A Bargaining Model of Collective Choice

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Abstract

We analyze sequential bargaining in general political and economic environments, where proposers are recognized according to a random recognition rule and a proposal is implemented if it passes under an arbitrary voting rule. We prove existence of stationary equilibria, upper hemicontinuity of equilibrium proposals in structural and preference parameters, and core equivalence under certain conditions.

1 Introduction

One of the central findings in the spatial model of social choice is that in two or more dimensions the majority rule core, consisting of those alternatives unbeaten in pairwise majority comparisons, is empty for "most" profiles of individual preferences.[1] This result extends beyond majority rule to include almost all voting rules (short of ones such as dictatorship and unanimity), depending on a critical dimension of the policy space.[2] Non-emptiness can be obtained by taking the transitive closure of the majority (or other) "social preference" relation — the resulting choice set is called the "top cycle" and consists of those alternatives that are socially preferred directly or indirectly, through a chain of preferred alternatives, to all others. But the top cycle often comprises the entire space of alternatives.[3] As the basis for a general positive model of collective decision making, therefore, simply invoking majority (or some other) rule is not sufficient to generate a useful prediction for many situations; rather, additional description and analysis of the underlying decision process is required.

In this paper, we assume that the decision process takes the form of bargaining. Specifically, we analyze bargaining among two or more agents who must select an alternative from a compact, convex subset of multidimensional Euclidean space. Agents' preferences are represented by continuous and concave utility functions but are otherwise quite arbitrary; thus, besides the classic spatial model of politics, we admit "divide the dollar" problems, exchange economies, and public goods economies (with or without private goods). In common with most of the bargaining literature, we model the decision process as one of discrete time and infinite horizon, where each period a single proposal is made and is either accepted (thereby stopping the process) or rejected. Agents may or may not discount future utility, and discount factors are allowed to vary among the agents. In contrast to the models of Rubinstein (1982) and others, the identity of the proposer does not alternate in a fixed sequence; rather, in the manner of Binmore (1987) and Baron and Ferejohn (1989), the proposer is randomly selected, or "recognized," in each period. The recognition probabilities are time-invariant but

[1] Plott (1967), Davis, DeGroot, and Hinich (1972), Rubinstein (1979), Schofield (1983), McKelvey and Schofield (1987), Cox (1984), Le Breton (1987).

[2] Greenberg (1979), Schofield (1984), Strnad (1985), Banks (1995), Saari (1997).

[3] Cohen (1979), McKelvey (1976, 1979), Austen-Smith and Banks (1997).

otherwise unconstrained. In contrast to the models of Binmore, who assumes unanimity rule, and Baron and Ferejohn, who assume majority rule, passage of a proposal requires the assent of one from a collection of "decisive coalitions." This collection is time-invariant and is arbitrary save for the minimal conditions of non-emptiness and monotonicity.

We prove the existence of stationary "no delay" equilibria, in which proposals and voting decisions are independent of the time period and history of the game, and where the first proposal is always accepted. Further, we show that when discount factors are all less than one, delay cannot occur in any stationary equilibrium. Our existence result requires that agents be allowed to mix over their proposals; however, for a restricted class of environments we prove the existence of pure strategy stationary equilibria. The sufficient conditions for pure strategy existence turn out to be well-known conditions sufficient for non-emptiness of the core of the voting rule: we can assume either that the policy space is one-dimensional (in which case, under majority rule, the core consists of the median policy or policies) or that there is a privileged group (an "oligarchy") with the power to pass a proposal, each member of which possesses a veto. As a special case of the latter assumption, we obtain pure strategy equilibria under unanimity rule. We also provide a two-dimensional, three-person, majority rule example with impatient agents where multiple equilibrium payoffs arise, thereby precluding any universal uniqueness result. Finally, we show that the set of equilibrium outcomes varies upper hemicontinuously in the recognition probabilities and the agents' discount factors, as well as in various utility-specific parameters (e.g., ideal points in the case of quadratic utilities).

We next turn to the relationship between the equilibrium outcomes in the bargaining game and the core outcomes (if any) of the underlying voting rule as described by the set of decisive coalitions. We show that when the agents are perfectly patient and utility functions are strictly quasi-concave, equivalence of stationary equilibrium outcomes and core outcomes obtains if the policy space is one-dimensional or if some agent possesses a veto, a condition weaker than the existence of an oligarchy but also known to be sufficient for non-emptiness of the core. In both cases, our result gives a non-cooperative story for how outcomes in the core can be realized; in the former case, our bargaining model provides an alternative to Black's (1958) model of committee decision-making and to Downs's (1957) model of competition between political parties, both of which also predict the median policy. Using our upper hemicontinuity result, we conclude that if preferences are "close" to admitting a core and agents are sufficiently patient, then the outcomes of the bargaining process will be "close" to core outcomes; and if there is a *unique* core outcome, then all equilibrium outcomes will be close to one another as well. Thus, we offer a result on the robustness of core outcomes, an issue that has not previously received rigorous treatment. On the other hand, we also provide a two-dimensional, four-person, majority rule example in which non-core outcomes exist despite the non-emptiness of the core, thereby precluding any universal core equivalence result.

In related work, Baron and Ferejohn (1989) analyze a model in which the alternatives are divisions of a "dollar," agents have selfish preferences, and majority rule is used. Where they assume utility equal to amount of the dollar received, Harrington (1989, 1990a,b) extends the model to allow for risk aversion; and Calvert and Dietz (1996) allow for externalities in consumption of the dollar. Baron (1991) examines the case of a two-dimensional set of alternatives, three voters with quadratic preferences, and voting by majority rule. Winter (1996) analyzes the divide the dollar problem assuming at least one agent has a veto, focusing on the advantage of veto players over others; McCarty (1998) takes up the same subject, allowing for variable veto power across agents. Jackson and Moselle (1998) consider the case in which a one-dimensional public decision is selected along with a division of a dollar using majority rule; they investigate the nature of coalitions "formed" in equilibrium. Merlo and Wilson (1995) study the divide the dollar problem under unanimity rule, allowing the size of the dollar to change stochastically over time and generating (efficient) equilibrium delay as a consequence. They also prove the uniqueness of stationary equilibrium payoffs.

Other work on bargaining and the core stems from Selten's (1981) model in which a proposal to a coalition consists of a vector of payoffs "feasible" for the coalition. Chatterjee, Dutta, Ray, and Sengupta (1993) analyze a transferable utility game with discounting and a fixed "protocol" describing the order of proposers and respondents, with the first "rejector" becoming the new proposer. They show how delay is possible even in stationary equilibria, and they provide conditions under which no delay occurs. Moldovanu and Winter (1995) analyze a non-transferable utility game with no discounting and with a similar "rejector becomes proposer" feature, and they prove an equivalence between the core and stationary "order independent" equilibrium outcomes. Okada (1996) studies a transferable utility game with equal discount factors and a random recognition rule with equal recognition probabilities, showing that no delay occurs in equilibrium and providing conditions sufficient for the existence of efficient equilibria as the common discount factor goes to one.

2 The Model

Let $X \subseteq \Re^d$ denote a non-empty, compact, convex set of alternatives, with $d \geq 1$. Let $N = \{1, \ldots, n\}$ denote the set of agents, $n \geq 2$, and assume each $i \in N$ has preferences described by a von Neumann-Morgenstern utility function $u_i \colon X \to \Re$. We assume throughout that each u_i is continuous and concave and that $u_i(x) \geq 0$ for all $x \in X$; moreover, we assume there is some $x \in X$ such that, for all $i \in N$, $u_i(x) > 0$. We describe an additional restriction, required for some of our results, at the end of section. The voting rule is given by a collection $\mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$ of coalitions, called *decisive*.[4] We assume that $\mathcal{D}$ is non-empty and *monotonic:* $C \in \mathcal{D}$ and $C \subseteq C'$ imply $C' \in \mathcal{D}$. (Together these two assumptions imply $N \in \mathcal{D}$.) Save for Section 5, we do not assume that $\mathcal{D}$ is *proper:* $C \in \mathcal{D}$ implies $N \setminus C \notin \mathcal{D}$.

The timing of the interaction is as follows: (a) at $t = 1, 2, \ldots$, agent $i \in N$ is recognized with probability ρ_i, where $\rho = (\rho_1, \ldots, \rho_n) \in \Delta$, the unit simplex in $\Re^n$; (b) when recognized, i makes a proposal $p_i^t \in X$; (c) all $j \in N$ simultaneously vote to either *accept* or *reject* the proposal; if $\{j \in N \mid j \text{ accepts }\} \in \mathcal{D}$, then the proposal p_i^t is the chosen alternative and the game ends; otherwise, the process moves to period $t+1$ and it repeated. If $x \in X$ is accepted in period $t \in \{1, 2, \ldots\}$, i's payoff is given by $\delta_i^{t-1} u_i(x)$, $\delta_i \in [0, 1]$, where we implicitly assume here that the status quo (the outcome prevailing in the absence of agreement) has utility zero; and by our normalization of utilities, it follows that the alternatives in X are unanimously weakly preferred to the status quo.[5] If no alternative is ever accepted, each agent receives a utility of zero.

Complete information of preferences, the structure of the game form, etc. is assumed throughout. A *history of length l* in the game describes all that

[4] In the terminology of cooperative game theory, this is a simple game and the elements of $\mathcal{D}$ are winning coalitions.

[5] In Section 6, we explain that this restriction can be relaxed if the discount rates of the agents are identical.

has transpired in the first l periods (who the previous proposers were, what they proposed, how agents voted) as well as whether in the current period we stand prior to the proposer being recognized, after such recognition but prior to the proposal being made, or after the proposal but prior to the vote. Therefore, in general a *strategy* for an agent would be a mapping specifying her intended action (what to propose, how to vote) as a function of all histories of all lengths. Since our focus in this paper is only on equilibria in stationary strategies, we avoid unneeded generality and provide a formal definition only of such strategies. A (pure) *stationary strategy* for $i \in N$ consists of a proposal $p_i \in X$ offered anytime i is recognized; and a measurable decision rule $r_i\colon X \to \{\text{accept, reject}\}$, or equivalently an acceptance set $A_i = r_i^{-1}(\text{accept})$.

It turns out that mixtures over proposals are required for our most general existence result, so let $\mathcal{P}(X)$ denote the set of Borel probability measures on X and endow $\mathcal{P}(X)$ with the topology of weak convergence. Given measurable $Y \subseteq X$, let $\mathcal{P}(Y)$ denote the subset of probability measures on X that place probability one on Y. Let $\pi_i \in \mathcal{P}(X)$ denote a mixed stationary proposal for i, and let $\pi = (\pi_1, \ldots, \pi_n)$ denote a profile of mixed stationary proposals. A *mixed stationary strategy* for i is a pair $\sigma_i = (\pi_i, A_i)$, and we let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a profile of mixed stationary strategies. It is important to note that randomization takes place *before* voting: the agents know which alternative has been proposed at the time they cast their votes.

Given a mixed stationary proposal π_i for agent i, let $S(\pi_i)$ denote the support of π_i. For a profile σ in which agents use mixed stationary proposals $\pi = (\pi_1, \ldots, \pi_n)$, let $S(\pi) = \bigcup_{i=1}^n S(\pi_i)$. We say σ has *finite support* if $S(\pi)$ is finite. Let

$$x(\pi) \quad = \quad \sum_{i \in N} \rho_i \Big[\int_X x \pi_i(dx)\Big]$$

denote the *ex ante* expected value of the agents' proposals. Given a profile of mixed stationary proposals with acceptance sets $(A_1, \ldots, A_n)$ and given $C \subseteq N$, define the set

$$A_C \quad = \quad \bigcap_{i \in C} A_i$$

of proposals acceptable to all members of C, and define the *social acceptance* *set*

$$A \;=\; \bigcup_{C\in\mathcal{D}} A_C,$$

consisting of proposals passed in any and all periods. The profile is a *no-delay* profile if $\pi_i(A) = 1$ for all $i \in N$ (implying, of course, that $A \neq \emptyset$).

Informally, a profile σ^* constitutes a *stationary equilibrium* if, for all $i \in N$, (1) π_i^* is optimal given the acceptance sets $(A_1^*, \ldots, A_n^*)$ of the other agents; and (2) A_i^* is optimal given that σ^* describes what would happen if current proposal were rejected. To formalize these conditions, note first that any strategy profile σ defines in an obvious (if notationally dense) manner a probability distribution over the outcome space $(X \times \{1, 2, \ldots\}) \cup \{\emptyset\}$, and with it an expected utility $v_i(\sigma)$ for each $i \in N$ as evaluated at the beginning of the game; by stationarity this is also i's *continuation value* throughout the game, i.e., her expected utility as evaluated next period if the current period's proposal is rejected. In a no-delay stationary equilibrium, we can write the agent i's continuation value as

$$v_i(\pi) = \sum_{j\in N} \rho_j [\int_X u_i(x)\pi_j(dx)],$$

a continuous function of π.

Formally, we require that the agents' proposals satisfy *sequential rationality* and that their acceptance sets satisfy *weak dominance,* with the latter eliminating nasty equilibria in which, for instance, under majority rule everyone accepts $x \in X$ independently of preferences. The equilibrium condition on the acceptance sets is that

$$[u_i(x) > \delta_i v_i(\sigma^*) \text{ implies } x \in A_i^*] \quad \text{and} \quad [u_i(x) < \delta_i v_i(\sigma^*) \text{ implies } x \notin A_i^*]$$

for all $i \in N$.[6] As for proposals, if A is non-empty then agent i, when recognized as proposer, either chooses utility-maximizing outcomes from within A or chooses an outcome that will be rejected, thereby generating a payoff of $\delta_i v_i(\sigma)$. Thus, the equilibrium condition on proposals is that, for all $i \in N$,

$$\pi_i^*(\arg\max\{u_i(y) \mid y \in A^*\}) \;=\; 1$$

[6]Baron and Kalai (1993) refer to such strategies as "stage-undominated."

when $\sup\{u_i(y) \mid y \in A^*\} > \delta_i v_i(\sigma^*)$; that $\pi_i^*(X \backslash A^*) = 1$ when the inequality is reversed; and that π_i^* place positive mass only on the union of these two sets when equality holds.

[Figure 1 about here.]

Figure 1, from Baron (1991), gives a visual example of no-delay stationary equilibria, where $n = 3$, majority rule is used, X is the unit simplex in $\Re^3$, utility functions are quadratic with ideal points at the respective unit coordinate vectors, and $\rho_i = 1/3$ and $\delta_i = 1$ for all $i \in N$. One mixed strategy equilibrium is given by $\pi_1^*(a) = \pi_1^*(b) = 1/2$, $\pi_2^*(c) = \pi_2^*(d) = 1/2$, and $\pi_3^*(e) = \pi_3^*(f) = 1/2$, with acceptance sets given by the appropriate weak upper contour sets; other distributions over these six alternatives work as well. Indeed, one *pure* strategy equilibrium is $\pi_1^*(a) = \pi_2^*(c) = \pi_3^*(e) = 1$; another is $\pi_1^*(b) = \pi_2^*(d) = \pi_3^*(f) = 1$.

To state our additional restriction on preferences, define i's weak and strict upper contour sets at x, respectively, as

$$\begin{aligned} R_i(x) &= \{y \in X \mid u_i(y) \geq u_i(x)\} \\ P_i(x) &= \{y \in X \mid u_i(y) > u_i(x)\}, \end{aligned}$$

and define $R_C(x)$, $P_C(x)$, $R(x)$, and $P(x)$ following the conventions on acceptance sets. We say that *limited shared weak preference* (LSWP) holds if, for all $C \subseteq N$ and all $x \in X$,

$$|R_C(x)| > 1 \Rightarrow R_C(x) \subseteq \overline{P_C(x)}.$$

That is, if y (distinct from x) weakly dominates x from the perspective of a coalition C, it can be approximated by alternatives that strictly dominate x for C. In Appendix A, we present two general models in which LSWP holds. Special cases include the following familiar environments.

- *Classical spatial model/Pure public goods.* Each u_i is strictly quasi-concave, as when there exists x^i such that $u_i(x) = -||x - x^i||$ or $u_i(x) = -||x - x^i||^2$. Or, if alternatives represent public goods, each u_i may be monotonic as well. If a restriction on shared indifference is imposed, strict quasi-concavity may be dropped.

- *Public decisions with transfers.* $X = Z \times T$, $T \subseteq \Re^n$, and each u_i is quasi-linear: $u_i(z, t) = \phi_i(z) + t_i$, ϕ_i strictly quasi-concave.

- *Exchange economy.* Alternatives are allocations of private goods, and each u_i is strictly quasi-concave and strictly monotonic in i's consumption.

- *Divide the dollar.* $X = \Delta$ and $u_i(x) = x_i$.

Though LSWP is satisfied in divide the dollar environments, it does not hold for general linear u_i: for example, let $n = 2$, X the unit ball in $\Re^2$, where one agent has gradient $(1, 0)$ and the other has gradient $(-1, 0)$.

3 Existence

Theorem 1 *(i) If $\delta_i < 1$ for all $i \in N$, then there exists a no-delay stationary equilibrium, and every stationary equilibrium is a no-delay equilibrium. (ii) If LSWP holds, then there exists a no-delay stationary equilibrium. (iii) If each u_i is strictly quasi-concave, then every no-delay stationary equilibrium has finite support.*

Proof: For all $i \in N$, define $A_i(\pi) = \{x \in X \mid u_i(x) \geq \delta_i v_i(\pi)\}$. By concavity of u_i, by $u_i(x) \geq 0$ for all $x \in X$, and by $\delta_i \leq 1$, we have $x(\pi) \in A_i(\pi)$, and therefore $A_i(\pi)$ is non-empty. It is compact by the continuity of u_i, and it is convex by the concavity of u_i. For all $C \in \mathcal{D}$, define $A_C(\pi) = \bigcap_{i \in C} A_i(\pi)$, also non-empty, compact, and convex; and define $A(\pi) = \bigcup_{C \in \mathcal{D}} A_C(\pi)$, which is non-empty and compact, but not necessarily convex. Theorem 7, in Appendix B, establishes that A is continuous as a correspondence on $[\mathcal{P}(X)]^n$ if either $\delta_i < 1$ for all $i \in N$ or LSWP holds.

For all $i \in N$, define $M_i(\pi) = \arg\max\{u_i(x) \mid x \in A(\pi)\}$. By the Theorem of the Maximum (Aliprantis and Border, 1994, Theorem 14.30), $M_i : [\mathcal{P}(X)]^n \to\to X$ has non-empty and compact values and is upper hemicontinuous; however, it is not necessarily convex-valued since $A(\pi)$ is not necessarily convex. Let $B_i(\pi) = \mathcal{P}(M_i(\pi))$ denote the set of mixtures of optimal proposals, which defines a non-empty, compact- and convex-valued, upper hemicontinuous correspondence (Aliprantis and Border, 1994, Theorem 14.14). Define the correspondence $B\colon [\mathcal{P}(X)]^n \to\to [\mathcal{P}(X)]^n$ by $B(\pi) = (B_1(\pi), \ldots, B_n(\pi))$; since $[\mathcal{P}(X)]^n$ is compact and convex, by Glicksberg (1952) B has a fixed point, say $\pi^* = (\pi_1^*, \ldots, \pi_n^*)$. Then π^*, together with acceptance sets $A_i^* = A_i(\pi^*)$, $i = 1, \ldots, n$, constitutes an equilibrium. This completes the proof of (ii).

To complete the proof of (i), let σ^* be a stationary equilibrium. In step (7) of the proof of Theorem 7, we prove the existence of $\hat{x} \in X$ such that, for all $i \in N$, $u_i(\hat{x}) > \delta_i v_i(\pi^*)$. By weak dominance, $\hat{x} \in A^*$. Then

$$\sup\{u_i(y) \mid y \in A^*\} \quad > \quad \delta_i v_i(\pi^*),$$

and sequential rationality requires π_i^* put probability one on $\arg\max\{u_i(y) \mid y \in A^*\}$ for all $i \in N$. In particular, $\pi_i^*(A^*) = 1$ for all $i \in N$.

To prove (iii), let σ^* be a no-delay stationary equilibrium. Each A_i^* is convex by strict quasi-concavity and weak dominance, and therefore each A_C^* is convex. Optimality of i's proposals requires that π_i^* put probability one on $\arg\max\{u_i(y) \mid y \in A^*\}$, which can be rewritten

$$\arg\max\{\arg\max\{u_i(y) \mid y \in A_C^*\} \mid C \in \mathcal{D}\}.$$

By convexity of A_C^* and strict quasi-concavity of u_i, $\arg\max\{u_i(y) \mid y \in A^*\}$ is a singleton. Since $\mathcal{D}$ is finite, the result is proved. ∎

A few remarks are in order. By definition, in no-delay equilibria the first proposal is always accepted, so there is no inefficiency due to delay. However, if agents' proposals differ and utility functions are strictly concave, there will be *ex ante* inefficiency due to uncertainty.

The definitions of Section 2 and the proof of Theorem 1 would require only notational changes if we generalized the voting rule to a collection $\{\mathcal{D}_i\}_{i \in N}$, where $\mathcal{D}_i$ is the non-empty, monotonic collection of coalitions that can pass a proposal from agent i. We would let $A^i = \bigcup_{C \in \mathcal{D}_i} A_C$ consist of the proposals i can pass. In the proof, we would define $A^i(\pi)$ accordingly, and we would define $M_i(\pi) = \arg\max\{u_i(x) \mid x \in A^i(\pi)\}$, restricting i's proposals to those the agent can pass; we again find a fixed point of the correspondence B, defined as before, giving us a no-delay stationary equilibrium.

The fixed points of B are *all* the stationary equilibrium proposals when each $\delta_i < 1$. Consider a stationary equilibrium with proposals given by profile π° and acceptance sets $\{A_i^\circ\}$. By part (i) of Theorem 1, $\pi_i^\circ(A^\circ) = 1$ for all $i \in N$; by weak dominance, $A_i^\circ \subseteq A_i(\pi^\circ)$ for all $i \in N$, implying $A^\circ \subseteq A(\pi^\circ)$. In equilibrium, agent i must put probability one on the utility maximizing proposals in A°, but it is conceivable that these proposals are not utility maximizing in $A(\pi^\circ)$. Let x' and C' satisfy $x' \in A_{C'}(\pi^\circ)$, $C' \in \mathcal{D}$, and $u_i(x') = \max\{u_i(x) \mid x \in A(\pi^\circ)\}$. In step (7) of the proof of Theorem 7, in Appendix B, we prove the existence of $\hat{x} \in X$ such that $u_j(\hat{x}) > \delta_j v_j(\pi^\circ)$ for all $j \in N$. That is, $\hat{x} \in A_j^s(\pi^\circ) = \{x \in X \mid u_j(x) > \delta_j v_j(\pi^\circ)\}$ for all $j \in N$, and therefore $\hat{x} \in A_{C'}^s(\pi^\circ) = \bigcap_{j\in C'} A_j^s(\pi^\circ)$. By concavity, $\alpha\hat{x} + (1-\alpha)x' \in A_{C'}^s(\pi^\circ)$ for all $\alpha \in (0,1)$. By weak dominance, $\alpha\hat{x} + (1-\alpha)x' \in A_{C'}^\circ$. By choosing $\alpha > 0$ low enough, i can pass a proposal arbitrarily close to x' and, by continuity of u_i, with utility arbitrarily close to $u_i(x')$. Therefore, in equilibrium, π_i° must put probability one on $\arg\max\{u_i(x) \mid x \in A^\circ(\pi^\circ)\}$.

In contrast, when $\delta_i = 1$ for some agents, there can exist stationary equilibria with delay: $n = 2$, unanimity, $x \in X$ Pareto optimal for agents 1 and 2, $y \in X$ such that $u_2(x) > u_2(y)$, $\rho_1 = 1$. In each period agent 1 proposes x with probability 1/2, and y with probability 1/2; and agent 2 rejects any $z \in X$ such that $u_2(z) < u_2(x)$. Then since x is Pareto optimal x is the best 1 can do given 2's acceptance rule; and with probability one x is proposed in finite time, so i receives utility of $u_i(x)$.

Because the set A of socially acceptable proposals need not be convex, even in equilibrium (as in Figure 1), mixed strategies are required in the proof of Theorem 1 to convexify an agent's best responses. This is not a problem if we impose certain restrictions on either the voting rule or the dimension of the set of alternatives. Say that $\mathcal{D}$ is a *filter* (or is *oligarchic*) if $K(\mathcal{D}) \equiv \bigcap_{C\in\mathcal{D}} C \in \mathcal{D}$: two polar examples are unanimity rule ($\mathcal{D} = \{N\}$) and dictatorship (filter with $K(\mathcal{D}) = \{i\}$).[7]

Theorem 2 *Assume either that $\delta_i < 1$ for all $i \in N$ or that LSWP holds. Then there exists a pure strategy no-delay stationary equilibrium if either $\mathcal{D}$ is a filter or $d = 1$.*

Proof: If $\mathcal{D}$ is a filter, then $A(\pi)$, defined in the proof of Theorem 1, is just $\bigcap_{i\in K(\mathcal{D})} A_i(\pi)$, a convex set. Therefore, revising the argument there using only pure strategies, we find that $M_i(\pi)$ is convex-valued for all $i \in N$, so there exists a pure strategy stationary equilibrium.

We claim $A(\pi)$ is also convex if $d = 1$. Let $\overline{x} = \max A(\pi)$ and $\underline{x} = \min A(\pi)$, which exist by compactness of X and continuity of the u_i, and let $\overline{C}$ and $\underline{C}$ satisfy $\overline{x} \in A_{\overline{C}}(\pi)$ and $\underline{x} \in A_{\underline{C}}(\pi)$. Clearly $x(\pi) \in [\underline{x}, \overline{x}] = \mathrm{conv}A(\pi)$ and, by concavity, $x(\pi) \in A_{\overline{C}}(\pi) \cap A_{\underline{C}}(\pi)$. Take any $y \in \mathrm{conv}A(\pi)$. Suppose

[7] See Section 5, footnote 8, for other examples of filters.

$y \in [\underline{x}, x(\pi)]$. Since $A_{\underline{C}}(\pi)$ is convex, by concavity, we have $y \in A_{\underline{C}}(\pi) \subseteq A(\pi)$. A symmetric argument addresses the case $y \in [x(\pi), \overline{x}']$. Therefore, the proof of Theorem 1, revised for pure strategies, establishes a pure strategy equilibrium. ∎

If $n = 2$, then all proper voting rules are filters: either unanimity or dictatorship. Thus, from Theorem 2 we obtain the existence of pure strategy stationary equilibria for the case of two-agent bargaining under all proper voting rules.

Finally, the high degree of symmetry in the example of Figure 1 generates a multiplicity of equilibria, as Baron (1991) notes, but the equilibria are payoff equivalent. This raises the issue of uniqueness of stationary equilibrium payoffs. While this property may hold in special environments, it does not hold generally: in Section 5, we give an example with each $\delta_i = 1$ in which equilibria are not payoff equivalent; and we will now show that non-unique equilibrium payoffs may occur even when the agents' discount rates are below one. Assume $n = 3$, majority rule, X the unit simplex in $\Re^3$, $\rho_i = 1/3$ and $\delta_i = .95$ for all $i \in N$. We give agent i an ideal point at e^i, the ith unit vector, and we define u_i by a monotonic transformation of Euclidean distance from e^i. Specifically, define the piecewise linear function f by

$$\begin{array}{llllllll} f(0) & = & 14,900 & f(.25) & = & 14,894 & f(.2762) & = & 14,890 \\ f(1.138) & = & 12,100 & f(1.1642) & = & 12,000 & f(1.2983) & = & 11,220 \\ f(1.3072) & = & 11,000 & f(1.4142) & = & 0, & & & \end{array}$$

as in Figure 2, and define $u_i(x) = f(||e^i - x||)$, which is concave, strictly quasi-concave, and continuous.

[Figure 2 about here.]

Consider a strategy profile where agent 1 proposes $a = (.8232, .1768, 0)$, 2 proposes $b = (0, .8232, .1768)$, and 3 proposes $c = (.1768, 0, .8232)$. Note that

$$\begin{array}{lllll} u_1(a) & = & f(.25) & = & 14,894 \\ u_1(b) & = & f(1.3072) & = & 11,000 \\ u_1(c) & = & f(1.1642) & = & 12,000 \end{array}$$

and that, by symmetry, the payoffs of agents 2 and 3 are merely permutations of those for agent 1. Agent 1's continuation value is

$$v_1 = \frac{1}{3}(14,894) + \frac{1}{3}(11,000) + \frac{1}{3}(12,000) = 12,631,$$

and, by symmetry, $v_1 = v_2 = v_3$. Noting that $\delta_1 v_1 = 12,000$, we define agent 1's acceptance set as

$$A_1 \quad = \quad \{x \in X \mid u_1(x) \geq 12,000\},$$

with A_2 and A_3 defined symmetrically. Proposal a gives agent 2 a payoff of exactly $12,000$, which is the minimum needed to acquire 2's approval. Thus, agent 1's strategy is a best response, as are 2's and 3's.

Now consider a strategy profile where agent 1 proposes $a' = (.8047, .1953, 0)$, 2 proposes $b' = (0, .8047, .1953)$, and 3 proposes $c' = (.1953, 0, .8047)$. Note that

$$\begin{array}{rclcl} u_1(a') & = & f(.2762) & = & 14,890 \\ u_1(b') & = & f(1.2983) & = & 11,220 \\ u_1(c') & = & f(1.138) & = & 12,100 \end{array}$$

and that, again, the payoffs of agents 2 and 3 are merely permutations of those for agent 1. Agent 1's continuation value is

$$v_1' = \frac{1}{3}(14,890) + \frac{1}{3}(11,220) + \frac{1}{3}(12,100) = 12,737,$$

and $v_1' = v_2' = v_3'$. Noting that $\delta_1 v_1' = 12,100$, we define agent 1's acceptance set as

$$A_1' \quad = \quad \{x \in X \mid u_1(x) \geq 12,100\},$$

with A_2' and A_3' defined symmetrically. Proposal a' gives agent 2 a payoff of exactly $12,100$, which is the minimum needed to acquire 2's approval. Thus, agent 1's strategy is a best response, as are 2's and 3's, giving us a stationary equilibrium with continuation payoffs distinct from those in the preceding example.

4 Continuity

The equilibria identified in Theorem 1 are parametrized by the agents' recognition probabilities, $\rho = (\rho_1, \ldots, \rho_n) \in \Delta$, and their discount factors, $\delta = (\delta_1, \ldots, \delta_n) \in [0,1]^n$. To these we add information about their utility functions: let $\Lambda \subset \Re^k$ be a set parameterizing profiles of utility functions, so that agent i's preferences can be represented as $u_i(x) = u_i(x, \lambda)$ for some

$\lambda \in \Lambda$. For instance, we could have $\Lambda \subseteq \Re^{nd}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$, each λ_i representing the ideal point of a quadratic utility function for agent i; or more generally, λ_i might be the matrix defining weighted Euclidean distance utilities (cf. Hinich and Munger, 1997); alternatively, preferences could be represented in a general separable form (cf. Caplin and Nalebuff, 1991)

$$u_i(x, \lambda) \quad = \quad \sum_{h=1}^{m} \lambda_i^h t^h(x) + t^{m+1}(x),$$

where $t^h \colon X \to \Re$ for $h = 1, 2, \ldots, m+1$. Assume u_i is jointly continuous; for all $\lambda \in \Lambda$, assume $u_i(\cdot, \lambda)$ is concave and non-negative; moreover, assume there is some $x \in X$ such that, for all $j \in N$, $u_j(x, \lambda) > 0$. For parameters ρ, δ, and λ, let $E(\rho, \delta, \lambda)$ denote the set of no-delay stationary equilibrium proposals given by the fixed points of the correspondence B, defined in the proof of Theorem 1. As discussed in the previous section, when discount rates are less than one $E(\rho, \delta, \lambda)$ consists of *all* stationary equilibrium profiles of proposals; when some discount rates are equal to one, E may exclude some profiles that cannot be obtained as limits of equilibria as discount rates approach one.

Theorem 3 *If either $\delta_i < 1$ for all $i \in N$ or LSWP holds at λ, then E is upper hemicontinuous at (ρ, δ, λ).*

Proof: Given parameters ρ, δ, and λ satisfying the assumptions of the theorem and given a profile π of mixed stationary proposals, let θ denote the vector $(\rho, \delta, \lambda, \pi)$. Define

$$A_i(\theta) \quad = \quad \{x \in X \mid u_i(x, \lambda) \geq \delta_i v_i(\pi, \rho, \lambda)\},$$

where $v_i(\pi, \rho, \lambda)$ is i's continuation value, defined as in Section 2 but using $u_i(\cdot, \lambda)$. Theorem 7, in Appendix B, establishes that A is continuous as a correspondence at θ. Hence, by the Theorem of the Maximum,

$$M_i(\theta) \quad \equiv \quad \arg\max\{u_i(x, \lambda) \mid x \in A(\theta)\}$$

is upper hemicontinuous at θ, and therefore so is $B_i(\theta) \equiv \mathcal{P}(M_i(\theta))$. Since B_i has closed values and regular range as well, it has closed graph (Aliprantis and Border, 1994, Theorem 14.11). Now, let $(\rho^m, \delta^m, \lambda^m) \to (\rho^\circ, \delta^\circ, \lambda^\circ)$, and take any sequence $\{\pi^m\}$ such that $\pi^m \in E(\rho^m, \delta^m, \lambda^m)$ for all m. Suppose $\pi^m \to \pi^\circ$. Since $\pi_i^m \in B_i(\theta^m)$ for all m and since B_i has closed graph, we see that $\pi_i^\circ \in B_i(\theta^\circ)$ for all $i \in N$. Therefore, $\pi^\circ \in E(\rho^\circ, \delta^\circ, \lambda^\circ)$, and we conclude that E has closed graph at θ. Since it has compact Hausdorff range space as well, it is upper hemicontinuous at θ (Aliprantis and Border, 1994, Theorem 14.12). ∎

As applications, note that when $\delta_i = 0$ for all $i \in N$, the agents will propose their (unique, say) ideal points when recognized; by Theorem 3, therefore, when discount factors are all close to zero, each proposal will be close to the proposer's ideal point. Similarly, when $\rho_i = 1$, as in the model of Romer and Rosenthal (1978a,b), we can easily solve for the equilibrium proposals for i; and when ρ_i is close to one the agent's proposals will be close to that when ρ_i equals one and hence the expected outcome will be close to the original. Finally, suppose all equilibrium proposals actually coincide at the point x^* for some λ^*; then for all values of λ close to λ^* all equilibrium proposals will be close to x^*.

5 Core Equivalence

In this section, we asume throughout that each u_i is strictly quasi-concave, so LSWP holds; $\rho_i > 0$, implying that all agents have some chance of being recognized; and $\delta_i = 1$, implying that all agents are perfectly patient. In addition, we assume that $\mathcal{D}$ is proper. Let $\mathcal{C}(\mathcal{D}) \subseteq X$ denote the set of *core outcomes* associated with the decisive coalitions $\mathcal{D}$:

$$\mathcal{C}(\mathcal{D}) \;\; = \;\; \{x \in X \mid y \in P_C(x) \text{ for no } C \in \mathcal{D}, y \in X\}.$$

In what follows we will denote pure strategy profiles of proposals by $p = (p_1, \ldots, p_n)$. Note that the next lemma requires strict quasi-concavity only for one direction.

Lemma 1 *$p_1^* = \cdots = p_n^* = x^*$ are no-delay stationary equilibrium proposals if and only if $x^* \in \mathcal{C}(\mathcal{D})$.*

Proof: Assume $p_1^* = \cdots = p_n^* = x^*$ are no-delay stationary equilibrium proposals. If $x^* \notin \mathcal{C}(\mathcal{D})$ then there exists $C \in \mathcal{D}$ and $y \in P_C(x^*)$, implying

$u_i(y) > u_i(x^*) = \delta_i v_i(\pi^*)$ for all $i \in C$. By weak dominance, $y \in A_i^*$ for all $i \in C$, and hence $y \in A^*$. Thus, when $i \in C$ is the proposer she gets a strictly higher payoff from proposing y than she does from proposing x^*, a contradiction.

Assume $x^* \in \mathcal{C}(\mathcal{D})$. Setting $A_i^* = A_i(x^*)$, we claim that $((A_1^*, p_1^*), \ldots, (A_n^*, p_n^*))$, where $p_1^* = \cdots = p_n^* = x^*$, is a no-delay stationary equilibrium. The acceptance sets clearly satisfy weak dominance, so if this is not an equilibrium, there must exist an agent with a better acceptable proposal. But if there exists $z \neq x^* \in X$ such that $z \in A_C^*$ for some $C \in \mathcal{D}$, then $\frac{1}{2}x^* + \frac{1}{2}z \in P_C(x^*)$ by strict quasi-concavity, contradicting $x^* \in \mathcal{C}(\mathcal{D})$. ∎

The next lemma reveals a connection between the structure of the set of no-delay stationary equilibrium proposals and the structure of the voting rule. Say that $\mathcal{D}$ is a *pre-filter* (or is *collegial*) if $K(\mathcal{D}) \neq \emptyset$, i.e., if at least one agent possesses a veto. Note that a filter is necessarily a pre-filter.[8] Define the *Nakamura number* of $\mathcal{D}$, denoted $\mathcal{N}(\mathcal{D})$, as

$$\mathcal{N}(\mathcal{D}) \;=\; \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq \mathcal{D}, K(\mathcal{S}) = \emptyset\}$$

if $\mathcal{D}$ is not collegial, and define $\mathcal{N}(\mathcal{D}) = \infty$ if it is. In words, when $\mathcal{D}$ is not collegial the Nakamura number of $\mathcal{D}$ is the size of the smallest collection of decisive coalitions having empty intersection; when $\mathcal{D}$ is collegial, it suffices here to assign it any infinite cardinality. If $\mathcal{D}$ is proper, as we now assume, $\mathcal{N}(\mathcal{D}) \geq 3$; and if $\mathcal{D}$ is non-collegial, $\mathcal{N}(\mathcal{D}) \leq n$.

Given a subset $Y \subseteq X$ and $x \in Y$, say that x is an *extreme point* of Y if there do not exist $y, z \in Y$ (distinct from x) and $\alpha \in (0, 1)$ such that $x = \alpha y + (1 - \alpha)z$.

Lemma 2 *Let $\sigma^* = ((A_1^*, \pi_1^*), \ldots, (A_n^*, \pi_n^*))$ be a no-delay stationary equilibrium. If $S(\pi^*)$ has more than one extreme point, then the set of extreme points of $S(\pi^*)$ has cardinality at least $\mathcal{N}(\mathcal{D})$.*

[8] For a "one parameter" example of the difference between pre-filters and filters, let $n = 5$, and consider a *weighted q-rule* with the following weights assigned to the agents: $w_1 = w_2 = .35$, $w_3 = w_4 = w_5 = .1$. The set of decisive coalitions is then $\mathcal{D}(q) = \{C \subseteq N \mid \sum_{i \in C} w_i \geq q\}$, where q ranges from .5 to 1. For $q \in [.65, .70]$ the rule is a filter, since a decisive coalition requires both large agents to be members, and these agents together constitute a decisive coalition. Similarly, for $q \in (.90, 1]$ the rule is a filter, since the only decisive coalition is N. Finally, for $q \in (.70, .90)$ the rule is a pre-filter, as the presence of the two large agents is necessary for a coalition to be decisive, but is not sufficient. When $q \in (.5, .65)$ the rule is neither.

Proof: Suppose that $S(\pi^*)$ has more than one extreme point yet the cardinality of the set of extreme points is less than $\mathcal{N}(\mathcal{D})$. From Theorem 1, strict quasi-concavity implies that $S(\pi^*)$ is finite and, therefore, has a finite number of extreme points. Denumerate the set of extreme points of $S(\pi^*)$ as x_1, x_2, …, x_m, and define $C_h = \{i \in N \mid x_h \in A_i^*\}$, $h = 1, \ldots, m$. Since σ^* is a no-delay equilibrium, we have $C_h \in \mathcal{D}$ for all h. Then, by definition of the Nakamura number, there is some $j \in \bigcap_{h=1}^m C_h$. That is, there is some j such that $x_h \in A_j^*$ for all h.

We claim that $u_j(x) = u_j(y)$ for all $x, y \in S(\pi^*)$. Otherwise,

$$\overline{u} = \max\{u_j(x) \mid x \in S(\pi^*)\} > \min\{u_j(x) \mid x \in S(\pi^*)\} = \underline{u},$$

and furthermore, by the Bauer Maximum Principle (Aliprantis and Border, 1994, Theorem 4.104), $\underline{u}$ is achieved at an extreme point of $S(\pi^*)$, say x_k. But, since $\delta_j = 1$, this means that j's acceptance set violates weak dominance: $\rho_i > 0$ for all $i \in N$ implies $\underline{u} < v_j(\pi^*)$, so in equilibrium we must have $x_k \notin A_j^*$. This contradiction establishes the claim.

Note that, by concavity of the u_i, $u_i(x(\pi^*)) \geq v_i(\pi^*)$. Since $x_1 \in A_{C_1}^*$, strict quasi-concavity and the weak dominance condition on $A_{C_1}^*$ imply that the convex hull of $\{x_1, x(\pi^*)\}$ is included in $A_{C_1}^*$. Then, because π_j^* is a best response proposal strategy for j, continuity of u_j implies that there exists $x^* \in A^*$ such that $u_j(x^*) \geq u_j(x(\pi^*))$.

By the Krein-Milman Theorem (Aliprantis and Border, 1994, Theorem 4.103), we know that $x(\pi^*)$, since it is an element of the convex hull of $S(\pi^*)$, can be written as a convex combination of the extreme points of $S(\pi^*)$, which, we may suppose, places positive weight on the first h extreme points. Since $S(\pi^*)$ contains more than one element, $x(\pi^*)$ cannot itself be an extreme point, so $h > 1$. Then strict quasi-concavity and $u_j(x_1) = \cdots = u_j(x_h)$ yield $u_j(x(\pi^*)) > u_j(x_1)$. In equilibrium, j must propose alternatives with utility at least $u_j(x^*) \geq u_j(x(\pi^*)) > u_j(x_1)$. But $\rho_j > 0$, so j's proposals are elements of $S(\pi^*)$, contradicting the claim that j's utility is constant on $S(\pi^*)$. ∎

Nakamura (1979) introduces the concept of the Nakamura number in his analysis of core non-emptiness in the absence of convexity restrictions, while Schofield (1984) and Strnad (1985) assume convexity of X and impose a weak convexity condition on preferences: they prove that the core is non-empty if $d \leq \mathcal{N}(\mathcal{D}) - 2$; and that, otherwise, there exist profiles of preferences for which the core is empty. As special cases, the core is non-empty if $d = 1$ (since $\mathcal{N}(\mathcal{D}) \geq 3$) or if $\mathcal{D}$ is a pre-filter (since then $\mathcal{N}(\mathcal{D}) = \infty$). These two cases play analogous roles in the analysis of this section, providing sufficient structure for the existence of pure strategy stationary equilibria and equivalence with core outcomes.

If $\mathcal{D}$ is a pre-filter, so that $\mathcal{N}(\mathcal{D})$ is infinite, it is clear that the set of extreme points of $S(\pi^*)$, a finite set, cannot exceed $\mathcal{N}(\mathcal{D})$. From Lemma 2, therefore, $S(\pi^*)$ has at most one extreme point. Since it is finite, from part (ii) of Theorem 1, and non-empty, it has exactly one, say x^*. From Lemma 1, therefore, we have $x^* \in \mathcal{C}(\mathcal{D})$, giving us the following result.

Theorem 4 *If $\mathcal{D}$ is a pre-filter, then all no-delay stationary equilibria are in pure strategies and are of the form $p_1^* = \cdots = p_n^* = x^*$ for some $x^* \in \mathcal{C}(\mathcal{D})$.*

Using Lemma 1, we have an equivalence between the core outcomes of the underlying voting rule and the equilibrium bargaining outcomes when agents are perfectly patient and the decisive coalitions $\mathcal{D}$ constitute a pre-filter. Combine this with Theorems 2 and 3 when $\mathcal{D}$ is a *filter*: pure strategy equilibria exist for all discount factors; the equilibrium outcomes coincide with the core outcomes when discount factors all equal one; and the equilibrium outcomes converge to the core when these factors all converge to one.

If $d = 1$, the set of extreme points of $S(\pi^*)$ has at most two elements. Since $\mathcal{D}$ is proper, however, $\mathcal{N}(\mathcal{D}) \geq 3$, and from Lemma 2 we then know that $S(\pi^*)$ has at most one extreme point. Again, because $S(\pi^*)$ is finite and non-empty, we conclude that it has exactly one, say x^*. From Lemma 1, therefore, we have $x^* \in \mathcal{C}(\mathcal{D})$, giving us the following result.

Theorem 5 *If $d = 1$, then all no-delay stationary equilibria are in pure strategies and are of the form $p_1^* = \cdots = p_n^* = x^*$ for some $x^* \in \mathcal{C}(\mathcal{D})$.*

When the alternative space has but a single dimension and agents are perfectly patient, we again get an equivalence between the core outcomes of the underlying voting game and the equilibrium bargaining outcomes. Furthermore, by Theorems 2 and 3 we know that pure strategy equilibrium outcomes exist for all discount factors, and that they converge to core outcomes when discount factors converge to one.

Now suppose in addition that the core of $\mathcal{D}$ is actually a *singleton*; then we can also conclude that, as the discount factors converge to one, all of the equilibrium bargaining proposals collapse down to a single point in the space. Therefore, even if there are multiple equilibria when discount factors are less than one, when these factors are close to one the equilibrium proposals will be close to one another. A sufficient condition for the core (when non-empty) to be a singleton, given our assumption of strictly quasi-concave utilities, is that $\mathcal{D}$ is *strong:* $C \notin \mathcal{D}$ implies $N \setminus C \in \mathcal{D}$.[9] Majority rule is an example of a strong rule when n is odd, so by Theorem 5 the current bargaining model replicates the predictions of the Median Voter Theorem (Black, 1958; Downs, 1957) when agents are perfectly patient; and with Theorem 3, bargaining outcomes will all be close to the median voter's ideal point when discount factors are close to one. (Analogous results employing Theorem 4 are less interesting, as the only strong pre-filter turns out to be dictatorship.)

Given Theorems 4 and 5 and the results of Schofield (1984) and Strnad (1985), one might conjecture that core equivalence holds whenever $d \leq \mathcal{N}(\mathcal{D}) - 2$. To see that conjecture is false, consider the following example: $n = 4$, $d = 2$, $X = [-1, 1]^2$, $\rho_i = .25$ for all $i \in N$, and agent i's utility function is quadratic on X with ideal point x^i: $u_i(x) = -||x^i - x||^2$ (plus a constant to keep utilities non-negative, a term we can ignore since $\delta_i = 1$). Let the ideal points of agents 1 through 4 be given by $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$, respectively. Under majority rule there is a unique core point at $(0, 0)$; however, consider the proposals $p_1 = (\alpha, 0)$, $p_2 = (0, \alpha)$, $p_3 = (-\alpha, 0)$, and $p_4 = (0, -\alpha)$, where $\alpha \in (0, 1]$. Quadratic utility implies mean-variance analysis, so given that the mean of these four (equally weighted) proposals is $(0, 0)$ and the variance is α^2, agent i's continuation value is equal to $-1 - \alpha^2$. Consider the utility to agents 2 and 4 from the proposal p_1; by the Pythagorean Theorem this is equal to $-1 - \alpha^2$, and so these two agents are indifferent between accepting and rejecting p_1 and thus accepting is a best response. A similar logic holds for proposals p_2, p_3 and p_4, thereby guaranteeing that each proposal can be accepted. Further, given the continuation values is it clear that these are utility-maximizing choices for the proposers. Thus, we have a stationary equilibrium away from the unique core outcome

[9] The weighted q-rule of footnote 8 is strong when $q \in (.5, .55]$, since a coalition or its complement must include either both "large" agents or one "large" and two "small" agents; whichever does is a decisive coalition.

at $(0, 0)$ (in fact, a continuum of them, parametrized by α).

The "problem" in this example is that the core is not associated with any one agent's ideal point and so is not offered as a proposal. To illustrate, suppose there is now a fifth agent with quadratic preferences and ideal point at $(0, 0)$. Under majority rule the core remains at $(0, 0)$; however now, as long as ρ_5 is strictly positive, this core point can have some influence on the agents' behavior. In particular, the above proposals for agents 1 through 4, along with $p_5 = (0, 0)$, do *not* constitute an equilibrium, even with ρ_5 arbitrarily small (and the other recognition probabilities equal to, say, $.25 - \rho_5/4$). To see this, note that the mean of the proposals has not changed; the variance has strictly decreased below α^2, however, and by offering $p_1 = (\alpha, 0)$ agent 1 will acquire the votes of no other agent.[10]

The only equilibrium in this augmented example is, as in Theorems 4 and 5, at the core point. The next theorem subsumes this observation as a special case of a general result on quadratic preferences and strong voting rules.

Theorem 6 *If $\mathcal{D}$ is strong, u_i is quadratic for all $i \in N$, and $\{x^*\} = \mathcal{C}(\mathcal{D})$ with $x^* \in intX$, then all no-delay stationary equilibria are in pure strategies and are of the form $p_1^* = \cdots = p_n^* = x^*$.*

Proof: Let $u_i(x) = -||x - x^i||^2$ for all $i \in N$, and let σ^* be a no-delay stationary equilibrium. The proof proceeds in a number of steps.

(1) $\mathcal{D}$ strong, u_i quadratic and $\{x^*\} = \mathcal{C}(\mathcal{D})$ with $x \in \text{int}X$ imply $x^* = x^i$ for some $i \in N$: if not, $\nabla u_i(x^*) \neq 0$ for all $i \in N$, so let H be a hyperplane, with normal p, through zero containing no gradient vector. Since $\mathcal{D}$ is strong either

$$\{i \in N \mid \nabla u_i(x^*) \cdot p > 0\} \text{ or } \{i \in N \mid \nabla u_i(x^*) \cdot p < 0\}$$

is decisive. Suppose the former, without loss of generality. Since $x^* \in \text{int}X$, there exists $\epsilon > 0$ such that $x^* + \epsilon p \in X$ and for all members i, $u_i(x^* + \epsilon p) > u_i(x^*)$. Thus, $x^* \notin \mathcal{C}(\mathcal{D})$, a contradiction.

(2) $\{i \in N \mid u_i(x^*) \geq u_i(x(\pi^*))\} \in \mathcal{D}$: if not, since $\mathcal{D}$ is strong, its complement is decisive, implying $x^* \notin \mathcal{C}(\mathcal{D})$, a contradiction.

(3) $x^* \in A^*$: if not $C = \{i \in N \mid x^* \in A_i^*\} \notin \mathcal{D}$, and by $\mathcal{D}$ strong $N \setminus C \in \mathcal{D}$. By weak dominance, $u_i(x^*) \leq v_i(\pi^*)$ for all $i \in N \setminus C$; then concavity yields

[10] Note that this example demonstrates a violation of lower hemicontinuity of E in ρ.

$u_i(x^*) \leq u_i(x(\pi^*))$ for all $i \in N \setminus C$. If $|S(\pi)| > 1$, these inequalities hold strictly, but then $x \notin \mathcal{C}(\mathcal{D})$, a contradiction. The remaining case is $|S(\pi)| = \{y\}$ for some $y \in X \setminus x$, but then $\frac{1}{2}x^* + \frac{1}{2}y \in P_{N\setminus C}(x^*)$ and $x \notin \mathcal{C}(\mathcal{D})$, a contradiction.

(4) Let agent 1 be such that $x^* = x^1$; then $S(\pi_1^*) = \{x^*\}$: follows directly from (3) and the definition of stationary equilibrium.

Let $\underline{x} \in \arg\min\{u_1(x) \mid x \in S(\pi^*)\}$, which is well-defined by Theorem 1, parts (i) and (ii). Suppose, in order to deduce a contradiction, that $\underline{x} \neq x^*$, i.e., $|S(\pi^*)| > 1$.

(5) $u_1(\underline{x}) < v_1(\pi^*)$: follows from $p_1^* = x^*$ and $\rho_1 > 0$.

(6) x^* is a total median; in particular, the coalition

$$D \quad = \quad \{i \in N \mid (x^i - x^*) \cdot (x(\pi^*) - \underline{x}) \geq 0\}$$

is decisive: otherwise, by $\mathcal{D}$ strong, $N \setminus D \in \mathcal{D}$; but because $x^* \in \mathrm{int}X$ and the u_i are quadratic, we could then take $\epsilon > 0$ small enough that

$$x^* + \epsilon(\underline{x} - x(\pi^*)) \in P_{N\setminus D}(x^*)$$

and $x^* \notin \mathcal{C}(\mathcal{D})$, a contradiction.

(7) For all $i \in D$,

$$\begin{aligned}
&||x^i - x(\pi^*)||^2 - ||x^i - \underline{x}||^2 \\
&\quad = \quad -2x^i \cdot (x(\pi^*) - \underline{x}) + x(\pi^*) \cdot x(\pi^*) - \underline{x} \cdot \underline{x} \\
&\quad \leq \quad -2x^* \cdot (x(\pi^*) - \underline{x}) + x(\pi^*) \cdot x(\pi^*) - \underline{x} \cdot \underline{x} \\
&\quad = \quad ||x^* - x(\pi^*)||^2 - ||x^* - \underline{x}||^2.
\end{aligned}$$

(8) For all $i \in D$, $u_i(\underline{x}) < v_i(\pi^*)$: quadratic utility implies mean-variance analysis; letting v^* denote the variance of π^*, we have

$$\begin{aligned}
v_i(\pi^*) - u_i(\underline{x}) &= u_i(x(\pi^*)) - v^* - u_i(\underline{x}) \\
&= -||x^i - x(\pi^*)||^2 + ||x^i - \underline{x}||^2 - v^* \\
&\geq -||x^* - x(\pi^*)||^2 + ||x^* - \underline{x}||^2 - v^* \\
&= u_1(x(\pi^*)) - v^* - u_1(\underline{x}) \\
&= v_1(\pi^*) - u_1(\underline{x}) \\
&> 0,
\end{aligned}$$

where the first inequality follows from (7) and the second from (5).

(9) $\underline{x} \notin A^*$: given (8), the equilibrium condition of Section 2 implies that $\underline{x} \notin A_i^*$ for all $i \in D$; since $D \in \mathcal{D}$, by (6), and $\mathcal{D}$ is proper and monotonic, we have $\{i \in N \mid \underline{x} \in A_i^*\} \notin \mathcal{D}$.

Because $\underline{x}$ is, by definition, proposed with positive probability, (9) contradicts our assumption that σ^* is a no-delay equilibrium. Thus, we conclude that $\underline{x} = x^*$.

(10) $S(\pi^*) = \{x^*\}$: follows from $\underline{x} = x^*$ and uniqueness of agent 1's ideal point. ∎

Thus, under majority rule (n odd) and quadratic utilities, if the agents' ideal points satisfy the "Plott conditions" (Plott, 1967) and agents are perfectly patient, the unique equilibrium outcome of the bargaining game coincides with the majority rule core point. Furthermore, by Theorem 3, if we perturb the ideal points, the equilibrium proposals stay near to one another, as well as close to their original position. Just this sort of continuity has been witnessed experimentally: in Fiorina and Plott (1978) five-person, two-dimensional experiments were run in which at times the utilities were such that a majority rule core existed, while at other times the required conditions were not quite satisfied. The experimental outcomes tended to cluster around the core in the former, and did not stray very far in the latter.

The assumption of quadratic utility functions allows us to use mean-variance analysis in the proof of Theorem 6. To see that the theorem does not generalize to all utility functions based on Euclidean distance, return to our earlier example: $n = 5$, $d = 2$, $X = [-1, 1]^2$, $\rho_i = .2$ for all $i \in N$, ideal points x^i are at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(0, 0)$; we now modify the example by assuming $u_i(x) = -||x^i - x||^4$. Given the profile of proposal strategies where all agents propose their ideal points, the continuation values of the first four agents are -5, and the continuation value of agent 5, the core agent, is $-.8$. The utility to agents 2 and 4 from the proposal $p_1 = (1, 0)$ is -4, which is greater than their continuation values, and accepting is a best response. A similar logic holds for the remaining proposals. Thus, we have a stationary equilibrium in which the unique core outcome occurs with probability one fifth.

6 Discussion

We have analyzed a non-cooperative model of multi-agent bargaining over a multidimensional alternative space, proving existence of stationary equilibria, and relating equilibrium outcomes to the core outcomes of the underlying voting game. While our bargaining model replicates the predictions of the core under certain conditions, it avoids the principle "negative" result of social choice theory, the instability of social choices when the core is empty. Furthermore, we establish a robustness property of equilibrium predictions, namely, that the set of outcomes does not "blow up" upon perturbation of parameters. Our existence result relies on the use of mixed strategies, a maneuver with an extensive pedigree in game theory. One can see this as an advantage of the non-cooperative approach in contrast to the cooperative, in that an accepted (or, at least, standard) way out of the existence problem is readily at hand. Alternatively, it should be noted that the necessity of allowing for mixed strategies is being driven here by our quest for *stationary* equilibria. If instead we only required equilibria that were (say) subgame perfect, then presumably the existence of such equilibria in pure strategies for finite-horizon versions of the game, along with a limiting argument, would have sufficed to generate a pure-strategy equilibrium existence result. In any event, we *are* able to prove existence of pure strategy stationary equilibria for the special cases of two-agent bargaining (common in economic models), for n-agent bargaining with unanimous consent, and for one-dimensional spaces of alternatives.

The current model implicitly assumes that each agent's utility from the (unseen) "status quo" alternative is zero, while alternatives in X give the agents non-negative utility; equivalently, everyone weakly prefers all the alternatives in X to the status quo. If we drop this assumption, the proof of part (i) of Theorem 1 no longer goes through, but we regain existence if (a) we explicitly specify the status quo, q, as an alternative in X; (b) we impose the requirement that the discount rates of the agents are identical, i.e., there exists some $\delta \in [0,1]$ such that, for all $i \in N$, $\delta_i = \delta$; and (c) we define agent i's payoff from outcome (x,t) as $(1-\delta^{t-1})u_i(q) + \delta^{t-1}u_i(x)$.[11] If we then assume $u_i(q) = 0$ for all $i \in N$ (now just a normalization), agent i's payoff

[11]It need no longer be the case that $u_i(x(\pi)) \geq \delta_i v_i(\pi)$, so that $x(\pi)$ need not be an element of $A_i(\pi)$, invalidating our argument that $A_C(\pi)$ is non-empty. When all agents have discount rate δ, however, concavity of u_i yields $(1-\delta)q + \delta x(\pi) \in A_i(\pi)$ for all $i \in N$.

from outcome (x, t) would be defined just as before. As a special case, we obtain the model of Romer and Rosenthal (1978a,b), where $d = 1$, majority rule, $\rho_i = 1$ for some $i \in N$ (the "agenda setter") and $\delta_1 = \cdots = \delta_n = 0$. The core equivalence results are unaffected by the introduction of the status quo.

Once the status quo is explicitly brought into the model, an obvious but difficult extension would be to allow for bargaining to continue after passage of a proposal in period t, that proposal being the new status quo for period $t+1$, and so on. Baron (1996) characterizes the stationary equilibria of such a model when the set of alternatives is one-dimensional, and Baron and Herron (1998) investigate a finite horizon version of the model with three agents, two-dimensional policy space, and quadratic utility functions. With the addition of this state variable, stationary strategies must be conditional on the status quo: proposal strategies would be mappings from X (possible status quo outcomes) to probability measures on X, and acceptance strategies would be correspondences from X to X. It may be possible to modify the techniques of this paper to the analysis of this complex but realistic setting, but that is a matter for future research.

We have taken recognition probabilities as exogeneously given, without offering an explanation of their possible origins. One interpretation, appropriate when bargaining takes place within a parliamentary or legislative body, is that the agents represent parties and that the recognition probability of a party represents the number of seats it holds, proxying for the party's influence in lawmaking. An interesting extension would be to explicitly model the voters who elect the members of parliament, providing an alternative to Austen-Smith and Banks (1988), Coate (1997), Schofield (1998), Schofield and Sened (1998), and Baron and Diermeier (1998).

While we have investigated a model of bargaining, other types of institutional structures have been modelled non-cooperatively to obtain equilibrium existence in the absence of core outcomes. These approaches include sophisticated voting under various types of agendas;[12] mixed strategy equilibria in two-party spatial competition games;[13] and the "structure-induced equilibrium" model of committees (Shepsle, 1979), in which each committee is

[12]Farquharson (1969), Miller (1977,1980), McKelvey and Niemi (1978), Shepsle and Weingast (1984), Banks (1985).

[13]Kramer (1978), McKelvey (1986), Laffond, Laslier, and Le Breton (1993), Banks, Duggan, and Le Breton (1998), Laslier and Picard (1998).

assigned to a dimension of the alternative space and is given sole jurisdiction over the location of the policy along that dimension. In contrast, to the latter approach, our bargaining model might be thought of as a model of legislatures in the *absence* of a formal committee system, providing a useful benchmark against which a general model of institutional choice might be constructed, and offering a prediction in situations where no such institutional arrangements are to be found. An important question is then the extent to which our model gives different predictions for the same underlying preferences, and how well these predictions square with empirical observations.

An alternative within the cooperative paradigm is to consider solutions other than the core or top cycle set. Numerous alternative solution concepts have been suggested, such as the *von Neumann-Morgenstern solution* (von Neumann and Morgenstern, 1944), the *Nash bargaining solution* (Nash, 1950), the *bargaining set* (Aumann and Maschler, 1964), the *competitive solution* (McKelvey, Ordeshook, and Winer, 1978), the *uncovered set* (Miller, 1980; McKelvey, 1986; Cox, 1987), and the *heart* (Schofield, 1996, 1998). A question we have not addressed in this paper is the relationship between the equilibrium outcomes of our non-cooperative bargaining model and the above solutions, but the results of Section 5 do inform us that assuming unanimity rule and setting discount rates equal to one in our model does *not* yield the Nash bargaining solution (or any other selection from the Pareto optimals): by Lemma 1, every Pareto optimal alternative could be supported as a stationary equilibrium outcome for this specification of the model. It may be, however, that equilibrium outcomes *do* converge to the Nash solution as discount rates increase to one, giving us a robust selection from the limit equilibrium outcomes.

A Limited Shared Weak Preference

In the subsections below, we construct two general models in which the LSWP restriction holds: one in which limited forms of strict quasi-concavity and monotonicity (satisfied in most economic models) are imposed, another in which a limitation on shared indifference is imposed. In both cases, we will take arbitrary $C \subseteq N$ and $x \in X$ such that $|R_C(x)| > 1$. For any $x' \in R_C(x)$, we must then construct a sequence in $P_C(x)$ converging to x'. We note here that, since $|R_C(x)| > 1$, we can take $x'' \in R_C(x)$ distinct from x'.

A.1 Economic Environments

Assume $X \subseteq Z \times W \times T$, where $Z \subseteq \Re^l$, $W \subseteq \Re_+^{kn}$, and $T \subseteq \Re_+^n$, with elements written $x = (z, w, t)$, $x' = (z', w', t')$, and so on. Here, l represents a number of public goods, k represents a number of private goods, and elements of T represent allocations of an additional private good (distinguished notationally to facilitate the analysis when preferences are quasi-linear). Let $w(i)$ denote the components of w representing i's consumption of the first k private goods. To the assumptions on X in Section 2, we add *transferability* of private goods: given $x \in X$ and $x' \geq 0$, if $z' = z$, $\sum_{i=1}^n w(i) = \sum_{i=1}^n w'(i)$, and $\sum_{i=1}^n t_i = \sum_{i=1}^n t'_i$, then $x' = (z', w', t') \in X$.

In addition to continuity, we assume that u_i is strictly monotonic in private goods, and we assume u_i is quasi-concave, strictly so in the public good and strictly so in i's consumption of the first k private goods. Formally, our monotonicity assumption is that, for all $x, x' \in X$ with $z = z'$ if $(w(i), t_i) \geq (w'(i), t'_i)$ with strict inequality in at least one component, then $u_i(x) > u_i(x')$. Our convexity assumption is that, for all $x, x' \in X$ and all $\alpha \in (0, 1)$, $u_i(\alpha x + (1-\alpha)x') \geq \min\{u_i(x), u_i(x')\}$, with strict inequality if $z \neq z'$ or if $w(i) \neq w'(i)$. These assumptions generalize many economic models: setting $k = 0$ and noting that we do not require strict quasi-concavity in t, we encompass the standard quasi-linear model of public decisions with transfers; setting $l = 0$ and $T = \{0\}$, we encompass private good exchange economies; setting $k = 0$ and $T = \{0\}$, we have a pure public good economy with strictly convex preferences (this case is also treated in the next subsection); or setting $l = k = 0$, we can obtain the divide the dollar model. In fact, because we do not require monotonicity in the public goods, we allow for the possibility of public bads or, generalizing the classical spatial model, for ideal (or "satiation") points.

Now take C, x', and x'' as at the beginning of the section, and define $\hat{x} = \frac{1}{2}x' + \frac{1}{2}x''$. By quasi-concavity, we have $\hat{x} \in R_C(x)$. We consider four cases. First, if $z' \neq z''$, we have $\hat{x} \in P_C(x)$ by our convexity assumption. That assumption also yields $\frac{1}{m}\hat{x} + (1 - \frac{1}{m})x \in P_C(x)$ for all non-negative integers m. Letting m go to infinity, we have the desired sequence.

Second, if $z' = z''$ and $w'(i) \neq w''(i)$ for some $i \in C$, then, by our convexity assumption, $u_i(\hat{x}) > \min\{u_i(x'), u_i(x'')\} \geq u_i(x)$. Since $w'(i) \neq w''(i)$, it follows that $\hat{w}(i)$ is non-zero in some component, say the hth. Define $\tilde{x} \in X$ from $\hat{x}$ by taking a small enough amount, say $\epsilon > 0$, of the hth good and distributing it to the other members of C. By continuity, we can take ϵ small enough that $u_i(\tilde{x}) > u_i(x)$. By strict monotonicity in private goods, we have $\tilde{x} \in P_C(x)$. That assumption also yields $\frac{1}{m}\tilde{x} + (1 - \frac{1}{m})x \in P_C(x)$ for all non-negative integers m. Letting m go to infinity, we have the desired sequence.

Third, suppose $z' = z''$, $w'(i) = w''(i)$ for all $i \in C$, and $t'_i \neq t''_i$ for some $i \in C$. Without loss of generality, suppose $t'_i > t''_i$. By strict monotonicity, $u_i(\hat{x}) > u_i(x'') \geq u_i(x)$. Thus, following the argument of the second case, we can define $\tilde{x} \in P_C(x)$ from $\hat{x}$ by distributing a small enough amount of the distinguished private good from i to the other members of C. The desired sequence is defined as in the second case.

Fourth, suppose $z' = z''$, $w'(i) = w''(i)$ for all $i \in C$, and $t'_i = t''_i$ for all $i \in C$. Since $x' \neq x''$, there is some $j \notin C$ such that $(w'(j), t'_j) \neq 0$ or $(w''(j), t''_j) \neq 0$. Therefore, $(\hat{w}(j), \hat{t}_j) \neq 0$, and we can define $\tilde{x} \in P_C(x)$ from $\hat{x}$ by taking a small enough amount of some private good from j and distributing it to the members of C. The desired sequence is defined in the now familiar way.

A.2 Limited Shared Indifference

Assume now, as in Section 2, only that X is compact and convex. As a special case of the last subsection, we know that strict quasi-concavity is sufficient for the condition of interest. In this subsection, we drop strict quasi-concavity by imposing, along with two other weak conditions, a condition limiting the extent to which agents may share indifference. Let $I_i(x) = \{y \in X \mid u_i(y) = u_i(x)\}$. In addition to continuity and concavity,[14] we impose three conditions on the agents' utility functions: for all $x \in X$,

(i) $R_i(x) = \overline{P_i(x)}$ if $P_i(x) \neq \emptyset$;

(ii) $P_i(x) = \emptyset$ implies $R_i(x) = \{x\}$;

(iii) for all $j \neq i$, $I_i(x) \cap I_j(x)$ contains no line segment.

[14] In this subsection, a condition weaker than concavity is sufficient for our arguments. It is called "semistrict quasiconcavity" by Aliprantis and Border (1994, p.175): for all $x, y \in X$ and all $\alpha \in (0,1)$, $u_i(x) > u_i(y)$ implies $u_i(\alpha x + (1-\alpha)y) > u_i(y)$.

Condition (i) is the requirement that the agents' indifference curves be "thin." Condition (ii) requires that an agent has at most one ideal point. Condition (iii) limits the extent of shared indifference and is satisfied if each u_i is strictly quasi-concave. Moreover, it is enough that u_i be strictly quasi-concave for $n-1$ agents. If X is a subset of $\Re^2$, the condition is also satisfied when the agents' utility functions are linear with pairwise linearly independent gradients.[15]

Take C, x', and x'' as at the beginning of the section. Use condition (iii) to find $\hat{x}$ in the line segment $[x', x''] = \{\alpha x' + (1-\alpha)x'' \mid \alpha \in [0,1]\}$ such that $u_i(\hat{x}) = u_i(x')$ for at most one agent $i \in C$. By concavity, $u_j(\hat{x}) > \min\{u_j(x'), u_j(x'')\} \geq u_j(x)$ for all other $j \in C$. If $u_i(\hat{x}) = u_i(x')$ for no $i \in C$, then it follows that $\hat{x} \in P_C(x)$. By concavity, $\frac{1}{m}\hat{x} + (1-\frac{1}{m})x \in P_C(x)$ for all non-negative integers m. Letting m go to infinity, we have the desired sequence.

Suppose $u_i(\hat{x}) = u_i(x')$ for one $i \in C$. By continuity, there exists an open set G containing $\hat{x}$ such that $G \subseteq P_j(x)$ for all members $j \neq i$ of C. Since $\hat{x} \neq x'$, it follows from condition (ii) that $P_i(x) \neq \emptyset$. Then condition (i) implies that $\hat{x} \in R_i(x) = \overline{P_i(x)}$, so there exists $\tilde{x} \in G \cap P_i(x) \subseteq P_C(x)$. Again, we use concavity to construct the desired sequence.

B Continuity of Acceptance Correspondences

As in Section 4, let $\rho = (\rho_1, \ldots, \rho_n) \in \Delta$ and $\delta = (\delta_1, \ldots, \delta_n) \in [0,1]^n$ be the profiles of agents' recognition probabilities and discount rates. Index profiles of utility functions by $\lambda \in \Lambda \subseteq \Re^k$, and assume each $u_i(\cdot, \lambda)$ is concave and non-negative; moreover, assume there is some $x \in X$ such that, for all $j \in N$, $u_j(x, \lambda) > 0$. Lastly, assume each u_i is jointly continuous. Let $\Theta = \Delta \times [0,1]^n \times \Lambda \times \mathcal{P}(X)$, let $\theta = (\rho, \delta, \lambda, \pi)$ as in the proof of Theorem 3, and define

$$r_i(\theta) \;=\; \delta_i v_i(\pi, \rho, \lambda),$$

where v_i is defined as in Section 2 but using $u_i(\cdot, \lambda)$. Define the correspondence $A_i \colon \Theta \rightarrow\!\!\!\rightarrow X$ by

$$A_i(\theta) \;=\; \{x \in X \mid u_i(x, \lambda) \geq r_i(\theta)\},$$

[15] The example at the end of Section 2 illustrates the role of the linear independence condition.

and let $A_C(\theta) = \bigcap_{i \in C} A_i(\theta)$ and $A(\theta) = \bigcup_{C \in \mathcal{D}} A_C(\theta)$. We can now state the main result of this section.

Theorem 7 *A has non-empty, compact values. If either $\delta_i < 1$ for all $i \in N$ or LSWP holds at θ, then A is continuous at θ.*

Proof: The proof proceeds in a series of steps.

(1) A_C has non-empty values: by concavity and non-negativity of $u_i(\cdot, \lambda)$ and $\delta_i \leq 1$, it follows that $u_i(x(\pi), \lambda) \geq r_i(\theta)$; therefore, $x(\pi) \in A_C(\theta)$.

(2) A_C is compact-valued: follows from the continuity of $u_i(\cdot, \lambda)$, the compactness of X, and the fact that compactness of the $A_i(\theta)$ sets is preserved by intersections and (finite) unions.

(3) A_i is upper hemicontinuous: Take any θ and any open $V \subset X$ such that $A_i(\theta) \subseteq V$. Suppose there is a sequence $\{\theta^m\}$ converging to θ such that, for all m, $A_i(\theta^m) \setminus V \neq \emptyset$. For all m, let $x^m \in A_i(\theta^m) \setminus V$; then $\{x^m\}$ lies in $X \cap V^c$, which is compact since V is open and X is compact. Thus, without loss of generality we can assume $\{x^m\}$ converges to some $x \in X \cap V^c$. Since $u_i - r_i$ is jointly continuous, $(\theta^m, x^m) \to (\theta, x)$ implies $[u_i(x^m, \lambda^m) - r_i(\theta^m)] \to [u_i(x, \lambda) - r_i(\theta)]$. Then $x^m \in A_i(\theta^m)$ implies $u_i(x^m, \lambda^m) - r_i(\theta^m) \geq 0$ for all m, so $u_i(x, \lambda) - r_i(\theta) \geq 0$. But then $x \in A_i(\theta) \subseteq V$, contradicting the assumption that $x \in V^c$.

(4) A_C is upper hemicontinuous: upper hemicontinuity follows from Theorem 14.24 in Aliprantis and Border (1994) which states that the intersection of compact-valued, upper hemicontinuous correspondences is upper hemicontinuous.

Now define the correspondence $A_i^s \colon \Theta \to\!\!\to X$ by

$$A_i^s(\theta) \quad = \quad \{x \in X \mid u_i(x, \lambda) - r_i(\theta) > 0\},$$

and let $A_C^s(\theta) = \bigcap_{i \in C} A_i^s(\theta)$.

(5) A_i^s has open graph: take $(\theta, y) \in \mathrm{Gr} A_i^s$, i.e., $u_i(y, \lambda) - r_i(\theta) > 0$. Since $u - r_i$ is continuous on $\Theta \times X$, there exists an open set $V \subseteq \Theta \times X$ such that $(\theta, y) \in V$ and $u_i(y', \lambda') - r_i(\theta') > 0$ for all $(\theta', y') \in V$, implying $\mathrm{Gr} A_i^s$ is open.

(6) A_C^s has open graph: follows from (5), since the finite intersection of open sets is open.

(7) If $\delta_i < 1$ for all $i \in N$, then $A_C^s(\theta) \neq \emptyset$: let $x \in X$ be such that $u_i(x, \lambda) > 0$ for all $i \in N$. If $r_i(\theta) = 0$ then, since $u_i(\cdot, \lambda)$ is concave and non-negative, $u_i(\alpha x + (1-\alpha)x(\pi), \lambda) > 0 = r_i(\theta)$ for all $\alpha \in (0,1)$. If $r_i(\theta) > 0$ then, since $u_i(\cdot, \lambda)$ is concave and non-negative and $\delta_i < 1$, $u_i(x(\pi), \lambda) > \delta_i v_i(\pi, \lambda) = r_i(\theta)$. Thus, by continuity, $u_i(\alpha x + (1-\alpha)x(\pi), \lambda) > r_i(\theta)$ for $\alpha > 0$ low enough. Taking $\alpha > 0$ low enough, therefore, $\alpha x + (1-\alpha)x(\pi) \in A_C^s(\theta)$.

(8) If $\delta_i < 1$ for all $i \in N$, then A_C is lower hemicontinuous at θ: take any $x \in A_C(\theta)$; from (7), there exists $y \in A_C^s(\theta)$; by concavity of the $u_i(\cdot, \lambda)$, $\frac{1}{m}x + (1 - \frac{1}{m})y \in A_C^s(\theta)$ for all non-negative integers m. Letting m go to infinity, $A_C(\theta) \subseteq \overline{A_C^s(\theta)}$. With (6) and (7), it follows that A_C is lower hemicontinuous at θ (see Lemma 14.21 in Aliprantis and Border (1994)).

(9) If LSWP holds at θ, then either $|A_C(\theta)| = 1$ or $A_C(\theta) \subseteq \overline{A_C^s(\theta)}$: suppose $|A_C(\theta)| > 1$, take any $x \in A_C(\theta)$, and note that, by concavity, $x(\pi) \in A_i(\theta)$ for all $i \in C$. Partition C into two sets:

$$\begin{aligned} I &= \{i \in N \mid u_i(x(\pi), \lambda) = r_i(\theta)\} \\ J &= \{i \in N \mid u_i(x(\pi), \lambda) > r_i(\theta)\}. \end{aligned}$$

Note that, for all $i \in J$, $x(\pi) \in A_i^s(\theta)$, an open set. Thus, we can find an open set V such that $x(\pi) \in V \subseteq A_J^s(\theta)$. Note also that

$$|R_I(x(\pi))| = |A_I(\theta)| \geq |A_C(\theta)| > 1,$$

so, by LSWP, there is a sequence $\{x^k\}$ in $P_I(x(\pi))$ converging to $x(\pi)$. Picking k high enough, therefore, we have

$$x_k \in V \cap P_I(x(\pi)) \subseteq P_C(x(\pi)) \subseteq A_C^s(\theta).$$

By concavity, $\frac{1}{m}x_k + (1 - \frac{1}{m})x \in A_C^s(\theta)$ for all non-negative integers m. Letting m go to infinity, we have $x \in \overline{A_C^s(\theta)}$.

(10) If LSWP holds at θ, then A_C is lower hemicontinuous at θ: from (9) there are two cases to consider. If $|A_C(\theta)| = \{x'\}$ for some $x' \in X$ then, for every open set $V \subseteq X$, $V \cap \{x'\} \neq \emptyset$ implies $\{x'\} \subseteq V$, in which case lower hemicontinuity at θ follows from (1) and (4). If $A_C(\theta) \subseteq \overline{A_C^s(\theta)}$ then, by (6), A_C is lower hemicontinuous at θ.

We can now complete the proof. That A has non-empty, compact values follows from (1), (2), and the arbitrary choice of C. If $\delta_i < 1$ for all $i \in N$, then continuity of A_C at θ follows from (4) and (8); continuity of A at θ then follows from Theorem 14.26 in Aliprantis and Border (1994). If LSWP holds at θ, continuity of A_C at θ follows from (4) and (10), and continuity of A at θ again follows. ∎

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