Bargaining Foundations of the Median Voter Theorem

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Outline of talk

• social choice approach

• bargaining model

• stationary equilibria
  – existence, continuity, delay, core equivalence in one dimension

• voting subgames

• main result

• intuition for proof

• technical point

• extensions

• quick conclusion
Social choice approach

\( N \) \hspace{1cm} set of \( n \) agents (\( n \) odd)*

\( X \subseteq \mathbb{R}^m \) \hspace{1cm} set of alternatives
\hspace{1cm} (compact, convex)

\( u_i : X \rightarrow \mathbb{R} \) \hspace{1cm} \( i \)'s utility function
\hspace{1cm} (cont., strictly concave)*

\( \tilde{x}_i \) \hspace{1cm} \( i \)'s ideal point

\( M \) \hspace{1cm} collection of majority coalitions

\( M \) \hspace{1cm} majority dominance relation:
\hspace{1cm} \( xM y \) means that
\hspace{1cm} \( \{ i \mid u_i(x) > u_i(y) \} \in M \)
The Social choice approach (cont.)

- The **core**, denoted $K$, consists of the “unbeaten” alternatives: $x \in K$ if and only if there is no $y \in X$ such that $y M x$.

- Under our maintained assumptions, if the core is non-empty, then it is a singleton.

- **Median Voter Theorem**: Assume $X \subseteq \mathbb{R}$, and let $k$ satisfy
  \[
  \#\{i \mid \tilde{x}_i < \tilde{x}_k\} < \frac{n}{2}
  \]
  and
  \[
  \#\{i \mid \tilde{x}_i > \tilde{x}_k\} < \frac{n}{2}.
  \]
  Then $K = \{\tilde{x}_k\}$. Moreover, for every $y \in X \setminus K$, $\tilde{x}_k M y$. 

Social choice approach (cont.)

- Picture of median voter theorem:

- Problems in multiple dimensions:
The goal

• We want a general non-cooperative theory of collective decision-making that generates equilibrium predictions even in multiple dimensions.

• Even better if consistent with the median voter theorem in one dimension.
  – characterization of equilibrium outcomes
  – non-cooperative underpinning of cooperative solution
Bargaining literature

• Rubinstein (1982)

• Binmore (1987)

• Baron and Ferejohn (1989)

• Banks and Duggan (2000, 2003)
Bargaining framework

\( N \) \hspace{0.5cm} \text{set of } n \text{ agents } \left( n \text{ odd} \right) *

\( X \subseteq \mathbb{R}^m \) \hspace{0.5cm} \text{set of alternatives}
\hspace{0.5cm} \text{(compact, convex)}

\( u_i : X \rightarrow \mathbb{R} \) \hspace{0.5cm} i’s stage utility function
\hspace{0.5cm} \text{(cont., strictly concave)} *

\( q \) \hspace{0.5cm} \text{status quo}

\( \delta \) \hspace{0.5cm} \text{discount factor}
\hspace{0.5cm} \text{(common, less than one)} *

\( \rho_i \) \hspace{0.5cm} \text{recognition probability}
\hspace{0.5cm} \text{(positive, fixed) *}
Bargaining protocol in period $t$

- an agent $i$ is selected with probability $\rho_i$ to propose, say, $x$

- each agent $j$ votes accept ($a$) or reject ($r$)

- if $\{j \mid v_j = a\} \in \mathcal{M}$, then the game ends with outcome $(x, t)$, and payoffs are
  $$ (1 - \delta)^{t-1} u_j(q) + \delta^{t-1} u_j(x) $$

- otherwise, the game continues to period $t + 1$ and is repeated

- if the game continues ad infinitum, then payoffs are $u_j(q)$
Bargaining protocol (cont.)

- Think of payoffs from \((x, t)\) as the discounted sum of the flow of payoffs

\[
\sum_{t=1}^{\infty} \frac{u_i(q)}{1 + \delta} \frac{u_i(q)}{1 + \delta} \cdots \frac{u_i(x)}{1 + \delta} \frac{u_i(x)}{1 + \delta} \cdots
\]

Two models of status quo

(A) the status quo is just an alternative, \(q \in X\)

(B) the status quo is “bad,” i.e., for all \(i \in N\) and all \(x \in X\), \(u_i(x) \geq u_i(q)\).
Stationary strategies

\[ p_i \in X \quad \text{pure proposal strat.} \]

\[ \pi_i \in \mathcal{P}(X) \quad \text{mixed proposal strat.} \]

\[ A_i \subseteq X \quad \text{acceptance set} \]

\[ A_C = \bigcap_{i \in C} A_i \quad \text{accept. set for } C \]

\[ A = \bigcup_{C \in \mathcal{M}} A_C \quad \text{social accept. set} \]

\[ v_i(\sigma) \quad \text{continuation value} \]

\[ (1 - \delta)u_i(q) + \delta v_i(\sigma) \quad \text{reservation value} \]
Stationary strategies (cont.)

- We say $\sigma$ exhibits delay if, for some $i \in N$,
  \[ \pi_i(X \setminus A) > 0. \]
  Otherwise, it is no delay.

- Under (A), we say $\sigma$ is static if, for all $i \in N$,
  \[ \pi_i(A \setminus \{q\}) = 0. \]
Stationary equilibria

- Voting strategies are stage-game undominated: for all \( i \in N \),

\[
A_i^* = \{ x \mid u_i(x) \geq (1 - \delta)u_i(q) + \delta v_i(\sigma^*) \}.
\]

- Proposal strategies are optimal: for all \( i \in N \),

\[
\pi_i^*(\text{arg max}\{u_i(y) \mid y \in A^*\}) = 1
\]

whenever

\[
\sup\{u_i(y) \mid y \in A^*\} > (1 - \delta)u_i(q) + \delta v_i(\sigma^*),
\]

and so on.
One-dimensional example

• Let $n = 3$, even chance recognition probabilities, quadratic utilities.

\[ \Delta = \sqrt{\frac{(1 - \delta)(\tilde{x}_2 - q)^2}{1 - 2\delta/3}}. \]

• Note
  
  – unique stationary equilibrium, no-delay, pure strategies
  
  – $\delta \to 1$ implies $\Delta \to 0$
  
  – $q \to \tilde{x}_2$ implies $\Delta \to 0$
Existence and continuity

- **Theorem:** There exists a no-delay stationary equilibrium.

- **Theorem:** If $X \subseteq \mathbb{R}$, then every no-delay stationary equilibrium is in pure strategies.

- **Theorem:** The correspondence of no-delay stationary equilibrium mixed proposal strategies is upper hemicontinuous in the parameters of the model.
Delay

• **Theorem:** Under (B), all stationary equilibria are no-delay.

• Remark: The above result does not rely on strict concavity.

• **Theorem:** Assume \( \delta > 0 \). Under (A), if \( \sigma \) is a stationary equilibrium, then \( \sigma \) is static or no-delay.

• Remark: Under (A), delay may occur if \( \delta = 0 \) or if agents are risk neutral.
Delay (cont.)

- **Theorem:** Under (A), there exists a static stationary equilibrium if and only if $q \in K$.

- Thus,
  - when the core is empty (in multiple dimensions), every stationary equilibrium is no-delay.
  - in one dimension, unless $q \in K$, no stationary equilibrium is static, so all stationary equilibria are no-delay.

- **Theorem:** Assume $X \subseteq \mathbb{R}$ and $\delta > 0$. If $q \notin K$, then $q \notin A^*$ in every stationary equilibrium.
Uniqueness in one dimension

- **Theorem:** Assume $X \subseteq \mathbb{R}$ and each $u_i$ is quadratic. There is a unique no-delay stationary equilibrium.

- **Theorem:** Assume $X \subseteq \mathbb{R}$. If $\hat{\sigma}$ and $\tilde{\sigma}$ are stationary equilibria, then either $\hat{A} \subseteq \tilde{A}$ or $\tilde{A} \subseteq \hat{A}$.

- Thus, there is a unique minimal and a unique maximal no-delay stationary equilibrium.

- **Remark:** With non-quadratic utilities, there may be multiple no-delay stationary equilibria.
Core equivalence in one dimension

- **Asymptotic Median Voter Theorem I:**
  Assume $X \subseteq \Re$. Let $\delta^m \to 1$, and let
  \{$\sigma^m$\} be a sequence of stationary equilibria. Then $A^m \to K$.

- **Asymptotic Median Voter Theorem II:**
  Assume $X \subseteq \Re$ and $\delta > 0$. Let $\sigma^*$ be any
  stationary equilibrium. Under (A),
  
  – if \{$q$\} $\neq K$, then $q \notin A^*$
  
  – if \{$q$\} = $K$, then $A^* = \{$q$\}$, so the status quo is the unique stationary equilibrium outcome

  – if $q \to K$, then $A^* \to K$. 

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Folk theorems in bargaining

• Let $X$ be the unit simplex in $\mathbb{R}^n$, so $x = (x_1, \ldots, x_n)$ is an allocation of a “dollar” to the agents. Let $u_i(x) = x_i$. Then every allocation can be supported as a subgame perfect equilibrium outcome when the agents are patient enough in...

  – the multi-agent Rubinstein bargaining model

  – the Baron-Ferejohn model of majority bargaining.

• Is stationarity critical for the asymptotic median voter results? Is there a folk theorem for the one-dimensional bargaining model?
Continuation lotteries

• We can summarize future play by a “continuation lottery.”

• For example, suppose in odd periods, there is a $1/3$ chance that $x$ passes and $2/3$ chance of delay; in even periods, there is a $1/4$ chance of $x'$ and $3/4$ chance of delay.

• Then agent $i$’s expected payoff $U_i$ at the beginning of period 1 satisfies:

$$\frac{U_i}{1 - \delta} = \frac{1}{3} [u_i(x) + \delta u_i(x) + \cdots]$$
$$+ \frac{2}{3} [u_i(q) + \frac{\delta}{4} [u_i(x') + \delta u_i(x') + \cdots]]$$
$$+ \frac{3\delta}{4} [u_i(q) + \frac{\delta U_i}{1 - \delta}]]$$
Continuation lotteries (cont.)

- Then agent $i$'s expected payoff $U_i$ at the beginning of period 1 can be written as

$$U_i = \alpha u_i(x) + \beta u_i(x') + \gamma u_i(q)$$

where

$$\alpha = \frac{2}{6 - 3\delta^2}$$
$$\beta = \frac{\delta}{6 - 3\delta^2}$$
$$\gamma = \frac{4 - \delta - 3\delta^2}{6 - 3\delta^2}$$

and

$$\alpha + \beta + \gamma = 1.$$  

- That is, $U_i = E_\lambda[u_i]$, where $\lambda$ is the continuation lottery on $X$. 
Voting subgames

• In stationary equilibria, the voting game reduces to a binary vote, and stage-game weak dominance is effective.

• Suppose $x$ has been proposed.

\[
\begin{array}{c|cc}
 & a & r \\
\hline
a & x & x \\
r & x & \lambda \\
a & & \\
\end{array}
\quad
\begin{array}{c|cc}
 & a & r \\
\hline
a & x & \lambda \\
r & \lambda & \lambda \\
\end{array}
\]

If row strictly prefers $x$ to continuing, then voting $r$ is weakly dominated in the stage game.

• In fact, $x$ survives elimination if and only if it is weakly majority-preferred to continuing.
Voting subgames (cont.)

- Not so when stationarity is dropped.

- Suppose payoffs are as follows.

<table>
<thead>
<tr>
<th>row</th>
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<th>matrix</th>
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<tbody>
<tr>
<td>$\lambda_1$</td>
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<td>$\lambda_3$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>$\lambda_4$</td>
<td>$\lambda_4$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$\lambda_3$</td>
<td>$\lambda_1$</td>
</tr>
</tbody>
</table>

Then $\lambda_4$ is a strict (therefore undominated) Nash equilibrium outcome, but all agents strictly prefer $x$. 
Voting subgames (cont.)

- We model voting as **sequential**: this allows us to drop the stage-game weak dominance refinement and leaves stationary equilibrium outcomes unchanged.

- A **key property** we want: the median $\tilde{x}_k$ passes with positive probability if it is proposed.

- For this, we assume:
  - The order of voting is determined randomly after the proposer is determined.
  - Every order of voting has a positive, fixed* probability.
Subgame perfect equilibria

• A strategy $\sigma$ is a pure strategy subgame perfect equilibrium if there do not exist an agent, a history, and a deviation from $\sigma$ that increases that agent’s expected payoff following that history.

• Let $X(\delta)$ consist of alternatives $x$ such that:

  there exists a proposer history $h$ and a pure strategy subgame perfect equilibrium $\sigma$ such that $x$ passes with positive probability from $h$ in $\sigma$.

• Let $V_i(\delta)$ consist of payoffs $r$ such that:

  there exists a proposer history $h$ and a pure strategy subgame perfect equilibrium $\sigma$ such that $i$’s expected payoff is $r$ from $h$ in $\sigma$. 
Main result

• Given a sequence of sets \( \{Y^m\} \) in Euclidean space and an element \( x \), we write \( Y^m \to x \) if \( \sup_{y \in Y^m} ||y - x|| \to 0 \).

• **Asymptotic MVT I:** Let \( \{\delta^m\} \) be a sequence of discount factors converging to one. Then
  
  - \( X(\delta^m) \to \tilde{x}_k \),
  
  - for all \( i \in N \), \( V_i(\delta^m) \to u_i(\tilde{x}_k) \).

• That is,
  
  - subgame perfect equilibrium outcomes converge to the median ideal point,
  
  - equilibrium delay becomes negligible.
Intuition for proof

• Suppose $X(\delta^m)$ does not converge to the median. Letting*

\[
\begin{align*}
x^m &= \min X(\delta^m) \\
\overline{x}^m &= \max X(\delta^m),
\end{align*}
\]

suppose $x^m \to x$ and $\overline{x}^m \to \overline{x}$.

• A typical situation is below.

• For each $m$, concavity implies that there is a majority $C^m$ such that either $x^m$ is the worst thing in the interval $[x^m, \overline{x}^m]$ for every member of $C^m$ or $\overline{x}^m$ is worst for all.
Intuition for proof (cont.)

- **Lemma:** Suppose $x \in X(\delta^m)$ is proposed in some equilibrium and passes with positive probability. Then there is an agent $i^m$ in $C^m$ who weakly prefers $x$ to the continuation lottery from rejecting the proposal.

- Apply the lemma with $x = \bar{x}^m$ to get an agent $i^m$ from each $C^m$ who weakly prefers $\bar{x}^m$ to the continuation lottery from rejection. But...

- **Lemma:** In every subgame perfect equilibrium, if the median $\tilde{x}_k$ is proposed, then it passes with positive probability (bounded above zero).
Intuition for proof (cont.)

- As a consequence, agent $i^m$ can obtain the median with positive probability when selected to propose.

- So why would the agent $i^m$ be willing to vote for his/her worst alternative rather than reject it and at least obtain the median $\tilde{x}_k$ with positive probability?

- It must be that the status quo is inferior to $\bar{x}^m$, and that delay makes the agent worse off.

- But we show that, as the agent becomes patient, delay becomes inconsequential, a contradiction.
Subtle technical point

- Let ideal points of 1, 2, 3, 4, 5 be in order of voter indices. Suppose the median $\tilde{x}_3$ is proposed and 3 has voted $a\ldots$
<table>
<thead>
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<th></th>
<th>1</th>
<th>2</th>
<th>4</th>
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<tr>
<td>$\lambda'$</td>
<td>$\lambda'$</td>
<td>$\tilde{x}_3$</td>
<td>$\lambda$</td>
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<tr>
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<td>$\tilde{x}_3$</td>
<td>$\lambda'$</td>
<td>$\tilde{x}_3$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\lambda$</td>
<td></td>
</tr>
</tbody>
</table>
Technicality (cont.)

- Now suppose the voting order alternates...
Technicality (cont.)

• By concavity, if agent 5 weakly prefers $\lambda$ to the median $\tilde{x}_3$, and if agent 2 has the same weak preference, then the mean of $\lambda$ must be equal to $\tilde{x}_3$.

• By strict concavity, the continuation lottery $\lambda$ must have zero variance, so it is the point mass on $\tilde{x}_3$.

• Therefore, the only possible equilibrium outcome of the first two 2:5 voting subgames is the median.
Technicality (cont.)

• Then the game reduces to...

\[ \tilde{x}_3 \]

... and the same logic applies.

• Therefore, the median voter can (and will) obtain her ideal point by voting \( a \) in equilibrium.
Extensions of Main Result

- Even number of agents: okay, but we need one of the median voters to "break ties."

- Concavity: strict concavity can be weakened to allow for piece-wise linear.

- Heterogeneous discount factors: we just need discount factors to converge to one at a "uniform enough" rate, i.e.,
  \[
  \frac{\ln(\delta^m_i)}{\ln(\delta^m_j)} \to 1
  \]
  for all \(i, j \in N\).
Extensions (cont.)

- Variable recognition probabilities: These can vary arbitrarily with histories, as long as each agent’s probability is bounded strictly above zero.

- Variable probability of voting orders: Can vary arbitrarily with histories, as long as voting alternates with probability bounded strictly above zero.

- Mixed strategy equilibria: mixed proposal strategies are not a problem; mixed voting strategies introduce some difficulties.

- Asymptotic MVT II...
Quick conclusion

- In contrast to the “divide the dollar” model of bargaining, an anti-folk theorem holds for the one-dimensional model.

- The asymptotic MVT I for stationary equilibria extends quite generally.

- Delay becomes negligible as agents become patient.