A Note on Backward Induction, Iterative Elimination of Weakly Dominated Strategies, and Voting in Binary Agendas

John Duggan
Caltech and University of Rochester

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Finite perfect information extensive (FPIE) games are quite well-understood: backward induction yields all of the pure strategy subgame perfect equilibria of such games. Iterative elimination of weakly dominated strategies (IEWDS), which can be hazardous in general games due to order-dependence of outcomes, can be much better behaved. In fact, for a large class of FPIE games, the outcomes of IEWDS match quite well with the outcomes of backward induction.

Still, the details can be a little bothersome. In this note, my goal is to bring together some of what’s known about these details and to fill in some of the gaps. You’ll notice that I’m only making “claims” here, and that the only proof I give is a bit loose. Proceed at your own risk!

I will focus on the connections between IEWDS and backward induction in five classes of games:

- arbitrary FPIE games,
- binary voting agendas in which voters may have any weak orders over the set of alternatives, allowing indifferences,
- FPIE games satisfying the condition of transference of decision-maker indifference (TDI) due to Marx and Swinkels (1997),
- binary voting agendas in which voters have linear orders over the set of alternatives,
• FPIE games with “generic” payoffs, i.e., no player receives equal payoffs at any two distinct terminal nodes.

I will consider them in reverse order.

1 FPIE Games with Generic Payoffs

In this class of games, it is known that backward induction yields a unique strategy profile. Moreover, Moulin’s (1979) condition (1) holds: letting $N$ denote the set of players and letting $u_i$ denote player $i$’s payoff function defined on the set $T$ of terminal nodes, we trivially have

\[
\forall t, t' \in T, \forall i \in N, (u_i(t) = u_i(t')) \Leftrightarrow (u_j(t) = u_j(t')).
\]

From Moulin’s Proposition 2, therefore, the normal representation is dominance solvable. This means the following: Suppose we simultaneously remove all weakly dominated strategies for all players, then repeat the process in the reduced game, and so on, until no further reduction is possible. I will call this order of elimination “exhaustive,” since at each step we remove all possible weakly dominated strategies. Letting $S^\infty$ denote the set of strategy profiles remaining, each player’s payoff function $u_i$ is constant on $S^\infty$. In an FPIE game with generic payoffs, of course, this means that the set $S^\infty$ of profiles remaining after IEWDS using this order of elimination all determine the same path of play.

What is the connection between backward induction and dominance solvability? In the following example, from Osborne and Rubinstein (1994, pp.108-109), we see that there may be orders of elimination that remove the unique backward induction strategy profile.

[ Figure 1 ]

In the above, if we first remove $AE$, and then $D$, then the backward induction strategy profile $(BE, D)$ is deleted. Three remarks in this regard are in order. First, the previous order of elimination still generates the backward induction play-path. Second, the backward induction strategy profile does survive IEWDS for some orders of elimination, and Osborne and Rubinstein argue that this is generally true. Third, the backward induction strategy profile actually survives IEWDS using the order of elimination specified by
Moulin — in fact, exhaustive elimination leaves four strategy profiles, all with the same payoff vector.

The next claim sheds some light on these observations.

Claim 1 In a F PIE game with generic payoffs, there is a unique backward induction strategy profile. There is an order of elimination such that IEWDS leaves the backward induction strategy profile and only profiles with the same play-path. If the backward induction strategy profile survives IEWDS using exhaustive elimination, then the unique play-path realized by IEWDS using exhaustive elimination is equivalent to the unique backward induction play-path.

The assumption of the last part of Claim 1 begs the following open question: (1) Is the backward induction strategy profile guaranteed to survive IEWDS using exhaustive elimination? My conjecture is that it is, but I leave it open. While Claim 1 only applies to a specific order of elimination, other orders of elimination are clearly possible. The claim is stated in a weak way in order to use Moulin’s result, but we will see that a stronger result holds for the broader class of F PIE games satisfying TDI: regardless of whether the backward induction strategy profile survives IEWDS using exhaustive elimination and regardless of the order of elimination, IEWDS produces a unique payoff vector, and this equals the unique backward induction payoff vector.

Before we move to that result, let’s consider binary voting agendas.

2 Binary Voting Agendas with No Indifferences

Now assume a finite number of alternatives are paired together to be voted on, where the schedule of votes is represented by a voting tree — each active (non-terminal) node of the voting tree representing a majority-rule contest between two alternatives, and each terminal node representing a final alternative chosen. We will assume an odd number of voters, so that no ties are possible.\(^1\) Suppose that each voter linearly orders the alternatives, and that \(v_i\) is a numerical representation of \(i\)’s ordering, and that \(u_i(t) = v_i(a(t))\), where \(a(t)\) is the alternative selected at terminal node \(t\) of the voting tree.

\(^1\)We can easily consider other voting rules, such as one that gives a particular voter the power to break ties. Then we can drop the assumption of an odd number of voters.
Once we describe the process of voting at each active node of the voting tree, this determines an extensive form game. There are basically two possibilities to consider: we may assume all votes at a given node are simultaneous, or we may assume that the votes are sequential, in some predetermined order.

If voting is simultaneous, then we have defined an extensive form game of imperfect information. First, suppose there are just two alternatives, $a$ and $b$. If there are more than three voters, then there are many Nash equilibria, and both $a$ and $b$ are Nash equilibrium outcomes. Note, however, that voting for the preferred alternative is a dominant strategy for a voter. Furthermore, voting for the less-preferred alternative is weakly dominated. Thus, dominant strategy equilibrium determines a unique Nash equilibrium strategy profile, and that profile is the unique survivor of IEWDS for at least one order of elimination. There may be other strategy profiles that survive IEWDS for other orders of elimination, but they all have the same outcome: the majority-preferred alternative. Applying this logic to a binary tree with more than two alternatives, we can work backward through the tree ("pruning the voting tree") to the initial node, where an outcome is determined. This is McKelvey and Niemi’s (1978) “multistage sophisticated solution,” which captures an idea informally expressed by Farquharson (1969). It is clear that several paths through the voting tree, which I refer to as “paths of winners,” may be determined by this procedure: consider the trivial voting tree with just one active node, with that followed by two terminal nodes, each representing alternative $a$. By the preceding observations, however, it is clear that a unique alternative is determined. I refer to this as the “realized alternative.”

If voting is sequential, then we have defined a FPIE game. Again starting with two alternatives, if there are more than three voters, then there are again many Nash equilibria, and $a$ and $b$ are both Nash equilibrium outcomes. A voter’s strategies are now more complex, and the voters do not all have dominant strategies: regardless of his/her preferences over $a$ and $b$, the first voter’s optimal vote will depend on the voting strategies of following voters. Going to subgame perfect equilibria, there will be many, but there is a unique backward induction outcome: the majority-preferred alternative. In a tree with multiple alternatives, backward induction again determines a unique outcome, and it is the outcome produced by McKelvey and Niemi’s (1978) multistage sophisticated solution. Thus, we have an equivalence between two approaches to solving binary voting games.
To relate backward induction to IEWDS in the sequential voting game, note that the assumption of linear orders implies Moulin’s condition (1), so his Proposition 2 shows that the game is dominance solvable: IEWDS using exhaustive elimination yields a unique payoff vector, and therefore a unique realized alternative. There may be multiple backward induction strategy profiles, and some may actually involve weakly dominated strategies, as in the following game.

[ Figure 2 ]

In the above, voter 1 votes for $a$, his worst alternative, in some backward induction strategy profiles, though it is a weakly dominated strategy to do so. Note, however, that there exists an order of elimination such that IEWDS leaves the strategy profile in question. In fact, there is an order of elimination that leaves exactly the set of backward induction strategy profiles.

One might wonder whether there is a backward induction strategy profile that survives every order of elimination. The answer for generic FPIE games was “no,” but we now have a different structure: in the case of just two alternatives, you might think that the strategy profile in which each voter follows “always vote for my preferred alternative” is especially robust. As shown in Figure 3, however, this profile does not generally survive IEWDS for all orders of elimination. To save space, I omit two of voter 3’s nodes (they are irrelevant). Since the voters have identical preferences, I just write one payoff.

[ Figure 3 ]

In the above, first remove $ab$ and $bb$ for voters 2 and 3. Then voting $b$ weakly dominates $a$ for voter 1, despite the fact that the voter prefers $a$. Alternatively, we can eliminate all but $aa$ for voter 3, in which case voting $a$ weakly dominates $b$. Thus, there may be a player who has no strategies that survive IEWDS for all orders of elimination.

Note that the strategy profile “always vote for my preferred alternative” does survive IEWDS using exhaustive elimination in the previous example. This is not always true of every backward induction strategy profile, however, as shown in Figure 4. Again, to save space, I omit two of voter 3’s nodes (they are irrelevant), and I only indicate voter 2’s payoffs.

[ Figure 4 ]

In the above, $(b, ab, bb)$ is a backward induction strategy profile, even though $ab$ is weakly dominated for voter 2. Thus, we have an **open question**: (2)
Is the backward induction strategy profile “always vote for my preferred alternative” guaranteed to survive IEWDS using exhaustive elimination when there are just two alternatives? I conjecture that it is, but this is open. If so, we should be able to use a sophisticated version of this strategy profile for the case of more than two alternatives.

The following claim generalizes some of these observations. The equivalence between backward induction and IEWDS is stated in Claim 2 for only certain orders of elimination, but we will see that it actually extends to IEWDS for all orders of elimination.

Claim 2 In a binary voting agenda with no indifferences, there is a unique alternative realized by backward induction, though there may be multiple backward induction paths of winners. It may be the case that, for every backward induction strategy profile, there is an order of elimination such that the profile fails to survive IEWDS for that order of elimination, and some backward induction strategy profiles may fail to survive IEWDS using exhaustive elimination. There is an order of elimination that leaves exactly the set of backward induction strategy profiles. Backward induction and IEWDS for that order of elimination realize the same alternative. If a backward induction strategy profile survives IEWDS using exhaustive elimination, then the backward induction alternative is the only alternative realized by strategy profiles that survive IEWDS using exhaustive elimination.

Simultaneous voting is, I think, usually implicitly assumed. This is the framework of Farquharson (1969), Kramer (1972), McKelvey and Niemi (1978), and Moulin (1979, Sect. 2; 1983, Ch. 5). Whereas Farquharson and Kramer consider IEWDS in the normal representation, somewhat informally, McKelvey and Niemi describe the multistage sophisticated solution, an algorithm for calculating the outcome of a particular way of eliminating weakly dominated strategies, which is also used by Moulin. McKelvey and Niemi conjecture that IEWDS realizes the same alternatives as their procedure. As discussed above for the case of just two alternatives, elimination of weakly dominated strategies in a simultaneous vote and backward induction in sequential votes lead to the same outcome: the majority-preferred alternative. Thus, the McKelvey-Niemi procedure leads to the same alternatives as backward induction — in fact, they generate the same paths of winners — and Claim 2 confirms their conjecture for certain orders of elimination. An implication of Claim 2', below, is that the conjecture of McKelvey and Niemi actually holds for all orders of elimination.
3 FPIE Games Satisfying TDI

Now suppose that Marx and Swinkel’s (1997) condition of transference of decision-maker indifference (TDI) is satisfied: for all players $i, j \in N$, all strategies $s_i, s_i' \in S_i$ and all profiles $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \Rightarrow u_j(s_i, s_{-i}) = u_j(s_i', s_{-i}).$$

That is, if any player is indifferent between one strategy and switching to another, then all players are indifferent. This is clearly satisfied in binary voting agendas with no indifference. An implication of their results is that, in any game satisfying TDI, all orders of IEWDS lead to the same set of payoff vectors. See the example in Figure 1. For another example, consider the following.

[ Figure 5 ]

Here, we may eliminate $d$ for row player, then $D$ for column player. Then we can eliminate redundant strategies to get the smaller game $\{a, b\} \times \{A, B\}$, after which no further elimination is possible. Or we can eliminate $a$ for row player, then $A$ for column player. Then we can eliminate redundant strategies to get the smaller game $\{b, c\} \times \{B, C\}$. Clearly, these two games are equivalent.

It is easy to see that TDI, or some such restriction, is needed for this result.

[ Figure 6 ]

In the above game, we can eliminate $U$, then $L$, then $M$, to leave $(D, R)$. Or we can eliminate $M$, then $R$, then $U$, to leave $(D, L)$. Even restricting payoffs to be zero or one, IEWDS in different orders can lead to different payoff vectors.

[ Figure 7 ]

In the above game, we can eliminate $U$, then $L$, then $M$, then $R$, to leave $(D, C)$. Or we can eliminate $R$, then $D$, then $C$, then $U$, to leave $(M, L)$.

Marx and Swinkels give the following informal argument, on pp.239-240, that, in any FPIE game satisfying TDI, backward induction and IEWDS yield the same payoff vector: in such a game, there is a unique backward induction payoff vector; any backward induction strategy profile corresponds to a particular order of IEWDS; since all orders lead to the same set of payoff vectors, all orders must generate the backward induction payoff vector. The second step in this argument is, however, incorrect: it is easy to
give examples where a strategy profile eliminated in the process of backward induction is not weakly dominated.

[ Figure 8 ]

In the above, for example, player 1’s strategy LR is eliminated by backward induction, but it is not weakly dominated.

Obviously, LR still generates the backward induction play-path. Marx and Swinkel’s claim is actually true, and it follows from their more general results for iterative elimination of nicely very weakly dominated strategies. As an application of this result, consider the example of Figure 1, where IEWDS in any order of elimination leaves only profiles with the payoff vector (3, 3).

**Claim 3 (Marx and Swinkels)** In a FPIE game satisfying TDI, there is a unique backward induction payoff vector, and IEWDS in every order leads only to strategy profiles with that payoff vector.

**Proof:** It is straightforward to check that the backward induction payoff vector is unique. Let $s$ be a backward induction strategy profile, so that $\hat{S} = \{s\}$ is a reduction of $S$ by nice very weak dominance. Let $S'$ be the result of IEWDS for some order of elimination, so it is a reduction of $S$ by nice very weak dominance. By Lemma C of Marx and Swinkels (2000), there is a subset of $S'$ equivalent to a subset of $\hat{S}$, so it consists of some $s' \in S'$, where $s'$ is obtainable from $S'$ by iterative removal of strategies that are either nicely weakly dominated or redundant. Since $S'$ is the result of IEWDS, the first step in this process cannot be the removal of a nicely weakly dominated strategy, so it is the removal of a redundant strategy. The second step cannot be removal of a nicely weakly dominated strategy either: since we have only removed a redundant strategy, say $\tilde{s}_j$ for some player $j$, any nice weak dominance over $S'_{-j,k} \times (S'_j \setminus \{\tilde{s}_j\})$ for any player $k$ would be a nice weak dominance over $S'_{-k}$, a contradiction. This argument inducts, so at each stage we may only remove redundant strategies. Thus, the payoff vector from every element of $S'$ is equal to the payoff vector from $s'$, which is identical to the backward induction payoff vector.

This allows us to state a more general version of Claim 1 for FPIE games with generic payoffs. There, of course, equivalence of payoffs in Claim 3 means equivalence of play-paths.
Claim 1' In a FPIE game with generic payoffs, there is a unique backward induction strategy profile. There is an order of elimination such that IEWDS leaves the backward induction strategy profile. The play-path realized by IEWDS, using any order of elimination, is unique, and it is equivalent to the unique backward induction play-path.

We can now strengthen our earlier result for binary voting agendas with no indifferences, where equivalence of payoffs now means equivalence of alternatives realized.

Claim 2' In a binary voting agenda with no indifferences, there is a unique alternative realized by backward induction, though there may be multiple backward induction paths of winners. It may be the case that, for every backward induction strategy profile, there is an order of elimination such that the profile fails to survive IEWDS for that order of elimination, and some backward induction strategy profiles may fail to survive IEWDS using exhaustive elimination. There is an order of elimination that leaves exactly the set of backward induction strategy profiles. The alternative realized by IEWDS, using any order of elimination, is unique and is equivalent to the unique alternative realized by backward induction.

We will see that allowing for voter indifferences undermines this result somewhat.

4 Binary Voting Agendas with Indifferences

Now assume only that each voter has a weak order of the set of alternatives, so that indifferences are allowed. Moulin’s (1979) condition (1) and Marx and Swinkel’s (1997) TDI no longer hold generally. An immediate contrast to the case of no indifferences is that sequential voting can be indeterminate: if an absolute majority (strictly more than half of the voters) prefer one alternative to another, then it will be the backward induction outcome; otherwise, if neither alternative has an absolute majority, then both alternatives are consistent with backward induction. Similar remarks hold for simultaneous voting. Thus, backward induction can yield strategy profiles with different outcomes and different payoff vectors.

Using these observations, we can completely describe the backward in-
duction paths of winners, but that is a topic for another note. Let me just say that, even in amendment agendas, it is possible that a covered alternative is realized. For each backward induction path of winners, there is an order of elimination such that the path is realized a strategy profile that survives IEWDS for that order of elimination. The same cannot be said for all backward induction strategy profiles, however, as demonstrated in the following example. Here, to save space, I omit two of player 3’s nodes (they are irrelevant) and depict only 2’s payoffs (1 and 3 are completely indifferent).

[ Figure 9 ]

In the above, the strategy profiles \((a, ba, aa)\), \((b, ab, ab)\), and \((b, bb, bb)\) are backward induction profiles. Because players 1 and 3 are completely indifferent, the only player with weakly dominated strategies is player 2. For every order of elimination, player 2’s strategies \(ba\), \(ab\), and \(bb\) are eliminated. Thus, in contrast to Claim 2’, it is not the case that each backward induction strategy profile survives IEWDS for some order of elimination.

This leads to several more open questions: (3) Does IEWDS, using every order of elimination, leave only strategy profiles that generate paths of realized winners consistent with backward induction? If not, (4) does IEWDS, using every order of elimination, leave at least one strategy profile that generates a path of realized winners consistent with backward induction? (5) Does there exist an order of elimination that leaves only strategy profiles that generate paths of realized winners consistent with backward induction? I conjecture that the answers are negative, but I do not have counterexamples.

Claim 4 In a binary voting agenda with indifferences, there can be multiple backward induction paths of winners and multiple realized alternatives. It may be the case that every backward induction strategy profile fails to survive IEWDS for some order of elimination, and some may fail, for every order of elimination, to survive IEWDS. For every path of votes realized by backward induction, there is an order of elimination such that IEWDS, using that order, includes that path of votes among the paths realized.

Note that the last part of Claim 4 weakens the last few sentences of Claim 2’ in two ways: as shown above, it is no longer true that each backward induction strategy profile survives, for some order of elimination, IEWDS; and I have not excluded the possibility that IEWDS may yield alternatives that are inconsistent with backward induction.


5 Arbitrary FPIE Games

As demonstrated in the following example, from Osborne and Rubinstein (1994), backward induction strategy profiles can be eliminated by IEWDS for some orders of elimination, and, unlike the example in Figure 2, IEWDS may leave no other strategy profiles with the same payoff vectors.

When AC, AD, and BD are eliminated for player 1, there is no strategy profile with the same payoff vector (1; 2) as the backward induction profile (AC, R). Notice that the order used to eliminate (AC, R) is exhaustive. Of course, (AC, R) does survive IEWDS for some orders of elimination. But the above example of Figure 9 shows even more: there may be backward induction strategy profiles that are always eliminated by IEWDS.

In the next example, also from Osborne and Rubinstein (1994), we see that IEWDS can leave a strategy profile with a payoff vector corresponding to no backward induction profile.

Here, the strategy profile (AC, L) survives every order of elimination, but there is not even a Nash equilibrium with its payoff vector (0, 0). Thus, there does not generally exist an order of elimination that leaves only strategy profiles with paths of play consistent with backward induction.

In the above example of Figure 10, while (AC, R) is eliminated by IEWDS, the backward induction strategy profile (BC, R) survives IEWDS using every order of elimination. In binary voting agendas, we saw that such strategy profiles need not exist, and for general FPIE games the same conclusion of course holds. For another example, consider the following.

Here, there are two backward induction strategy profiles: (A, C, F) and (B, C, F). If we first eliminate D and E for player 2, then H for player 3, we see that B weakly dominates A for player 1 in the reduced game. This eliminates the first equilibrium. A symmetric construction eliminates the second.

I end with a final open question: (6) Does IEWDS, using every order of elimination, leave at least one strategy profile that generates a path of
play consistent with backward induction? My guess is that the answer is negative, but that guess is not grounded in deep consideration.

References


