A Note on Uniqueness of Electoral Equilibrium
When the Median Voter Is Unobserved

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February 1, 2006

Abstract

This note considers the unidimensional electoral model in which two candidates seek to maximize their vote totals and do not observe the preferences of the voters. I show that there is a unique equilibrium, and in equilibrium both candidates locate at the median of the (normalized) sum of distributions of individual voter ideal policies. In fact, this equilibrium is unique among all mixed strategy equilibria.

JEL classification: C72, D72, D78.

Key words: median voter, probabilistic voting, mixed strategy, vote-maximization.

1 Introduction

This note considers an election between two office-motivated candidates who compete by locating in a unidimensional policy space. Voters have single-peaked preferences that are unobserved by the candidates. One formalization of office-motivation used in the literature is that candidates seek only to maximize their probabilities of winning. With this definition, Calvert (1985) proves that there is a unique pure strategy equilibrium, and that in equilibrium the candidates both locate at the median of the distribution of the median voter’s ideal point, generating a “median of medians” result. Bernhardt, Duggan, and Squintani (2005) prove that this equilibrium is actually unique in the class of all mixed

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strategy equilibria. This note provides a parallel result for the other formalization of office-motivation used in the literature, that of candidates who seek to maximize their vote totals. I show that there is a unique mixed strategy equilibrium, and that in equilibrium both candidates locate at the median of the (normalized) sum of distributions of individual voter ideal policies. Thus, we are back to a median-like result, but now the candidates locate at the median of the average distribution.

2 The Electoral Framework

We consider an election with just two political candidates, $A$ and $B$ (sometimes interpreted as parties), and we analyze an abstract model of campaigns: we assume that the candidates simultaneously announce policy positions $x_A$ and $x_B$ in a non-empty, closed, bounded interval $X \subseteq \mathbb{R}$, and we assume that the winning candidate is committed to his or her campaign promise. Following these announcements, a finite number $n$ of voters, denoted $i = 1, 2, \ldots, n$, cast their ballots, the winner being the candidate with the most votes. In case of a tie, the winner is determined by the flip of a fair coin. We do not allow abstention by voters.

Assume each voter $i$ has a single-peaked (continuous, strictly quasi-concave) utility function $u_i(\cdot, \theta_i)$, where $\theta_i$ is a preference parameter lying in a Euclidean space $\Theta$. Thus, for each $\theta_i$, $u_i(\cdot, \theta_i)$ admits a unique maximizer on $X$, which is $i$’s ideal policy. We assume that the vector $(\theta_1, \ldots, \theta_n)$ of parameters is a random variable from the candidates’ perspective, and we let $H_i$ denote the marginal distribution of voter $i$’s ideal policy. We do not assume that the random variables $\theta_i$ are independent, but we assume that $\sum_i H_i$ is continuous and strictly increasing on $X$. Of course, it follows that the average distribution $H_\alpha = \frac{1}{n} \sum_i H_i$ admits a unique median, which we denote $x_\alpha$. Assuming $n$ is odd, let $H_\mu$ denote the distribution of the median ideal policy, i.e., $H_\mu(x)$ is the probability that the median voter’s ideal policy is less than or equal to $x$. Under our assumptions, $H_\mu$ admits a unique median, denoted $x_\mu$.

To model voting behavior, let $P_i(x_A, x_B)$ denote the probability that voter $i$ votes for candidate $A$, given policy platforms $x_A$ and $x_B$. The probability of a vote for candidate $B$ is then $1 - P_i(x_A, x_B)$. We assume voters vote for the candidate with the preferred policy position, flipping a fair coin in case of indifference, so that

$$P_i(x_A, x_B) = \Pr(\{\theta_i \mid u_i(x_A, \theta_i) > u_i(x_B, \theta_i)\}) + \frac{1}{2} \Pr(\{\theta_i \mid u_i(x_A, \theta_i) = u_i(x_B, \theta_i)\}).$$

1 We also assume that for each policy $x$, the function $u_i(x, \cdot)$ from $\Theta$ to $\mathbb{R}$ is Borel measurable.
Assuming voting is costless, this is consistent with elimination of weakly dominated voting strategies in the voting game. Though it is customary to assume that the probability a voter is indifferent between distinct platforms is zero, that assumption is not needed here, and we do not impose it. Finally, let \( P(x_A, x_B) \) denote the probability that candidate \( A \) wins the election, given the individual vote probabilities. Of course, candidate \( B \)'s probability of winning is one minus this amount.

In the electoral game between the candidates, we endow the candidates with payoff functions, \( U_A \) and \( U_B \). We assume that the candidates are office-motivated, and we follow the literature in formalizing this in either of two ways. First, vote motivation is formalized as

\[
U_A(x_A, x_B) = \sum_i P_i(x_A, x_B),
\]

with \( U_B(x_A, x_B) \) equal to \( n \) minus the above quantity. Second, win motivation is formalized as

\[
U_A(x_A, x_B) = P(x_A, x_B),
\]

with candidate \( B \)'s payoff equal to one minus the above quantity. Even when voters are indifferent between distinct platforms with probability zero, the probability functions \( P_i(x_A, x_B) \) necessarily possess discontinuities along the “diagonal,” i.e., when one candidate’s platform crosses over the other’s.\(^2\) Thus, the electoral game is marked by discontinuous payoff functions.

We admit the possibility that candidates use mixed strategies, which formalize the idea that a candidate cannot precisely predict the policy position of his or her opponent at the time that campaign platforms are formed. We represent a mixed strategy for candidate \( A \) by a distribution \( F_A \) over policies, and likewise for candidate \( B \). Given a pair \((F_A, F_B)\) of mixed strategies, candidate \( A \)'s expected payoff is a double integral, which we denote by

\[
EU_A(F_A, F_B) = \int \int U_A(x_A, x_B) F_A(dx_A) F_B(dx_B),
\]

with a similar convention for \( B \).\(^3\) Given a mixed strategy \( F_B \) for candidate \( B \), we assume that candidate \( A \) seeks to maximize \( EU_A(x_A, F_B) \), and likewise for candidate \( B \).

Given these expected payoffs, we may subject the electoral model to an equilibrium analysis to illuminate the locational incentives of the candidates. We say a pair \((x_A^*, x_B^*)\) of policy platforms is a pure strategy equilibrium if

\(^2\)Specifically, if \( x \) is not the median of \( H_i \), then \( P_i \) is discontinuous at \((x, x)\). If \( x \) is less than the median, for example, then \( \lim_{x_A \downarrow x} P_1(x_A, x) > \frac{1}{2} = P_1(x, x) \).

\(^3\)If a distribution puts probability one on a policy, then we may simply substitute that policy in the argument of \( EU_A \), as in \( EU(x_A, F_B) \).
neither candidate can gain by unilaterally deviating, i.e., we have

\[ U^*_A(x_A, x_B^*) = \max_{x_A} U_A(x_A, x_B^*) \]

and likewise for candidate B. We say a pair \((F^*_A, F^*_B)\) of distributions over policy platforms is a mixed strategy equilibrium if neither candidate can gain by unilaterally deviating, i.e., we have

\[ EU^*_A(F^*_A, F^*_B) = \max_{x_A} EU_A(x_A, F^*_B) \]

and likewise for candidate B. An implication is that \(F^*_A\) puts probability one on the set of maximizers of \(EU_A(x_A, F^*_B)\).

### 3 Uniqueness of Equilibrium

Despite the presence of discontinuities in the candidates’ payoffs, we can show that mixed strategy equilibria exist under both formalizations of office motivation, that the mixed strategy equilibrium is unique in both models, and that in equilibrium the candidates actually use pure strategies that have a very simple characterization. For the case of win motivation, Calvert (1985), establishes that there is a unique pure strategy equilibrium and that in equilibrium the candidates must locate at the same policy, the “median of medians,” \(x_\mu\). Bernhardt, Duggan, and Squintani (2005) prove that this equilibrium is in fact unique among all mixed strategy equilibria.

**Theorem 1 (Calvert; Bernhardt, Duggan, Squintani)** Assume \(n\) is odd and win motivation. There is a unique mixed strategy equilibrium. In equilibrium, the candidates locate with probability one at the median \(x_\mu\) of the distribution of median ideal policies.

The case of vote motivation, while investigated thoroughly in other models of probabilistic voting, seems to have been overlooked in the class of models with stochastic voter preferences. The next result establishes that there is a unique mixed strategy equilibrium under vote motivation, and that in equilibrium the candidates must locate with probability one at the same policy, the median \(x_\sigma\) of the average distribution. Thus, we are back to a “median-like” result, but now the equilibrium is at the median of the average distribution. An advantage is that oddness of the number of voters no longer plays a role in the result.

**Theorem 2** Assume vote motivation. There is a unique mixed strategy equilibrium. In equilibrium, both candidates locate with probability one at the median \(x_\alpha\) of the average distribution.
To prove existence, define the pure strategies \( x_A^* = x_B^* = x_\alpha \). Then \((x_A^*, x_B^*)\) is a pure strategy equilibrium, for consider either candidate and any deviation, say candidate \(A\) and policy \( \hat{x} < x_\alpha \). Each voter with ideal policy greater than \( x_\alpha \) votes for \( B \), so we have

\[
U_A(\hat{x}, x_B^*) - U_A(x_A^*, x_B^*) = \sum_i [P_i(\hat{x}, x_\alpha) - P_i(x_\alpha, x_\alpha)]
\]

\[
\leq \sum_i [H_i(x_\alpha) - \frac{1}{2}]
\]

\[
= 0,
\]

where the inequality follows by definition of \( x_\alpha \). To prove uniqueness, let \((F_A^*, F_B^*)\) be a mixed strategy equilibrium, where for notational simplicity we let \( F = F_A^* \) and \( G = F_B^* \). Because the electoral game is symmetric, it follows that \((G, F)\) is also a mixed strategy equilibrium. Since the electoral game is constant-sum, interchangeability implies that \((F, F)\) is a mixed strategy equilibrium. I will argue that the support of \( F \) must be concentrated on \( x_\alpha \). Suppose not, so that without loss of generality \( \bar{x} < x_\alpha \) is the minimum of the support of \( F \). Let \( \hat{x} \) be a policy such that \( \bar{x} < \hat{x} < x_\alpha \) and \( F \) puts probability zero on \( \hat{x} \). Define the sequence \( \{x_m\} \) of policies so that if \( F \) puts positive probability on \( \bar{x} \), then \( x_m = \bar{x} \) for all \( m \); otherwise, let the sequence satisfy (i) \( \bar{x} < x_m < \hat{x} \) for all \( m \), (ii) \( EU_A(x_m, F) = EU_A(F, F) \) for all \( m \), and (iii) \( x_m \to \bar{x} \). Note that

\[
EU_A(\hat{x}, F) - EU_A(F, F)
\]

\[
= EU_A(\hat{x}, F) - EU_A(x_m, F)
\]

\[
= \int_{[\bar{x}, x_m]} \sum_i [P_i(\hat{x}, x_B) - P_i(x_m, x_B)] F(dx_B)
\]

\[
+ \int_{\{x_m\}} \sum_i [P_i(\hat{x}, x_B) - P_i(x_m, x_B)] F(dx_B)
\]

\[
+ \int_{(x_m, \hat{x})} \sum_i [P_i(\hat{x}, x_B) - P_i(x_m, x_B)] F(dx_B)
\]

\[
+ \int_{(\hat{x}, \infty)} \sum_i [P_i(\hat{x}, x_B) - P_i(x_m, x_B)] F(dx_B)
\]

for all \( m \). Here, the interval \([\bar{x}, x_m]\) is either empty, if \( x_m = \bar{x} \), or the probability of that interval goes to zero. When both candidates position at \( x_m \), each voter votes for \( A \) with probability \( \frac{1}{2} \). If candidate \( A \) switches to \( \hat{x} \), then each voter with ideal point greater than \( \hat{x} \) votes for \( A \), so we have \( P_i(\hat{x}, x_m) \geq 1 - H_i(\hat{x}) \). Therefore,

\[
\sum_i [P_i(\hat{x}, x_m) - P_i(x_m, x_m)] \geq \sum_i [1 - H_i(\hat{x}) - \frac{1}{2}] > 0,
\]

where the strict inequality follows from \( \hat{x} < x_\alpha \). Now take any \( x_B \in (x_m, \hat{x}) \). When candidate \( A \) positions at \( \hat{x} \), each voter with ideal policy greater than \( \hat{x} \)
votes for $A$, so we have $P_i(\hat{x}, x_B) \geq 1 - H_i(\hat{x})$. When candidate $A$ positions at $x_m$, each voter who votes for $A$ has ideal policy less than $\hat{x}$, so we have $P_i(x_m, x_B) \leq H_i(\hat{x})$. Therefore,

$$\sum_i [P_i(\hat{x}, x_B) - P_i(x_m, x_B)] \geq \sum_i [1 - H_i(\hat{x}) - H_i(\hat{x})] > 0,$$

where the strict inequality follows from $\hat{x} < x_\alpha$. Note that $F$ puts positive probability on the intervals $(x_m, \hat{x})$ and that this probability has a positive limit, so that

$$\lim_{m \to \infty} \int_{[x_m, \hat{x}]} \sum_i [P_i(\hat{x}, x_B) - P_i(x_m, x_B)] F(dx_B) > 0.$$

Finally, take any $x_B \in (\hat{x}, \infty)$. Each voter who weakly prefers $x_m$ to $x_B$ must strictly prefer $\hat{x}$ to $x_B$, and therefore $P_i(\hat{x}, x_B) > P_i(x_m, x_B)$. We conclude that $EU_A(\hat{x}, F) = EU_A(x_m, F) > 0$ for sufficiently high $m$, and therefore $EU_A(\hat{x}, F) > EU_A(F, F)$. This contradicts the premise that $F$, as an equilibrium mixed strategy, puts probability one on the maximizers of $EU_A(x_A, F)$ and completes the proof of the theorem.

References
