DOMINANCE-BASED SOLUTIONS
FOR STRATEGIC FORM GAMES

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INTRODUCTION

WE DO NOT ASSUME:

- players use mixed strategies/formulate probabilistic beliefs about opponents’ strategies;
- expected utility maximization;
- common knowledge of mixed strategies;

WE DO ASSUME:

- players’ decisions can be modeled by choice sets, rather than mixed strategies;
- common knowledge of choice sets, rather than of mixed strategies;
- choice sets adhere to some notion of dominance, which may or may not involve expected utility maximization.

WE GET:

- extension of the classical theory of choice/consumer theory to strategic situations;
- broader scope of application, including non-Bayesian players;
- generalization of Shapley’s approach to two-player zero-sum games;
- unification of results on rationalizability;
- unified approach to the theory of tournaments.
SHAPLEY’S SADDLES

SET-UP:

• Let \((X_1, X_2, u_1, u_2)\) be a finite two-player zero-sum game.

• A **generalized saddle** is a subset \(Y_1 \times Y_2 \subseteq X_1 \times X_2\) such that, for each \(x_1 \notin Y_1\), there exists \(y_1 \in Y_1\) satisfying

\[
(\forall x_2 \in Y_2)(u_1(y_1, x_2) > u_1(x_1, x_2)),
\]

and likewise for player 2.

• A **saddle** is a generalized saddle that is minimal with respect to set-inclusion: if \(Z_1 \times Z_2\) is a generalized saddle with \(Z_1 \subseteq Y_1\) and \(Z_2 \subseteq Y_2\), then \(Z_1 = Y_1\) and \(Z_2 = Y_2\).

• A **generalized weak saddle** is a subset \(Y_1 \times Y_2 \subseteq X_1 \times X_2\) such that, for each \(x_1 \notin Y_1\), there exists \(y_1 \in Y_1\) satisfying

\[
(\forall x_2 \in Y_2)(u_1(y_1, x_2) \geq u_1(x_1, x_2))
\]

with strict inequality for some \(x_2 \in Y_2\), and likewise for player 2.

• A **weak saddle** is a minimal generalized weak saddle.
SHAPLEY’S SADDLES (CONT.)

THEOREM (Shapley): There is exactly one saddle.

REMARKS:

- The weak saddle is not generally unique.
- The saddle possesses “internal” as well as “external” stability: if \(Y_1 \times Y_2\) is a saddle, there do not exist distinct \(x_1, y_1 \in Y_1\) such that

\[
(\forall x_2 \in Y_2)(u_1(y_1, x_2) > u_1(x_1, x_2)),
\]

and likewise for player 2.
- The same is not true of the weak saddle.

DIRECTIONS:

- multi-player, non-zero-sum games,
- non-minimal pairs \(Y_1 \times Y_2\) possessing “external” and “internal” stability properties,
- other dominance concepts,
- infinite games.
STRATEGIC FORM GAMES

$I$  
set of players (finite)

$i, j$  
elements of $I$

$X_i$  
$i$'s strategy set (finite)

$x_i, y_i, z_i$  
elements of $X_i$

$X = \prod_{i \in I} X_i$  
set of strategy profiles

$x, y, z$  
elements of $X$

$Y_i, Z_i$  
subsets of $X_i$

$Y = \prod_{i \in I} Y_i$  
product set

$Y_{-i} = \prod_{j \neq i} Y_j$  
partial product set

$x_{-i}, y_{-i}, z_{-i}$  
elements of $Y_{-i}$

$u_i(x)$  
i's payoff from $x$

$Q_i$  
binary relation on $X_i$
FORMULATION OF CHOICE SETS

DEFINITION: A set $Y_i$ possesses the maximality property with respect to $Q_i$ if

$$(x_i \in Y_i) \Leftrightarrow (\neg \exists y_i \in X_i)(y_i Q_i x_i \land \neg x_i Q_i y_i).$$

DEFINITION:

(i) A set $Y_i$ possesses the inner solution property with respect to $Q_i$ if

$$(x_i \in Y_i) \Rightarrow (\forall y_i \in Y_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

(ii) A set $Y_i$ possesses the outer solution property with respect to $Q_i$ if

$$(x_i \notin Y_i) \Rightarrow (\exists y_i \in Y_i \setminus \{x_i\})(y_i Q_i x_i).$$

(iii) A set $Y_i$ possesses the solution property with respect to $Q_i$ if

$$(x_i \in Y_i) \Leftrightarrow (\forall y_i \in Y_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

PROPOSITION 1: If $Q_i$ is transitive and irreflexive, then $Y_i$ possesses the solution property with respect to $Q_i$ if and only if it possesses the maximality property with respect to $Q_i$. 
DEFINITION: A dominance structure is a mapping $Q$ from players $i$ and partial product sets $Y_{-i}$ to binary relations $Q_i(Y_{-i})$ on $X_i$.

DEFINITION: A set $Y$ is a $Q$-solution if, for all $i$, $Y_i$ possesses the solution property with respect to $Q_i(Y_{-i})$.

DEFINITION: Let $Q$ be a dominance structure.

(i) $Q$ satisfies irreflexivity if, for all $i$ and all $Y_{-i}$, $Q_i(Y_{-i})$ is irreflexive.

(ii) $Q$ satisfies transitivity if, for all $i$ and all $Y_{-i}$, $Q_i(Y_{-i})$ is transitive.

(iii) $Q$ satisfies monotonicity if, for all $i$, all $x_i, y_i \in X_i$, and all $Y_{-i} \subseteq Z_{-i}$,

$$x_iQ_i(Z_{-i})y_i \Rightarrow x_iQ_i(Y_{-i})y_i.$$ 

PROPOSITION 2: Let $Q$ be transitive. A set $Y$ is a $Q$-solution if and only if

$(\star)$ for all $i$, $Y_i$ is a minimal subset of $X_i$ possessing the outer solution property with respect to $Q_i(Y_{-i})$. 


SOME DOMINANCE STRUCTURES

Shapley Dominance

\[ x_i S_i(Y_{-i}) y_i \iff (\forall x_{-i} \in Y_{-i})(u_i(x_i, x_{-i}) > u_i(y_i, x_{-i})) \]

Nash Dominance

\[ x_i N_i(Y_{-i}) y_i \iff (\forall x_{-i} \in Y_{-i})(u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i})) \]

weak Shapley Dominance

\[ x_i W_i(Y_{-i}) y_i \iff (x_i N_i(Y_{-i}) y_i) \wedge (\neg y_i N_i(Y_{-i}) x_i) \]

CLAIM:

- \( \{x\} \) is a \( N \)-solution if and only if \( x \) is a Nash equilibrium.

- \( \{x\} \) is a \( S \)-solution if and only if \( \{x\} \) is a \( W \)-solution if and only if \( x \) is a strict Nash equilibrium.

CLAIM:

- The dominance structures \( N, W, \) and \( S \) are transitive. \( W \) and \( S \) are irreflexive, while \( N \) is not.

- The dominance structures \( S \) and \( N \) are monotonic, while \( W \) is not.
DOMINANCE STRUCTURES (CONT.)

EXAMPLE (distinct solutions):

\[
\begin{array}{ccc}
\text{e} & \text{f} & \text{g} \\
\hline
\text{a} & (1,-1) & (-1,1) & (0,0) \\
\text{b} & (-1,1) & (1,-1) & (0,0) \\
\text{c} & (-1,1) & (1,-1) & (0,0) \\
\text{d} & (-1,1) & (0,0) & (0,0) \\
\end{array}
\]

CLAIM:

- \{a, b\} \times \{e, f, g\} is a \(N\)-solution but not a \(W\)-solution or a \(S\)-solution.

- \{a, b, c\} \times \{e, f, g\} is a \(W\)-solution but not a \(N\)-solution or a \(S\)-solution.

- \{a, b, c, d\} \times \{e, f, g\} is a \(S\)-solution, but not a \(N\)-solution or a \(W\)-solution.
EXISTENCE OF $Q$-SOLUTIONS

DEFINITION: A set $Y$ is an **outer $Q$-solution** if, for all $i$, $Y_i$ possesses the outer solution property with respect to $Q_i(Y_{-i})$.

DEFINITION: A set $Y$ is **minimal** among a class of sets if, for all $Z$ in that class, $Z \subseteq Y$ implies $Z = Y$. It is **maximal** if, for all $Z$ in that class, $Z \supseteq Y$ implies $Z = Y$.

PROPOSITION 3: Let $Q$ be transitive and monotonic. If $Y$ is a minimal outer $Q$-solution then it is a (minimal) $Q$-solution.

PROOF: It suffices to show that a minimal outer $Q$-solution satisfies

\[(\star) \text{ for all } i, \ Y_i \text{ is a minimal subset of } X_i \text{ possessing the outer solution property with respect to } Q_i(Y_{-i}).\]

Suppose there is some $i$ and $Z_i \subset \subset Y_i$ such that $Z_i$ possesses the outer solution property with respect to $Q_i(Y_{-i})$. Consider $j \neq i$ and $x_j \notin Y_j$. The outer solution property with respect to $Q_j(Y_{-j})$ implies the existence of $y_j \in Y_j$ such that $y_jQ_j(Y_{-j})x_j$. Then monotonicity implies $y_jQ_j(Z_i \times Y_{-i,j})x_j$, and $Z_i \times Y_{-i}$ is an outer $Q$-solution, contradicting minimality of $Y$. //
EXISTENCE (CONT.)

PROPOSITION 4: If $Q$ is transitive and monotonic then there is at least one $Q$-solution.

PROOF: $X$ is an outer $Q$-solution. Since $X$ is finite, a minimal outer $Q$-solution exists. //

COROLLARY 1: There exist a $S$-solution and a $N$-solution.

EXAMPLE (non-existence of $W$-solution):

<table>
<thead>
<tr>
<th></th>
<th>$c$</th>
<th>$d$</th>
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<tbody>
<tr>
<td>$a$</td>
<td>(2,1)</td>
<td>(1,2)</td>
</tr>
<tr>
<td>$b$</td>
<td>(1,2)</td>
<td>(1,1)</td>
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</tbody>
</table>

Note that $aW_1\{c,d\}b$ and $dW_2\{a\}c$, so player 2 cannot choose $\{c,d\}$; $aW_1\{c\}b$ and $dW_2\{a\}c$, so 2 cannot choose $\{c\}$; etc.
COMPARISONS OF $Q$-SOLUTIONS

DEFINITION: $Q$ is **stronger** than $Q'$ if, for all $i$, $x_i$, $y_i$, and $Y_{-i}$, $x_iQ_i(Y_{-i})y_i \Rightarrow x_iQ'_i(Y_{-i})y_i$.

RELATIONSHIPS FOR ABOVE $Q$’S:

PROPOSITION 5: Let $Q$ be stronger than $Q'$.

(i) If $Q'$ is transitive and monotonic then every $Q$-solution includes a $Q'$-solution.

(ii) If $Q$ is transitive and monotonic then every $Q'$-solution is included in a $Q$-solution.*

RELATIONSHIPS FOR ABOVE $Q$’S:
COMPARISONS (CONT.)

EXAMPLE (N-solution vs. Nash equilibrium):

\[
\begin{array}{cccc}
| & c & d & e & f \\
|---|---|---|---|
| a | (1,0) & (1,10) & (1,11) & (1,-1) \\
| b | (1,10) & (1,0) & (1,-1) & (1,11) \\
\end{array}
\]

The mixed strategies \((1/2, 1/2)\) and \((1/2, 1/2, 0, 0)\) form an equilibrium with minimal support, \(\{a, b\} \times \{c, d\}\). But the only \(N\)-solutions are \(\{a\} \times \{e\}\) and \(\{b\} \times \{f\}\).
MIXED STRATEGY NASH EQUILIBRIUM

NOTATION: If $p = (p_i)_{i \in I}$ is a profile of mixed strategies, let

$$\sigma(p) = \{x \in X | \Pi_{i \in I} p_i(x_i) > 0\}$$

denote the support of $p$.

DEFINITION:

(i) A set $Y_i$ possesses the **inner maximality property** with respect to $Q_i$ if

$$(x_i \in Y_i) \Rightarrow (\neg \exists y_i \in X_i)(y_i Q_i x_i \land \neg x_i Q_i y_i).$$

(ii) A set $Y$ is an **inner $Q$-maximum** if, for all $i$, $Y_i$ possesses the inner maximality property with respect to $Q_i(Y_{-i})$.

NOTATION: Given a mixed strategy profile $p$, write $x_i Q_i^p(Y_{-i}) y_i$ if the expected payoff of $x_i$ is greater than that of $y_i$, calculated with respect to $\Pi_i p_i$ conditional on $Y$.

THEOREM *: Mixed strategy profile $p$ is an equilibrium if and only if $\sigma(p)$ is an inner $Q^p$-maximum.
THE RATIONALIZABILITY LITERATURE

DEFINITION:

(i) $x_i$ is **ordinally rationalizable** if there is some monotonic transformation of $u_i$ such that it is a best response to some profile of pure strategies; for each such strategy there is a monotonic transformation of payoffs that makes it a best response to some profile of pure strategies; and so on.

(ii) $x_i$ is **correlated rationalizable** if it is a best response to some profile of (possibly correlated) mixed strategies; every strategy played with positive probability is a best response to a profile of (possibly correlated) mixed strategies; and so on.

(iii) $x_i$ is **rationalizable** if it is a best response to some profile of (independent) mixed strategies; every strategy played with positive probability is a best response to a profile of (independent) mixed strategies; and so on.

(iv) $x_i$ is **point rationalizable** if it is a best response to some profile of pure strategies; each such strategy is a best response to some profile of pure strategies; and so on.
PROPOSITION 6: Under each criterion, a set \( Y \) consists of the rationalizable strategy profiles if and only if it is the unique maximal solution for a corresponding dominance structure . . .

Börgers Dominance

\[ x_iB_i(Y_{-i})y_i \iff \text{for all } Z_{-i} \subseteq Y_{-i} \text{ there is some } z_i \text{ such that } x_iW_i(Z_{-i})y_i. \]

Mixed Shapley Dominance

\[ x_iS^*_i(Y_{-i})y_i \iff y_i \text{ is strictly dominated over } Y_{-i} \text{ by some mixed strategy}. \]

Rationalizable Dominance

\[ x_iR_i(Y_{-i})y_i \iff y_i \text{ is a best response to no mixed strategy profile with support in } Y_{-i}. \]

Point Rationalizable Dominance

\[ x_iP_i(Y_{-i})y_i \iff y_i \text{ is a best response to no pure strategy profile in } Y_{-i}. \]
COMMON FEATURES:

- The rationalizable strategy profiles can be found by iterative deletion of dominated strategies.
- Order of elimination is irrelevant.

DEFINITION: $Q$ is **weakly irreflexive** if, for all $i$, all $x_i, y_i \in X_i$, and all $Y_{-i}$, $x_i Q_i (Y_{-i}) x_i$ implies $y_i Q_i (Y_{-i}) x_i$.

PROPOSITION 6: If $Q$ is weakly irreflexive, transitive, monotonic, and hard*, then

(i) the maximal $Q$-solution is unique;

(ii) it can be found by iterative deletion of $Q$-dominated strategies;

(iii) the order of elimination is irrelevant.

COROLLARY 2: There is exactly one maximal $S$-solution, the strategy profiles remaining after iterative deletion of strictly dominated strategies. Similarly for $B$, $S^*$, $R$, and $P$. 
SHAPLEY SETS

DEFINITION: $Y$ is a $Q$-set if it is a minimal $Q$-solution.

DEFINITION: A game is **equilibrium safe** if there exists a mixed strategy Nash equilibrium $p^* = (p_1^*, \ldots, p_n^*)$ such that, for all mixed strategy equilibria $p = (p_1, \ldots, p_n)$ and all $i$, $p_i^*$ is a best response to $p_{-i}$.

CLAIM: A game is equilibrium safe if
- there is a unique mixed strategy Nash equilibrium;
- there is a dominant strategy equilibrium;
- mixed strategy equilibria are interchangeable;
- it is a two-player zero-sum game.

PROPOSITION 7:

(i) If the $R$-set is unique then so are the $S$-set and $S^*$-set.

(ii) In an equilibrium safe game, the $R$-set is unique.

EXTENSION: If a game is order equivalent to an equilibrium safe game, it has a unique $S$-set.
EXAMPLE (order equivalence):

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<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tbody>
<tr>
<td>a</td>
<td>(5,1)</td>
<td>(1,5)</td>
<td>(2,2)</td>
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<td>b</td>
<td>(1,5)</td>
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<td>(2,2)</td>
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<td>d</td>
<td>(2,2)</td>
<td>(2,2)</td>
<td>(1,5)</td>
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</table>

This game has multiple $R$-sets and is not equilibrium safe, but it is order equivalent to an equilibrium safe game. (Change 2’s to 3’s.) Therefore, it has a unique $S$-set.

PROPOSITION 8: In a two-player game with weakly Pareto optimal payoffs, the $S$-set is unique.
APPLICATION TO TOURNAMENTS

SET-UP:

• Two parties, 1 and 2, choose policy platforms from a finite set, \( A \).

• A vote between the two platforms is taken, the winner given by the majority relation, \( M \).

• \( M \) is asymmetric and total: if \( a \neq b \) then \( aMb \) or \( bMa \).

• Parties care only about winning the election.

DEFINITION:

(i) A policy \( a \) is a **Condorcet winner** if, for all \( b \neq a \), \( aMb \).

(ii) A **topcycle set** is a minimal set \( B \) such that, for all \( a \in B \) and all \( b \notin B \), \( aMb \).

(iii) For \( B \subseteq A \) and \( a,b \in B \), \( a \) **covers** \( b \) over \( B \) if \( aMb \) and, for all \( c \in B \), \( bMc \) implies \( aMc \). Let \( UC(B) \) denote the elements of \( B \) that are not covered over \( B \) by any other elements of \( B \).

(iv) \( B \) is a **covering set** if \( UC(B) = B \) and, for all \( a \in A \setminus B \), \( a \notin UC(B \cup \{a\}) \).
Q-SETS IN TOURNAMENTS:

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<th>unique max.</th>
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<td>$S$</td>
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<td>x</td>
<td>x</td>
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<td>topcycle</td>
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EXISTENCE IN INFINITE GAMES

FORMALITIES:

(i) Let $N$ be an arbitrary set.

(ii) Let $X_i$ be a compact space.

(iii) Let $X$ and $X_{-i}$ have the product topologies.

DEFINITION:

(i) A set $Y$ is a $Q$-solution if, for all $i$, $Y_i$ is a minimal compact subset of $X_i$ possessing the outer solution property with respect to $Q_i(Y_{-i})$.

(ii) $Q$ is transitive-continuous if, for all $i \in N$, all $x_i, y_i, z_i \in X_i$, all nets $x_i^\alpha \to x_i$, and all compact subsets $Y_{-i} \subseteq X_{-i}$,

$$
(y_i Q_i(Y_{-i}) z_i) \land (\forall \alpha)(x_i^\alpha Q_i(Y_{-i}) y_i) \Rightarrow (x_i Q_i(Y_{-i}) z_i).
$$

PROPOSITION 9: $B$, $S^*$, $R$, and $P$ are transitive-continuous. If each $u_i$ is upper semi-continuous in $x_i$ then $S$, $W$, and $N$ are transitive-continuous.
EXISTENCE (CONT.)

PROPOSITION 10: If $Q$ is transitive-continuous and monotonic then there exists at least one $Q$-solution.

COROLLARY 3: There exist $S$, $N$, $B$, $S^*$, $R$, and $P$-solutions.