: Equilibrium Bounds

- **Theorem 2:** In any SPESUV, $\xi_A$ and $\xi_B$ have support in $UDC$.

- Sketch: Suppose $\xi_A = \xi$ has support outside $UDC$. Note: $(\xi, \xi)$ is an equilibrium. By supposition, there exists $Y \subseteq X \setminus UDC$ such that $\xi(Y) > 0$ and $x'$ such that $x'DCY$.

Then $A$ can deviate to $x'$ and receive a higher payoff.
Results: Summary

• Existence: Yes.

• Characterization: Equilibrium platforms lie inside the deep uncovered set with probability one.

• Robustness: Yes.
Extension: Mixed Motives

- Instead of office-motivated candidates, let $u_A(C, x_C)$ be any continuous function, e.g.,
  
  $$u_A(C, x_C) = u_A(x_C) + w_C,$$

  where $w_A \geq w_B$.

- **Theorem 1'**: There exists a SPESUV.
Extension: Multiple Candidates

- Let the candidates be $A, B, \ldots, M$.

- Voting strategies are \textit{undominated*} if no voter votes for her strictly worst candidate.

- An equilibrium is \textit{Duvergerian} if there are two candidates such that, for every $(x_A, \ldots, x_M)$, only those two receive votes.

- \textbf{Theorem 1′′}: Assume there are at least four voters or there are only two candidates. There exists a Duvergian SPESU*V.
Extension: General Voting Behavior

- Let $\Pi: X \times X \rightarrow [0, 1]$ denote $A$'s probability of winning correspondence. Assume $\Pi$ has closed graph and convex values.

- Interpret $\Pi(x_A, x_B)$ as capturing equilibrium outcomes from the voting subgame.

- Examples:
  - Voters may care about names: $u_i(C, x_C) = u_i(x_C) + b_C$.
  - Voters may be probabilistic, e.g., bias term $b_C$ may be stochastic.
  - Abstention due to alienation, etc.

- **Theorem 1"":** There exists a $\Pi$-equilibrium.
Extension: Generalized Deep Covering

• In the Downsian model, $xDCy$ means: for all $z$, $zRx \Rightarrow zPy$.

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<td>$y$</td>
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• Let $U^\Pi_A(x_A, x_B)$ be the set of possible payoffs for candidate $A$ generated by voting equilibria:

$$U^\Pi_A(x_A, x_B) = \left\{ pu_A(A, x_A) + (1 - p)u_A(B, x_B) \mid p \in \Pi(x_A, x_B) \right\}.$$
Deep Covering (cont.)

- Say $x \Pi$-covers $y$ for $A$, $xC_A^\Pi y$, iff
  
  (i) for all $z$, $\min U_A^\Pi(x, z) \geq \max U_A^\Pi(y, z)$
  
  (ii) $\min U_A^\Pi(x, y) > \max U_A^\Pi(y, y)$.

- The $\Pi$-uncovered set for $A$ is
  
  $$UC_A^\Pi = \{x \mid \text{there is no } y \text{ s.t. } yC_A^\Pi x\}.$$  

- Facts:
  
  - $UC_A^\Pi$ is non-empty.
  
  - In Downsian model, $UC_A^\Pi = UDC$.
  
  - In non-Downsian models, it doesn’t have great continuity properties.
Extension: Back to Bounds

- General bounds are very difficult in non-Downsian models.

- Let $u^*_C$ and $\Pi^*$ be as in the Downsian model. Consider a sequence of models converging to Downsian:

  $$(u_A^m, u_B^m, \Pi^m) \to (u_A^*, u_B^*, \Pi^*).$$

  Let $(\xi_A^m, \xi_B^m)$ be a sequence of $\Pi^m$-equilibria from these games.

- **Claim:** If $(\xi_A^m, \xi_B^m)$ converges to $(\xi_A^*, \xi_B^*)$ in the weak* topology, then $(\xi_A^*, \xi_B^*)$ is a $\Pi^*$-equilibrium in the Downsian model.

- Implication: If we perturb candidate pay-offs or voting behavior in the Downsian model, resulting equilibria will be “close” to the deep uncovered set.
Conclusion

- We offer a game-theoretic solution to equilibrium existence problem by endogenizing behavior of indifferent voters.
  - Mixed strategy equilibria exist generally.
  - Policy platforms lie in the deep uncovered set.
  - Pure strategy equilibria, when they exist, are robust.

- Completes analysis of Downsian model, but leaves open question of general bounds for non-Downsian models.

- Approach can be extended to richer models of politics in which discontinuities arise from multiplicity of equilibria.