Collective Choice with Linear Utilities

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Abstract

This note considers a property, called "limited shared weak preferences" (LSWP), in the context of a model with linear utility functions defined on a polyhedron of alternatives. As special cases, we capture the distributive politics model with externalities in consumption, and we capture collective choice of lotteries by expected utility-maximizing agents. We give a simple sufficient condition for LSWP, which plays an important role in the analysis of stationary equilibria in spatial bargaining games and in the analysis of covering and dominance in social choice theory.

1 Introduction

A classical model of collective choice in political science and economics is the "divide-the-dollar" model, in which a set of agents must allocate a fixed amount of some resource among themselves and in which each agent is only concerned with maximizing his/her own consumption of the resource. The model plays an important role in social choice theory, where it is easily shown that the majority core – the allocations which cannot be improved on for a majority of agents – is empty. In contrast, a related solution imported from the theory of tournaments, called the "uncovered set," consists of all (or nearly all) allocations.[1] As well, the model is central in bargaining theory.

[1] See Epstein (1998).

It is, for example, one of the main building blocks in the analysis of Baron and Ferejohn (1989), who examine the properties of stationary equilibria to gain insight into legislative behavior.

Work on the uncovered set and on stationary bargaining equilibria has been generalized beyond the divide-the-dollar model. McKelvey (1986) analyzes the uncovered set in the context of the multidimensional spatial model from political science, and Banks, Duggan, and Le Breton (2003) extend the analysis to a general setting, including economic environments. Banks and Duggan (2000, 2003) extend the bargaining analysis of Baron and Ferejohn (1989) to the general setting. In moving beyond the divide-the-dollar model, a restriction on individual preferences, called "limited shared weak preference" (LSWP), plays a key role. The condition requires that, if an alternative y is weakly preferred to alternative x by the members of any coalition, then there exist alternatives arbitrarily close to x that are strictly preferred to x by the members of that coalition.

It is straightforward to verify that LSWP is satisfied in the divide-the-dollar model, and Banks and Duggan (1999) establish LSWP in relatively general classes of environments, including exchange economies and including environments with a convex set of public goods and an additively separable private good. In these environments, however, some form of strict convexity or monotonicity plays an important role in verifying LSWP. The extension to the general linear setting presents special technical problems that have not been addressed.

Two environments of special interest are omitted, as a consequence, from the domain of these collective choice theories. The first is the divide-the-dollar model with linear externalities in consumption, as analyzed in Calvert and Dietz (2004). Here, agent i's utility is given by a collection of coefficients, say $u_1^i, \ldots, u_n^i$, one for each agent, where u_j^i/u_k^i (when defined) measures i's fixed marginal rate of substitution for j's consumption relative to k's. In the standard divide-the-dollar model, of course, we have $u_i^i = 1$ and $u_j^i = 0$ for all $j \neq i$. The second is the model in which there is a finite number of "pure" alternatives, and the objects of collective choice are lotteries over pure alternatives. Here, an alternative is a vector of probabilities of pure alternatives, and linear utilities are interpreted in terms of von Neumann-Morgenstern expected utility.

The LSWP condition is not satisfied in the general linear setting, where individual preferences are given by gradient vectors: if the gradient vec-

tors of two agents point in opposite directions, for example, then LSWP is violated. This example is non-generic, however, raising the question of the restrictiveness of LSWP in linear environments. In this note, we offer a relatively transparent sufficient condition, called "limited positive linear dependence" (LPLD), for LSWP, and we apply it to the divide-the-dollar model with externalities and to the lottery model. We show that LSWP is generically satisfied in these environments.

We then discuss some implications of these observations for social choice and bargaining theory. When LSWP is satisfied, Banks, Duggan, and Le Breton (2003) show that the following correspondence, denoted R, is lower hemi-continuous: for each alternative x, let $R(x)$ denote the set of alternatives that are weakly majority-preferred to x. A further consequence is that the "undominated set" (a subset of the uncovered set) is non-empty. Also when LSWP is satisfied, Banks and Duggan (2000, 2003) prove a general existence result for stationary bargaining equilibria. We conclude that, in the divide-the-dollar model with externalities and in the lottery model, the undominated set and the set of stationary bargaining equilibria are generically non-empty.

2 Formalities

Let $X \subseteq \Re^d$ consist of the solutions to the finite system of weak inequalities, $p^j \cdot x \geq c_j$, where the index j ranges over the finite set J. Thus, X is closed and convex. Let $u^i \in \Re^d$ denote the gradient of a linear function, where i ranges over the finite set I of n elements. We interpret X as a set of alternatives to be chosen from, I as a set of agents, and u^i as the utility function of agent i. We denote i's weak and strict upper contour sets at x, respectively, by

$$\begin{aligned} R_i(x) &= \{y \in X \mid u^i \cdot y \geq u^i \cdot x\} \\ P_i(x) &= \{y \in X \mid u^i \cdot y > u^i \cdot x\}, \end{aligned}$$

and we let

$$R_C(x) = \bigcap_{i \in C} R_i(x) \quad \text{and} \quad P_C(x) = \bigcap_{i \in C} P_i(x).$$

That is, $R_C(x)$ consists of the policies that every member of C weakly prefers to x, with a similar interpretation of $P_C(x)$. Given $Y \subseteq X$, let $\overline{Y}$ denote the closure of Y.

We say *Limited Shared Weak Preference (LSWP)* is holds if, for all $C \subseteq I$ and all $x \in X$,

$$|R_C(x)| > 1 \text{ implies } R_C(x) \subseteq \overline{P_C(x)}.$$

That is, if y (distinct from x) is weakly preferred to x by all members of a group C, then it can be approximated by policies that all members of C strictly prefer to x. Banks and Duggan (1999) verify that LSWP holds in the classical *divide-the-dollar model* of distributive politics, described as follows. Here, the set of alternatives is the unit simplex in $\Re^n$,

$$X = \left\{(x_1, \ldots, x_n) \in \Re^n_+ \mid \sum_{i \in I} x_i \leq 1\right\},$$

representing allocations of a fixed resource across legislative districts, and each $u^i = e^i$, the ith unit coordinate vector: $u^i \cdot x = x_i$. While Banks and Duggan (1999) also allow for non-linear utilities and verify LSWP in a variety of environments, they leave open the case of general linear environments, where the set X may not be the unit simplex and utility vectors may be more general.

The general linear model of this note covers two applications of interest. First, the *divide-the-dollar model with externalities,* examined in Calvert and Dietz (2004), allows for externalities in consumption of money. Here, the set X is again the unit simplex in $\Re^n$, but we allow for arbitrary utility vectors $u^i = (u^i_1, \ldots, u^i_n)$, where u^i_j/u^i_k (when defined) measures i's marginal rate of substitution for j's consumption relative to k's. Second, the *lottery model* views vectors in X as probability distributions over a finite set of "pure" alternatives, $\{a_1, \ldots, a_d\}$. Here, X is the outer face of the unit simplex, i.e.,

$$X = \left\{(x_1, \ldots, x_d) \in \Re^d_+ \mid \sum_{k=1}^{d} x_k = 1\right\},$$

and x_k is the probability of alternative a_k. Assuming the agents' preferences over lotteries satisfy the von Neumann-Morgenstern axioms, they admit linear utility representations, given by u^i. We assume then without loss of generality (by a positive affine transformation) that each u^i lies in the outer face of the unit simplex, i.e., $u^i \in X$.

Given any set $Y \subseteq \Re^d$, we denote by $\text{lin}[Y]$ the linear hull of (or subspace spanned by) Y. Given a vector $x \in \Re^d$, we write $x = (x_1, \ldots, x_d) > 0$ if $x_j > 0$ for each component j, in which case we say x is "positive." We write $x \geq 0$ if $x_j \geq 0$ for each component, with strict inequality for at least one,

in which case x is "semi-positive." We say a set $\{y^1, \ldots, y^m\}$ is *positively linearly dependent* if there exist $a_1, \ldots, a_m \in \Re$ such that

$$\sum_{j=1}^{m} a_j y^j \quad = \quad 0,$$

where $a_j > 0$ for all $j = 1, \ldots, m$. In other words, $\{y^1, \ldots, y^m\}$ is positively linearly dependent if, when we form the matrix M with columns y^j ($j = 1, \ldots, m$), the system of equations $Ma = 0$ has a positive solution.

Finally, we say *Limited Positive Linear Dependence (LPLD)* holds if, for all $C \subseteq I$ and all $D \subseteq J$, if $\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is positively linearly dependent, then

$$\text{lin}[\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}] \quad \supseteq \quad \text{lin}[X - x],$$

where x is any element of X. In words, LPLD holds if, whenever the zero vector can be written as a convex combination of some subset of the u^i and p^j vectors, then that subset spans X, when translated to the origin.

3 Sufficient Conditions

The LPLD condition turns out to be a key sufficient condition for establishing LSWP. It is illustrated in Figure 1, where u^1 and u^2 point in opposite directions, creating a violation of LSWP: though there are points $y \in X$ such that $u^1 \cdot y \geq u^1 \cdot x$ and $u^2 \cdot y \geq u^2 \cdot x$, there is no element of X for which these inequalities hold strictly. Similarly, if u^1 and p^j point in opposite directions, there are points $y \in X$ such that $u^1 \cdot y \geq u^1 \cdot x$ and $u^3 \cdot y \geq u^3 \cdot x$, but x maximizes u^i within X.

[Figure 1 here.]

LPLD precludes the situations depicted above, for it requires that X, translated, lie within the line spanned by u^1 and u^2 (or p^j). In this case, there is no $y \in X$ distinct from x such that $u^1 \cdot y \geq u^1 \cdot x$ and $u^2 \cdot y \geq u^2 \cdot x$, fulfilling LSWP. The next lemma generalizes these insights.

Lemma 1 *If LPLD holds, then LSWP holds.*

Proof: Take any $C \subseteq I$, any $x \in X$, and any $y \in R_C(x) \setminus \{x\}$, which implies $y \cdot u^i \geq x \cdot u^i$ for all $i \in C$ and $y \cdot p^j \geq c_j$ for all $j \in J$. Let D consist of the indices $j \in J$ such that $x \cdot p^j = c_j$. Letting $e = |C| + |D|$, form the $d \times e$ matrix M by using u^i $(i \in C)$ and p^j $(j \in D)$ as columns. If $zM > 0$ has a solution, we are done: define $z_\epsilon = x + \epsilon z$, for arbitrarily small $\epsilon > 0$, to fulfill LSWP. We will see that the remaining case produces a contradiction. If $zM > 0$ has no solution, then the Theorem of the Alternative (Gale, 1960, Theorem 2.9) implies that $Mw = 0$ has a semi-positive solution, say $w = (w_1, \ldots, w_e)$. Let K be the subset of indices $k \in C \cup D$ such that $w_k > 0$, and note that LPLD implies that $\text{lin}[\{m^k \mid k \in K\}] \supseteq \text{lin}[X - x]$. Let K^* index a maximal linearly independent subset of $\{m^k \mid k \in K\}$. Letting M^* be the submatrix of M obtained by keeping just those columns, we see that $M^*w = 0$ has a positive solution. Thus, by another version of the Theorem of the Alternative (Gale, 1960, Corollary 2), $zM^* \geq 0$ has no solution. If $y \cdot m^k = x \cdot m^k$ for each $k \in K^*$, then linear independence of $\{m^k \mid k \in K^*\}$ implies $y = x$, a contradiction. Since $K^* \subseteq C \cup D$, we cannot have $y \cdot m^k < x \cdot m^k$ for any $k \in K^*$. Therefore, we have $y \cdot m^k \geq x \cdot m^k$ all $k \in K^*$, with strict inequality for at least one $k \in K^*$. This implies $(y - x)M^* \geq 0$, a final contradiction. ∎

To see that LPLD holds in the divide-the-dollar model, let $J = I \cup \{0\}$; let $p^j = e^j$ for each $j = 1, \ldots, n$; and let $p^0 = (-1, \ldots, -1)$. Let $u^i = e^i$ for each $i \in I$. If $\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is positively linearly dependent for some $C \subseteq I$ and $D \subseteq J$, then it is clear that $C \cup D$ must contain all unit coordinate vectors as well as p^0, i.e., $C \cup D = J$, and then it follows that the linear hull of $\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is indeed $\Re^n$, fulfilling LPLD.

In fact, LPLD holds in the divide-the-dollar model with externalities under the following restriction:

(C1) For all $i \in I$, $u^i \in \Re^n_+ \setminus \{0\}$, and $(1, \ldots, 1)$ cannot be written as a semi-positive combination of any $n - 1$ elements of $\{u^1, \ldots, u^n, p^1, \ldots, p^n\}$.

To see that this condition is sufficient for LPLD, suppose that the zero vector lies in the convex hull of

$$Z \quad = \quad \{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}.$$

Then $0 \in D$ and, by assumption, $|C| + |D| \geq n$. If Z has dimension n, then of course X is contained in its linear hull. Otherwise, it has lower

dimension. By Caratheodory's theorem, however, we may select $z^1, \ldots, z^n$ from Z so that 0 lies in the convex hull of $\{z^1, \ldots, z^n\}$, but then necessarily $z^k = (-1, \ldots, -1)$ for some k with positive weight, violating the proposed condition.

A perhaps simpler, yet still generic, sufficient condition is that no subset of $\{u^i \mid i \in I\} \cup \{p^j \mid j \in J\}$ containing n or fewer vectors is linearly dependent. To see sufficiency of this condition, suppose $Z = \{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is positively linearly dependent, and therefore linearly dependent. By assumption, then, $|C| + |D| \geq n + 1$. Furthermore, $\text{lin}[Z]$ has dimension n, for otherwise we could remove an element of Z to arrive at a smaller linearly dependent set. We conclude that $\text{lin}[Z] = \text{lin}[X]$, as required.

Because the set of alternatives in the lottery model is defined by the equality $\sum_{j=1}^{d} x_j = 1$, we would need two vectors, such as $p^j = (1, \ldots, 1)$ and $p^k = (-1, \ldots, -1)$, to describe X as the solution to a system of weak inequalities. Since these two vectors are positively linearly dependent, however, LPLD would require that X is one-dimensional. Therefore, we capture the lottery model with d pure alternatives indirectly, by translating X to

$$\hat{X} \quad = \quad \left\{ (x_1, \ldots, x_{d-1}) \in \Re_+^{d-1} \mid \sum_{k=1}^{d-1} x_k \leq 1 \right\},$$

which we define with vectors $\hat{p}^0 = (-1, \ldots, -1)$ and $\hat{p}^j = e^j$ $(j = 1, \ldots, d-1)$, and by translating u^i to

$$\hat{u}^i \quad = \quad (u^i_1 - u^i_d, \ldots, u^i_{d-1} - u^i_d).$$

Given $x \in X$, write $\hat{x} \in \hat{X}$ for the truncation of x, i.e., $\hat{x} = (x_1, \ldots, x_{d-1})$. It is straightforward to check that, for all $x, y \in X$,

$$u^i \cdot x \geq u^i \cdot y \quad \Leftrightarrow \quad \hat{u}^i \cdot \hat{x} \geq \hat{u}^i \cdot \hat{y},$$

so that $(x, u^1, \ldots, u^n) \mapsto (\hat{x}, \hat{u}^i, \ldots, \hat{u}^n)$ is a linear isomorphism that preserves preferences. Thus, if LSWP is satisfied in the new environment, then it is satisfied in the original.

As discussed with respect to the divide-the-dollar model with externalities, the LPLD condition will hold in the new environment under the following generic restriction:

(C2) No subset of $\{\hat{u}^i \mid i \in I\} \cup \{\hat{p}^j \mid j \in J\}$ containing $d-1$ or fewer vectors is linearly dependent.

Mapping back to the original model, we see that LSWP is once again generically satisfied.

4 Implications

Let $\mathcal{D}$ be a collection of subsets of I such that (i) if $C \in \mathcal{D}$ and $C \subseteq C'$, then $C' \in \mathcal{D}$, and (ii) if $C, C' \in \mathcal{D}$, then $C \cap C' \neq \emptyset$. Then define the corresponding weak and strict social preference relations, respectively, by

$$\begin{aligned} R(x) &= \bigcup_{C \in \mathcal{D}} R_C(x) \\ P(x) &= \bigcup_{C \in \mathcal{D}} P_C(x). \end{aligned}$$

In the linear environments examined above, it is clear that the relation R is closed and P is open. With Proposition 14 of Banks, Duggan, and Le Breton (2003), we also have the following proposition.

Proposition 1 *Assume LPLD holds. Then $R(\cdot)$ is lower hemi-continuous as a correspondence.*

We say x *dominates* y, written xDy, if $P(x) \subseteq P(y)$ and $R(x) \subseteq R(y)$, with at least one inclusion strict. The *undominated set,* denoted UD, is the maximal elements of the dominance relation, i.e.,

$$UD = \{x \in X \mid \text{ there is no } y \text{ such that } yDx\}.$$

While D is transitive, it does not generally possess desirable continuity properties, so non-emptiness of the undominated set does not follow from standard results. The set UD is interesting purely as a social choice solution, but it also has foundations from non-cooperative game theory: as discussed by McKelvey (1986), the undominated set corresponds to the undominated strategies of office-motivated candidates in the Downsian electoral model. With Proposition 23 of Banks, Duggan, and Le Breton (2003), we have the following result.

Proposition 2 *Assume X is compact and LPLD holds. Then $UD \neq \emptyset$.*

Banks and Duggan (2000, 2003) consider the following infinite-horizon model of bargaining, based on the closed-rule model of Baron and Ferejohn (1989). It is presented here in the context of linear environments. If no alternative has been accepted prior to period t, then (1) agent i is recognized with probability $\rho_i \geq 0$, where $\sum_{i \in N} \rho_i = 1$; then (2) the selected agent i makes a proposal $p^i \in X$; then (3) every agent $j \in N$ simultaneously votes to either *accept* or *reject* the proposal. The *recognition probabilities,* $\rho_1, \ldots, \rho_n$, are exogenously fixed throughout the game. Let $\mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$ be a collection satisfying (i) above. If the group of agents voting for the proposal is decisive, i.e., $\{j \in N \mid j \text{ accepts } \} \in \mathcal{D}$, then the proposal p^i is the chosen alternative and bargaining ends with outcome p^i in period t and in every subsequent period. Otherwise, the outcome in period t is the status quo, and steps 1–3 are repeated for period $t + 1$.

Each agent i's preferences over outcomes of the bargaining game are described by a von Neumann-Morgenstern linear utility representation $u^i \in \Re^d$ and a discount factor $\delta_i \in [0, 1)$ as follows. If alternative x is passed at time t, then agent i's payoff is

$$(1 - \delta_i^{t-1})\overline{u}^i + \delta_i^{t-1}(u^i \cdot x),$$

where $\overline{u}^i$ is a fixed status quo payoff. If no alternative is ever passed, then i's payoff is just $\overline{u}^i$. We maintain either of two assumptions.

(A1) There exists $q \in X$ such that, for all $i \in I$, $\overline{u}^i = u^i \cdot q$; and $\delta_i = \delta_j$ for all $i, j \in I$.

(A2) For all $i \in I$ and all $x \in X$, $\overline{u}^i \leq u^i \cdot X$.

The first assumption formalizes the idea that the status quo could be any alternative under consideration (but adds commonality of the discount factor), while the second formalizes the idea of a "bad" status quo (and allows heterogeneous discount factors).

Complete information of preferences, the structure of the game form, etc., is assumed, and stationary subgame perfect equilibrium is defined in the usual way. In addition, we impose the refinement of stage-game weak dominance: after an alternative, say x, is proposed, an agent i votes for x only if the utility from x weakly exceeds i's "reservation value," given by

$(1-\delta_i)\overline{u}^i + \delta_i v_i$, where v_i is i's continuation value from moving bargaining to the next period. This standard refinement eliminates implausible equilibria in which some agents, because their vote is not pivotal, best respond by voting for the worst of two choices.[2]

Proposition 3 *Assume X is compact and LPLD holds. Then there exists a stationary subgame perfect equilibrium in stage-undominated voting strategies.*

Thus, under (C1) in the divide-the-dollar model with externalities or under (C2) in the lottery model, existing results from social choice and bargaining extend to these two environments. We know that, generically, the undominated set (and therefore the uncovered set) is non-empty, and that, generically, stationary bargaining equilibria exist for the widely-used Baron-Ferejohn closed rule bargaining protocol.

References

[1] J. Banks and J. Duggan (1999) "A Bargaining Model of Collective Choice," Caltech Social Science Working Paper no. 1053.

[2] J. Banks and J. Duggan (2000) "A Bargaining Model of Collective Choice," *American Political Science Review,* 94:73-88.

[3] J. Banks and J. Duggan (2003) "A Bargaining Model of Legislative Policy-Making," mimeo.

[4] J. Banks, J. Duggan, and M. Le Breton (2003) "Social Choice and Electoral Competition in the General Spatial Model," mimeo.

[5] D. Baron and J. Ferejohn (1989) "Bargaining in Legislatures," *American Political Science Review,* 83:1181-1206.

[6] R. Calvert and N. Dietz (2004) "Legislative Coalitions in a Bargaining Model of Externalities," mimeo.

[7] D. Epstein (1998) "Uncovering Some Subtleties of the Uncovered Set: Social Choice Theory and Distributive Politics," *Social Choice and Welfare,* 15:81-93.

[2] See Banks and Duggan (2000, 2003) for other results on the possibility of delay, etc.

[8] D. Gale (1960) *The Theory of Linear Economic Models,* Chicago: University of Chicago Press.

[9] R. McKelvey (1986) "Covering, Dominance, and Institution-free Properties of Social Choice," *American Journal of Political Science,* 30:283-314.