Bargaining Foundations of the Median Voter Theorem

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Outline of talk

• brief introduction
• bargaining literature
• model
• voting subgames
• main result
• intuition for proof
• extensions
• quick conclusion
Brief introduction

• The Median Voter Theorem: If an odd number of voters have single-peaked preferences over a unidimensional space, then the median ideal point is strictly majority-preferred to every other point.

• Non-cooperative underpinnings:
  
  large groups $\rightarrow$ elections  
  (Downs)

  small groups $\rightarrow$ committees and legislatures  
  (bargaining)

• Do bargaining equilibria lead to the median ideal point? If so, how generally?
## The bargaining literature

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Rubinstein</th>
<th>Binmore</th>
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<tbody>
<tr>
<td>players</td>
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<td>( n = 2 )</td>
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<tr>
<td>alternatives</td>
<td>( \Delta/\mathcal{R} )</td>
<td>( \Delta/\mathcal{R} )</td>
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<td>random</td>
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<td>status quo</td>
<td>bad</td>
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<tr>
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<tr>
<td>proposer</td>
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<td>random</td>
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<td>status quo</td>
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<td>whatever</td>
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The bargaining literature (cont.)

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<th>B-F</th>
<th>B-D</th>
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<td>Y</td>
<td>Y</td>
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<td>N</td>
<td>N</td>
<td>N</td>
<td>N*</td>
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<td>uniqueness of stationary eq.</td>
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<td>Y</td>
<td>N</td>
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<td>n/a</td>
<td>n/a</td>
<td>Y</td>
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<td>folk thm</td>
<td>N*</td>
<td>N*</td>
<td>Y</td>
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Elements of the model

\[ N \] set of \( n \) agents (\( n \) odd)*

\[ X \] set of alternatives
(a compact interval)

\[ u_i : X \rightarrow \mathbb{R} \] stage utility
(cont., strictly concave)*

\[ \tilde{x}_i \] \( i \)'s ideal point
(median is \( \tilde{x}_k \))

\[ q \] status quo (in \( X \))*

\[ \delta \] discount factor
(common, less than one)*

\[ \rho_i \] recognition prob.
(positive, fixed)*
The bargaining protocol in period \( t \)

- an agent \( i \) is selected with probability \( \rho_i \) to propose, say, \( x \)

- each agent \( j \) votes accept (\( a \)) or reject (\( r \))

- if \( \#\{j \mid v_j = a\} > n/2 \), then the game ends with outcome \((x, t)\), and payoffs are
  \[
  (1 - \delta)^{t-1}u_j(q) + \delta^{t-1}u_j(x)
  \]

- otherwise, the game continues to period \( t + 1 \) and is repeated

- if the game continues ad infinitum, then payoffs are \( u_j(q) \)
The bargaining protocol (cont.)

- Think of payoffs from \((x, t)\) as the discounted sum of the flow of payoffs

\[
\underbrace{u_i(q) \ u_i(q) \ \cdots \ u_i(q)}_{t-1 \text{ periods}} \quad \underbrace{u_i(x) \ u_i(x) \ u_i(x) \ \cdots}_{\text{ad infinitum}}
\]

Continuation lotteries

- We can summarize future play by a “continuation lottery.”

- For example, suppose in odd periods, there is a 1/3 chance that \(x\) passes and 2/3 chance of delay; in even periods, there is a 1/4 chance of \(x'\) and 3/4 chance of delay.
Continuation Lotteries (cont.)

• Then agent $i$'s expected payoff $U_i$ at the beginning of period 1 can be written as

$$U_i = \alpha u_i(x) + \beta u_i(x') + \gamma u_i(q)$$

where

$$\alpha = \frac{2}{6 - 3\delta^2}$$
$$\beta = \frac{\delta}{6 - 3\delta^2}$$
$$\gamma = \frac{4 - \delta - 3\delta^2}{6 - 3\delta^2}$$

and

$$\alpha + \beta + \gamma = 1.$$ 

• That is, $U_i = E_\lambda[u_i]$, where $\lambda$ is the continuation lottery on $X$. 

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Voting subgames

• In stationary equilibria, the voting game reduces to a binary vote, and stage-game weak dominance is effective.

• Suppose $x$ has been proposed.

$$
\begin{array}{c|cc}
 & a & r \\
\hline
a & x & x \\
r & x & \lambda \\
\end{array}
\quad
\begin{array}{c|cc}
 & a & r \\
\hline
a & x & \lambda \\
r & \lambda & \lambda \\
\end{array}
$$

If row strictly prefers $x$ to continuing, then voting $r$ is weakly dominated in the stage game.

• In fact, $x$ survives elimination if and only if it is weakly majority-preferred to continuing.
Voting subgames (cont.)

- Not so when stationarity is dropped.

\[
\begin{array}{|c|c|}
\hline
a & x \\
\hline
r & \lambda_1 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|}
\hline
a & x \\
\hline
r & \lambda_2 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|}
\hline
a & x \\
\hline
r & \lambda_3 \lambda_4 \\
\hline
\end{array}
\]

- Suppose payoffs are as follows.

<table>
<thead>
<tr>
<th>row</th>
<th>column</th>
<th>matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>\lambda_1</td>
<td>\lambda_2</td>
<td>\lambda_3</td>
</tr>
<tr>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>\lambda_4</td>
<td>\lambda_4</td>
<td>\lambda_4</td>
</tr>
<tr>
<td>\lambda_2</td>
<td>\lambda_3</td>
<td>\lambda_1</td>
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</table>

Then \( \lambda_4 \) is a strict (therefore undominated) Nash equilibrium outcome, but all agents strictly prefer \( x \).
Voting subgames (cont.)

• We model voting as sequential: this allows us to drop the stage-game weak dominance refinement and leaves stationary equilibrium outcomes unchanged.

• A key property we want: the median $\tilde{x}_k$ passes if it is proposed.

• For this, we assume:
  
  -- The order of voting is determined randomly before and/or after the proposer is determined.

  -- With positive, fixed* probability, the median voter votes first, and voting then alternates from either side of the median.
The problem: Let ideal points of 1, 2, 3, 4, 5 be in order of voter indices. Suppose the median $\tilde{x}_3$ is proposed and 3 has voted $a$...
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>$\lambda'$</td>
<td>$\lambda'$</td>
<td>$\tilde{x}_3$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\tilde{x}_3$</td>
<td>$\tilde{x}_3$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda$</td>
<td></td>
</tr>
</tbody>
</table>
Voting subgames (cont.)

- Now suppose the voting order alternates...

```
  3
    a
  /
  1
  /  
 4  4
    / 
   a  a  a  a
  /   /   /   /
  r  r  r  r
 /     /     /
\x3  \x3  \x3  \x3
  /     /     /
 a     a     a     a
  /     /     /
\x3  \x3  \x3  \x3
  /
 a
```

\[ \tilde{x}_3 \]
Voting subgames (cont.)

- By concavity, if agent 5 weakly prefers \( \lambda \) to the median \( \tilde{x}_3 \), and if agent 2 has the same weak preference, then the mean of \( \lambda \) must be equal to \( \tilde{x}_3 \).

- By strict concavity, the continuation lottery \( \lambda \) must have zero variance, so it is the point mass on \( \tilde{x}_3 \).

- Therefore, the only possible equilibrium outcome of the first two 2:5 voting subgames is the median.
Voting subgames (cont.)

• Then the game reduces to...

\[ \tilde{x}_3 \]

... and the same logic applies.

• Therefore, the median voter can (and will) obtain her ideal point by voting \( a \) in equilibrium.
Subgame perfect equilibria

• A strategy $\sigma$ is a pure strategy subgame perfect equilibrium if there do not exist an agent, a history, and a deviation from $\sigma$ that increases that agent’s expected payoff following that history.

• Let $X(\delta)$ consist of alternatives $x$ such that:

  there exists a proposer history $h$ and a pure strategy subgame perfect equilibrium $\sigma$ such that $x$ passes with positive probability from $h$ in $\sigma$.

• Let $V_i(\delta)$ consist of payoffs $r$ such that:

  there exists a proposer history $h$ and a pure strategy subgame perfect equilibrium $\sigma$ such that $i$’s expected payoff is $r$ from $h$ in $\sigma$.
The main result

- Given a sequence of sets \( \{Y^m\} \) in Euclidean space and an element \( x \), we write \( Y^m \to x \) if \( \sup_{y \in Y^m} ||y - x|| \to 0 \).

- Theorem: Let \( \{\delta^m\} \) be a sequence of discount factors converging to one. Then
  
  - \( X(\delta^m) \to \tilde{x}_k \),
  
  - for all \( i \in N \), \( V_i(\delta^m) \to u_i(\tilde{x}_k) \).

- That is,
  
  - subgame perfect equilibrium outcomes converge to the median ideal point,
  
  - equilibrium delay becomes negligible.
Intuition for proof

• Suppose $X(\delta^m)$ does not converge to the median. Letting*

$$
\underline{x}^m = \min X(\delta^m) \\
\overline{x}^m = \max X(\delta^m),
$$

suppose $\underline{x}^m \to \underline{x}$ and $\overline{x}^m \to \overline{x}$.

• A typical situation is below.

• For each $m$, concavity implies that there is a majority $C^m$ such that either $\underline{x}^m$ is the worst thing in the interval $[\underline{x}^m, \overline{x}^m]$ for every member of $C^m$ or $\overline{x}^m$ is worst for all.
• Lemma: Suppose $x \in X(\delta^m)$ is proposed in some equilibrium and passes with positive probability. Then there is an agent $i^m$ in $C^m$ who weakly prefers $x$ to the continuation lottery from rejecting the proposal.

• Apply the lemma with $x = \bar{x}^m$ to get an agent $i^m$ from each $C^m$ who weakly prefers $\bar{x}^m$ to the continuation lottery from rejection. But...

• Lemma: In every subgame perfect equilibrium, if the median $\tilde{x}_k$ is proposed, then it passes with positive probability (bounded above zero).
Intuition for proof (cont.)

• As a consequence, agent $i^m$ can obtain the median with positive probability when selected to propose.

• So why would the agent $i^m$ be willing to vote for his/her worst alternative rather than reject it and at least obtain the median $\tilde{x}_k$ with positive probability?

• It must be that the status quo is inferior to $x^m$, and that delay makes the agent worse off.

• But we show that, as the agent becomes patient, delay becomes inconsequential, a contradiction.
Extensions

- Even number of agents: okay, but we need one of the median voters to “break ties.”

- Concavity: strict concavity can be weakened to allow for piece-wise linear.

- Bad status quo: we can take the status quo out of $X$ as long as every agent strictly prefers the median to the status quo.

- Heterogeneous discount factors: we just need discount factors to converge to one at a “uniform enough” rate, i.e.,

$$\frac{\ln(\delta^m_i)}{\ln(\delta^m_j)} \to 1$$

for all $i, j \in N$. 

Extensions (cont.)

• Variable recognition probabilities: These can vary arbitrarily with histories, as long as each agent’s probability is bounded strictly above zero.

• Variable probability of alternating voting order: Can vary arbitrarily with histories, as long as voting alternates with probability bounded strictly above zero.

• Mixed strategy equilibria: mixed proposal strategies are not a problem; mixed voting strategies introduce some complications, but these should not be a problem.
Quick conclusion

• In contrast to the distributive model of bargaining, an anti-folk theorem holds for the unidimensional model.

• The asymptotic median voter theorem for stationary equilibria extends quite generally.

• Delay becomes negligible as agents become patient.

• The median voter theorem for small groups is pretty “safe.”