A Slight Variation on Glicksberg’s Theorem

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Glicksberg’s (1952) fixed point theorem is stated for a correspondence defined on a convex, compact subset of a locally convex Hausdorff linear space. One of the main applications of this result, the existence of mixed strategy equilibria in compact, continuous games, focuses on a metrizable subset of the vector space of signed measures, endowed with the weak* topology. I thought it would be interesting to re-prove the fixed point theorem imposing restrictions only on the metric subspace, rather than the vector space in which it is imbedded. The result is a statement of Glicksberg’s fixed point theorem that doesn’t require knowledge of topological vector spaces. It is enough to understand the basics of vector spaces and metric spaces.

Assume $V$ is a vector space, and let $X$ be a convex subset endowed with a metric $d$. The key condition on this metric is that $d$ is compatible with $V$, in the sense that:

(i) $x^n \to x$ and $y^n \to y$ in $X$ and $\alpha^n \to \alpha$ in $[0,1]$ implies $\alpha^n x^n + (1 - \alpha^n)y^n \to \alpha x + (1 - \alpha)y$.

(ii) for all $x,y,z,w \in X$, all $\alpha \in [0,1]$, and all $\beta \in \mathbb{R}$, $d(x,y) \leq \beta$ and $d(w,z) \leq \beta$ implies $d(\alpha x + (1 - \alpha)w, \alpha y + (1 - \alpha)z) \leq \beta$.

I use the usual notation $B_r(x)$ for an open ball of radius $r$ around $x$.

A SLIGHT VARIATION ON GLICKSBERG’S THEOREM Let $V$ be a vector space, let $X \subseteq V$ be convex, and let $d$ be a metric on $X$ such that $X$ is compact and $d$ is compatible with $V$. If $\varphi : X \to X$ has nonempty, convex values and has closed graph, then there exists $x^* \in X$ such that $x^* \in \varphi(x^*)$.

The proof of the theorem proceeds in a few steps.

Finite approximation. For each natural number $m$, $\{B_{1/m}(y) \mid y \in X\}$ is an open cover of $X$, so it has a finite cover consisting of the open balls of radius $1/m$ centered at points $x_1(m), \ldots, x_k(m) \in X$. Let $Y(m) = \text{conv}\{x_1(m), \ldots, x_k(m)\}$.

Constructing a homeomorphism. Let $Z(m) = \text{span}\{x_1(m), \ldots, x_k(m)\}$, and let $\{z_1(m), \ldots, z_l(m)\} \subseteq \{x_1(m), \ldots, x_k(m)\}$ be a basis of $Z(m)$. Define the mapping
suppose nonempty values. It is clear that follows from (i).

To see that it is linear, 1-1, and onto. Let \( g(m) \) denote the restriction of \( f(m) \) to \( Y(m) \), and let \( g(m)_i \) denote the \( i \)th coordinate mapping corresponding to \( g(m) \). Thus, for each \( y \in Y(m) \), we have \( y = \sum_{i=1}^{\ell(m)} g(m)_i(y)z_i(m) \). Note that \( g(m)(Y(m)) = E(m) \), where \( E(m) \subseteq \mathbb{R}^{\ell(m)} \) is defined by

\[
E(m) = \text{conv}\{f(m)(x_1(m)), \ldots, f(m)(x_{k(m)}(m))\}.
\]

To see that \( g(m): Y(m) \to E(m) \) is continuous, suppose \( y^n \to y \) in \( Y(m) \). For each \( n \), we have \( y^n = \sum_{i=1}^{\ell(m)} g(m)_i(y^n)z_i(m) \) and \( y = \sum_{i=1}^{\ell(m)} g(m)_i(y)z_i(m) \). By (i), we have (1/2)\( y^n + (1/2)y \to y \). Suppose \( g(m)_j(y^n) \neq g(m)_j(y) \) for some coordinate \( j \). Then (1/2)\( g(m)_j(y^n) + (1/2)g(m)_j(y) \neq g(m)_j(y) \), and then

\[
(1/2)y^n + (1/2)y = (1/2) \left( \sum_{i=1}^{\ell(m)} g(m)_i(y^n)z_i(m) \right) + (1/2) \left( \sum_{i=1}^{\ell(m)} g(m)_i(y)z_i(m) \right)
\]

\[
= \sum_{i=1}^{\ell(m)} [(1/2)g(m)_i(y^n) + (1/2)g(m)_i(y)]z_i(m)
\]

\[
\neq \sum_{i=1}^{\ell(m)} g(m)_i(y)z_i(m)
\]

\[
= y,
\]

a contradiction. Therefore, \( g(m)(y^n) \to g(m)(y) \). To see that \( g(m)^{-1}: E(m) \to Y(m) \) is continuous, suppose \( \alpha^n \to \alpha \) in \( E(m) \). Then

\[
g(m)^{-1}(\alpha^n) = \sum_{i=1}^{\ell(m)} \alpha^n_i z_i(m) \to \sum_{i=1}^{\ell(m)} \alpha_i z_i(m) = g(m)^{-1}(\alpha)
\]

follows from (i).

**Constructing another correspondence.** Define \( \varphi(m): Y(m) \to Y(m) \) by

\[
\varphi(m)(x) = \text{conv}\{x_i(m) | \exists w \in \varphi(x) \text{ s.t. } d(x_i(m), w) = \min_{j=1, \ldots, k(m)} d(x_j(m), w)\}.
\]

Since \( \varphi \) has nonempty values and the minimization above is over a finite set, \( \varphi(m) \) has nonempty values. It is clear that \( \varphi(m) \) has convex values. To see that it has closed graph, suppose \( x^n \to x \) and \( y^n \to y \) in \( X \) and \( y^n \in \varphi(m)(x^n) \) for all \( n \). For each \( n \), there exists \( \beta^n \in \mathbb{R}^{k(m)}_+ \) such that \( \sum_{i=1}^{k(m)} \beta^n_i = 1 \), that \( y^n = \sum_{i=1}^{k(m)} \beta^n_i x_i(m) \), and that \( \beta^n_i > 0 \) implies
there exists \( w^*_i \in \varphi(x^n) \) such that \( d(x_i(m), w^*_i) = \min\{d(x_j(m), w^*_j) \mid j = 1, \ldots, k(m)\} \).

We may suppose, by compactness of the unit simplex, that \( \beta^n \to \beta \). By linearity of \( g(m) \), we have

\[
g(m)(y^n) = \sum_{i=1}^{k(m)} \beta^i g(m)(x_i(m)) = \sum_{i=1}^{k(m)} \beta g(m)(x_i(m)) = g(m)\left(\sum_{i=1}^{k(m)} \beta_i x_i(m)\right).
\]

Therefore, \( y = \sum_{i=1}^{k(m)} \beta_i x_i(m) \). Take any \( i \) such that \( \beta_i > 0 \), implying \( \beta^n_i > 0 \) for high enough \( n \), implying the existence of \( w^n_i \in \varphi(x^n) \) such that \( d(x_i(m), w^n_i) = \min\{d(x_j(m), w^n_j) \mid j = 1, \ldots, k(m)\} \) for high enough \( n \). By compactness, we may suppose \( w^n_i \to w_i \). Since \( \varphi \) has closed graph, we have \( w_i \in \varphi(x) \). Furthermore, \( d(x_i(m), w_i) = \min\{d(x_j(m), w_i) \mid j = 1, \ldots, k(m)\} \).

Finding a fixed point. Define the correspondence \( \Phi(m) : E(m) \longrightarrow E(m) \) by

\[
\Phi(m)(\alpha) = g(m)(\varphi(m)(g(m)^{-1}(\alpha))).
\]

Since \( g(m) \) is a linear homeomorphism, this defines a correspondence on a compact, convex subset of \( \mathbb{R}^{k(m)} \) that has nonempty, convex values and has closed graph. Therefore, there exists \( \alpha^*(m) \in E(m) \) such that \( \alpha^*(m) \in \Phi(m)(\alpha^*(m)) \). Define \( x^*(m) = g(m)^{-1}(\alpha^*(m)) \), and note that \( x^*(m) \in \varphi(m)(x^*(m)) \). For each \( m \), there exist \( \beta(m) \in \mathbb{R}^{k(m)} \) such that \( \sum_{i=1}^{k(m)} \beta_i(m) = 1 \), that \( x^*(m) = \sum_{i=1}^{k(m)} \beta_i(m) x_i(m) \), and that \( \beta_i(m) > 0 \) implies \( d(x_i(m), w_i(m)) = \min\{d(x_j(m), w_i(m)) \mid j = 1, \ldots, k(m)\} \) for some \( w_i(m) \in \varphi(x^*(m)) \). Thus, \( d(x_i(m), w_i(m)) \leq 1/m \). By (ii), we have \( d(x^*(m), w^*(m)) \leq 1/m \), where \( w^*(m) = \sum_{i=1}^{k(m)} \beta_i(m) w_i(m) \). Since \( \varphi(x^*(m)) \) is convex, we have \( w^*(m) \in \varphi(x^*(m)) \). By compactness, we may suppose that \( x^*(m) \to x^* \) and \( w^*(m) \to w^* \), and from the foregoing it follows that \( x^* = w^* \). Since \( \varphi \) has closed graph, \( x^* = w^* \in \varphi(x^*) \), completing the proof.

PURE STRATEGY EQUILIBRIUM EXISTENCE The above Theorem can be used to prove, using standard arguments, that compact, quasi-concave, continuous strategic form games have pure strategy equilibria. Basically, we let each of \( n \) players have a set \( X_i \) of pure strategies, assumed to lie in some ambient vector space \( V_i \) and to be endowed with a metric \( d_i \) compatible with \( V_i \). Obviously, the main application of interest is when \( X_i \) is a subset of finite-dimensional Euclidean space. We let \( X = \prod X_i \) denote the set of pure strategy profiles, endowed with the maximum of the individual metrics, i.e., \( d(x, y) = \max\{d_i(x_i, y_i) \mid i = 1, \ldots, n\} \), and we imbed \( X \) in the product of vector spaces \( V = \prod V_i \). It is easy to see that \( d \) is compatible with \( V \). Each player \( i \) has a payoff function \( u_i : X \to \mathbb{R} \). If each \( X_i \) is compact and if each \( u_i \) is continuous in \( (x_1, \ldots, x_n) \) and quasi-concave in \( x_i \), then the game \((X_i, u_i), i = 1, \ldots, n\) has a pure strategy Nash equilibrium. To prove this, we just define the product of best response correspondences. Our convexity assumptions ensure that this
is convex-valued, while our continuity assumptions ensure that it has nonempty values and closed graph. Thus, it has a fixed point, which yields the desired equilibrium point.

**MIXED STRATEGY EQUILIBRIUM EXISTENCE** If we drop the convexity assumptions from the above description of a strategic form game, pure strategy equilibria need not exist. Suppose that each \( X_i \) is a separable metric space, and let \( \mathcal{P}(X_i) \) denote the set of Borel probability measures on \( X_i \), denoted \( \sigma, \mu, \nu, \) etc. We view \( \mathcal{P}(X_i) \) as a subset of the vector space of signed measures on \( X_i \), say \( \mathcal{V}_i \), and we endow this subset with the Prohorov metric \( \rho_i \), defined by

\[
\rho_i(\mu, \nu) = \inf\{\varepsilon > 0 \mid \text{for all Borel } B, \mu(B) \leq \nu(B^c) + \varepsilon \text{ and } \nu(B) \leq \mu(B^c) + \varepsilon\},
\]

where \( B^c = \{x \in X \mid d(x, B) < \varepsilon\} \). Note that, if \( \rho_i(\mu^n, \mu) \leq \varepsilon \) and \( \rho_i(\nu^n, \nu) \leq \varepsilon \), then, for all Borel \( B \) and all \( \alpha \in [0, 1] \),

\[
\alpha \mu^n(B) + (1 - \alpha) \nu^n(B) \leq \alpha \mu(B^c) + (1 - \alpha) \nu(B^c) + \varepsilon,
\]

and visa versa. Thus, for all \( \alpha \in [0, 1] \), we have \( \rho_i(\alpha \mu^n + (1 - \alpha) \nu^n, \alpha \mu + (1 - \alpha) \nu) \leq \varepsilon \). Similarly, if \( |\alpha^n - \alpha| \leq \varepsilon \), then, for all \( \mu, \nu \in \mathcal{P}(X_i) \), we have \( \rho_i(\alpha^n \mu + (1 - \alpha^n) \nu, \alpha \mu + (1 - \alpha) \nu) \leq \varepsilon \). Therefore, if \( \mu^n \rightharpoonup \mu \), if \( \nu^n \rightharpoonup \nu \), and if \( \alpha^n \rightharpoonup \alpha \), then, for every \( \varepsilon > 0 \), we can choose \( n \) high enough that

\[
\rho_i(\alpha^n \mu^n + (1 - \alpha^n) \nu^n, \alpha \mu + (1 - \alpha) \nu) \\
\leq \rho_i(\alpha^n \mu^n + (1 - \alpha^n) \nu^n, \alpha^n \mu + (1 - \alpha^n) \nu) \\
\quad + \rho_i(\alpha^n \mu + (1 - \alpha^n) \nu, \alpha \mu + (1 - \alpha) \nu) \\
\leq (\varepsilon/2) + (\varepsilon/2) \\
= \varepsilon,
\]

and we conclude that \( \alpha^n \mu^n + (1 - \alpha^n) \nu^n \rightharpoonup \alpha \mu + (1 - \alpha) \nu \), as in (i). For all \( \mu, \nu, \mu', \nu' \in \mathcal{P}(X_i) \), all \( \alpha \in [0, 1] \), and all \( \beta \in \mathbb{R} \), if

\[
\mu(B) \leq \nu(B^c) + \beta \quad \text{and} \quad \mu'(B) \leq \nu'(B^c) + \beta,
\]

(and visa versa) for all Borel \( B \), then we have

\[
\alpha \mu(B) + (1 - \alpha) \mu'(B) \leq \alpha \nu(B^c) + (1 - \alpha) \nu'(B^c) + \beta
\]

(and visa versa), as required for (ii). Thus, \( \rho_i \) is compatible with \( \mathcal{V}_i \). We then endow \( \prod \mathcal{P}(X_i) \) with the maximum of the individual metrics, say \( \rho \), and we imbed it in the product of vector spaces \( \prod \mathcal{V}_i \). As above, \( \rho \) will be compatible with \( \prod \mathcal{V}_i \). Assuming each \( X_i \) is a compact metric space, then, because the topology induced by \( \rho_i \) is the weak* topology, each
$\mathcal{P}(X_i)$ is compact, and so is $\prod \mathcal{P}(X_i)$. Finally, we extend player $i$’s payoff function on pure strategy profiles to mixed strategy profiles $(\sigma_1, \ldots, \sigma_n)$ as

$$U_i(\sigma_1, \ldots, \sigma_n) = \int u_i d\sigma,$$

where $\sigma = \sigma_1 \times \cdots \times \sigma_n$ is the product measure. Of course $U_i$ is multi-linear in mixed strategies and, therefore, quasi-concave in $\sigma_i$. If $u_i$ is continuous, then so is $U_i$. Thus, we have defined a strategic form game $((\mathcal{P}(X_i), U_i), i = 1, \ldots, n)$, called the mixed extension of $((X_i, u_i), i = 1, \ldots, n)$, satisfying the conditions for existence of a pure strategy equilibrium. We conclude that the mixed extension has a pure strategy equilibrium, which corresponds to a mixed strategy equilibrium of the original game.

References