Chapter 2

Social Choice Theory

2.1 Collective Choice Environments

A collective choice *environment* is defined by four things: a set of social states, which represent information that we want to treat as variable; a set of individual decisionmakers; a set of alternatives from which they must make a collective choice; and individual preferences over alternatives.

Θ	set of social states θ, ϕ, etc.
$N(\theta)$	finite set of individuals i, j, etc.
$X(\theta)$	set of alternatives x, y, z, etc. $\lvert X(\theta)\rvert \geq 2$
$(P_i(\theta), R_i(\theta))$	i's preferences in state θ, satisfying Axioms 1-3.

By Axioms 1 and 2, $(P_i(\theta), R_i(\theta))$ is a dual pair. We also impose Axioms 3 on individual preferences, so $(P_i(\theta), R_i(\theta))$ is actually an ordering, i.e., $R_i(\theta)$ is a weak order and $P_i(\theta)$ is a strict order. As usual, $I_i(\theta)$ will denote i's indifference relation in state θ.

Recall that $R_i(\theta)$ and $P_i(\theta)$ contain the same information: $xR_i(\theta)y$ if and only if not $yP_i(\theta)x$, and $xP_i(\theta)y$ if and only if not $yR_i(\theta)x$. Whether we discuss weak or strict preference is entirely a matter of convenience.

The dependence of N, X, and (P_i, R_i) on θ indicates that, conceivably, these sets could be treated as variables. It is customary, however, to restrict the analysis to collective choice problems in which the sets of individuals and alternatives are fixed.[1] We therefore drop the

[1] This is certainly not universal. There is a nice version of Arrow's impossibility theorem (and others) that is stated in terms of a variable set of feasible alternatives. And Peyton Young, William Thomson, and

dependence on θ and write simply N and X. We assume the number of individuals is n, and we enumerate them $1, \ldots, n$.

Paralleling our earlier notation, let

$$\begin{aligned} R_i(x|\theta) &= \{y \in X \mid yR_i(\theta)x\} \\ P_i(x|\theta) &= \{y \in X \mid yP_i(\theta)x\} \end{aligned}$$

denote the upper sections of individual i's weak and strict preferences in state θ, and denote lower sections by $R_i^{-1}(x|\theta)$ and $P_i^{-1}(x|\theta)$.

Denote the profiles of individual preferences at θ by

$$PR(\theta) = ((P_1(\theta), R_1(\theta)), \ldots, (P_n(\theta), R_n(\theta)))$$

To save space, I may sometimes write just $(R_1, \ldots, R_n)$ for a profile of preferences.

The set $PR(\Theta) = \{PR(\theta) \mid \theta \in \Theta\}$ is the "domain of preferences," the set of possible preference profiles, depending on the state. I assume the existence of a *free pair,* i.e., we assume there exist $x, y \in X$ such that, for every partition $\{I, J\}$ of N, there exists $\theta \in \Theta$ such that

$$I = \{i \in N \mid xP_i(\theta)y\} \quad \text{and} \quad J = \{i \in N \mid yP_i(\theta)x\}.$$

In words, individual strict preferences over x and y are unrestricted. This is a very mild assumption satisfied in most all environments.

Some extra notation:

$$\begin{aligned} &R(x, y|\theta) = \{i \in N \mid xR_i(\theta)y\} \qquad &&P(x, y|\theta) = \{i \in N \mid xP_i(\theta)y\} \\ &r(x, y|\theta) = |R(x, y|\theta)| &&p(x, y|\theta) = |P(x, y|\theta)|. \end{aligned}$$

We will later consider different restrictions on the domain of preferences: in many applications there are some *a priori* restrictions that can be imposed on individual preferences by the analyst. We'll survey some examples in the next section.

We are interested in how individuals overcome possible conflicts of interest to collectively choose an alternative. Following the standard approach to the analysis of individual decision making, we view the group as making this decision on the basis of a "social preference." For

others have introduced several axioms of interest when the set of individuals may vary.

now, we suppose social preferences are determined arbitrarily by the social state according to the mapping F:

$$\theta \overset{F}{\mapsto} (P_F(\theta), R_F(\theta)).$$

We call F a *Preference Aggregation Rule,* or simply *PAR.*

Here, $R_F(\theta)$ is the weak social preference relation at θ and $P_F(\theta)$ is the strict social preference relation at θ. We maintain Axioms 1 and 2, so $(P_F(\theta), R_F(\theta))$ is a dual pair. We use $I_F(\theta)$ to denote the corresponding social indifference relation at θ.

Thus, strict preference $P_F(\theta)$ and weak preference $R_F(\theta)$ contain the same information:

$$xP_F(\theta)y \Leftrightarrow \text{ not } yR_F(\theta)x \quad \text{and} \quad xR_F(\theta)y \Leftrightarrow \text{ not } yP_F(\theta)x.$$

Thus, we can use $R_F(\theta)$ and $P_F(\theta)$ interchangeably, depending on which is more convenient. If the PAR F is understood, it may be suppressed. Thus, we may write simply $R(\theta)$ and $P(\theta)$.

Paralleling our earlier notation, let

$$\begin{aligned} R_F(x|\theta) &= \{y \in X \mid yR_F(\theta)x\} \\ P_F(x|\theta) &= \{y \in X \mid yP_F(\theta)x\} \\ I_F(x|\theta) &= \{y \in X \mid xI_F(\theta)y\} \end{aligned}$$

denote the upper sections of weak and strict social preference and social indifference, and denote lower sections by $R_F^{-1}(x|\theta)$ and $P_F^{-1}(x|\theta)$. (Because $I_F(\theta)$ is symmetric, we don't need notation for its lower sections.)

In the social choice theory approach, we apply the same theory of choice to groups that we do to individuals. Thus, given a PAR F, a state θ, and social preferences $(P_F(\theta), R_F(\theta))$, we posit that the plausible collective choices are just the socially maximal alternatives, if any. We refer to this set of alternatives as the *core* of F at θ and denote it by $C_F(\theta)$. Formally, it is defined as

$$C_F(\theta) = M(R_F(\theta)) = M(P_F(\theta)).$$

When the PAR F is understood, we write simply $C(\theta)$ for the core. Whether this set is nonempty will depend on continuity and transitivity properties of social preferences.

2.2 Examples of Environments

Example 1: The Discrete Choice Model. A classic example of a collective choice environment from political science is that of a group N of voters who must choose from a finite set X of candidates to fill a political office. Each voter has preferences over the candidates, and states simply index the set of possible preference profiles. Alternatively, a voting body (e.g., an electorate, a city council, a board of directors) must choose from a finite set of projects to undertake. Or a department faculty must choose from a finite set of job candidates to fill a position. In these kinds of environments, it is often difficult to impose *a priori* restrictions on individual preferences, and it is natural to allow for every profile of weak orders. Let

$$\mathbf{U} = \{((P_1,R_1),\ldots,(P_n,R_n)) \mid \forall i \in N : (P_i,R_i) \text{ is an order of } X\}$$

denote the set of all profiles of orders of X. We refer to the assumption that $PR(\Theta) = \mathbf{U}$ as *unrestricted domain.*

Sometimes, especially when the set of alternatives is small, we may wish to consider the possibility that individuals can always discern a strict preference between distinct candidates. Let

$$\mathbf{L} = \{((P_1,R_1),\ldots,(P_n,R_n)) \mid \forall i \in N : (P_i,R_i) \text{ is a linear order of } X\}.$$

We refer to the assumption that $PR(\Theta) = \mathbf{L}$ as *linear domain.*

There are special finite choice environments in which extra structure suggests intuitive preference restrictions. For example, suppose an electorate has several political positions to fill, as in multi-member district systems. Suppose there are k candidates and $m \leq k$ slots to fill. In this case, an alternative x would represent an assignment of m candidates to the m different slots, y would represent another assignment. We then take as primitive voter preferences over *assignments* of candidates to the slots — not actually over candidates themselves. In this way, the collective choice problem of filling multiple political offices is isomorphic to the problem of filling one.

Note, however, that preferences over assignments of candidates may implicitly be generated from (unmodelled) preferences over the candidates themselves. If so, we may be able to take advantage of this extra structure in formulating reasonable preference restrictions. For example, suppose there are three offices to fill; let $x = (a,b,c)$ denote the assignment of candidate a to the first office, b to the second, c to the third; let $y = (a,b,d)$; let $x' = (a',b',c)$; and let $y' = (a',b',d)$. If xP_iy, then we might infer that voter i prefers having c in the third office over d. Since x' and y' differ only in that way, it may be reasonable to infer $x'P_iy'$. This is a restriction on preferences that is not possible in the original model. It is not without loss of generality, however, for implicit in it is an idea of "separability," i.e., that i's preferences over the third slot are independent of which candidates fill the first two.

Similarly, we might suppose that a voting body, say a city council, has a fixed budget from which it can fund any number of projects, subject to budget balance. In this case, x would denote a *list* of projects funded, and we assume council members have preferences over lists of projects. Or in the example of an academic department, there may be several slots to fill, in which case an alternative x would be a list of hired job candidates.

Example 2: The One-Dimensional Spatial Model. A second classic example of a collective choice environment is that of a society that must choose a single proportional income tax rate between 0 and 1. Or that of a town that must choose the location of a library on its main street. Or, suppose that public policies can be ordered according to a single summary statistic that contains all information relevant for voter preferences. The standard interpretation is that policies corresponding to smaller values of the statistic are more "liberal," while those corresponding to greater values of the statistic are more "conservative." Thus, under this interpretation, we reduce policies to their ideological content, placing them in a one-dimensional space to represent their place in the liberal-conservative spectrum. If ideological content is indeed all that matters to voters, we can then model public policy as the choice of a point on the real line. The common element of these examples is the one-dimensional structure inherent in the set of alternatives.

This structure suggests an intuitive restriction on the domain of preferences. The idea is that each individual will have some ideal alternative (e.g., a favorite tax rate, location of a library, ideological mix); and that, if we move to the left or right from that ideal point, then the alternatives grow worse for the individual. Formally, we say a profile $((P_1, R_1), \ldots, (P_n, R_n))$ of individual preferences is *single-peaked* if there exists a weak linear ordering, say $\preceq$, of X such that, for each $i \in N$, there exists $\tilde{x}^i \in X$ satisfying the following conditions:

- for all $y \neq \tilde{x}^i$, $\tilde{x}^i P_i y$;
- for all $y, z \in X$, if $z \prec y \prec \tilde{x}^i$, then $y P_i z$;
- for all $y, z \in X$, if $\tilde{x}^i \prec y \prec z$, then $y P_i z$,

where $\prec$ is the asymmetric part of $\preceq$.[2] (Because $\preceq$ is complete, $x \prec y$ if and only if not $y \preceq x$.) In this case, we say $((P_1, R_1), \ldots, (P_n, R_n))$ is *single-peaked with respect to* $\preceq$. Let

$$\mathbf{S} \;=\; \{((P_1, R_1), \ldots, (P_n, R_n)) \mid ((P_1, R_1), \ldots, (P_n, R_n)) \text{ is single-peaked } \}.$$

We refer to the assumption that $PR(\Theta) = \mathbf{S}$ as *single-peaked domain.*

Given a weak order R_i for individual i, we call $\tilde{x}^i \in X$ the individual's *ideal point* if, for all other $x \in X$, $\tilde{x}^i P_i x$. Other terminology is "bliss point" or "satiation point."

For an example of single-peakedness, suppose that $n = 3$, that $X = \{x, y, z\}$, and that

[2] Many of our results for this model can be adapted to the more general setting in which we allow $\tilde{x}^i \in \{\infty, -\infty\}$, representing "extremists" who prefer alternatives as far as possible in one direction or the other.

individual preferences are as follows.

R_1	R_2	R_3
x	y	z
y	x	y
z	z	x

That (R_1, R_2, R_3) is single-peaked with respect to $x \prec y \prec z$ (with $\tilde{x}^1 = x,\ \tilde{x}^2 = y,\ \tilde{x}^3 = z$) can be easily seen by drawing utility representations of these preferences. See Figure 2.1. Note that these representations really do look "single-peaked."

Figure 2.1: A Single-peaked profile

It's important to note that, in the definition of single-peakedness, the same weak linear ordering $\preceq$ of X must be used for all individual preferences. But different $\preceq$'s can be used for different profiles. For example, consider the following preferences.

R_1	R_2	R_3
y	x	z
x	z	x
z	y	y

This profile is single-peaked with respect to $y \prec x \prec z$, but it is not single-peaked with respect to the ordering used above, $x \prec y \prec z$.

Note that the domain of single-peaked preference profiles is not generally "rectangular," i.e., there is no set D of weak orders such that the domain of single-peaked profiles is just $D \times \cdots \times D$ (n times). To see this, note that

R_1'	R_2'	R_3'
z	z	z
y	y	y
x	x	x

is clearly single-peaked with respect to $x \prec z \prec y$. But, using R_1 and R_2 from the previous example, (R_1, R_2, R_3') is not single-peaked with respect to any weak linear ordering of $\{x, y, z\}$. (Do you see why?)

Note that (R_1', R_2', R_3') is also single-peaked with respect to $x \prec y \prec z$, so the weak linear ordering on X in the definition of single-peakedness is not unique, even up to inversions.

To elaborate on non-rectangularity: single-peakedness is not a property of a weak order considered in isolation, but rather it is a property of a profile of weak orders taken together.

Now that I've emphasized that point, I'll backtrack a bit. In some models, there is a particular weak linear order of X that suggests itself: when X is a subset of the real line, for example, we will usually want to consider profiles that are single-peaked with respect to the usual less-than-or-equal-to relation, i.e., $\preceq\, = \,\leq$. When the weak linear order $\preceq$ is fixed, I may want to consider only preferences single-peaked with respect to $\preceq$. You can check that the set of profiles single-peaked with respect to a given $\preceq$ is, indeed, rectangular. Thus, single-peakedness with respect to $\preceq$ can be considered a property of weak orders by themselves.

This observation is borne out in the following proposition, which characterizes the profiles of weak orders single-peaked with respect to the usual $\leq$ relation on $\mathbb{R}$. Note that the two-part condition characterizing single-peakedness is stated for each R_i separately.

Proposition 2.1 *Let $X \subseteq \mathbb{R}$ be convex. The profile $((P_1, R_1), \ldots, (P_n, R_n))$ is single-peaked with respect to $\leq$ if and only if, for all $i \in N$, (i) (P_i, R_i) is strictly convex, and (ii) R_i has a maximal element, i.e., there exists $\tilde{x}^i \in X$ such that, for all $y \in X$, we have $\tilde{x}^i R_i y$.*

Proof: Assume $((P_1, R_1), \ldots, (P_n, R_n))$ is single-peaked with respect to $\leq$, and take any $i \in N$. Then i's ideal point is maximal. Take $x \in X$ and distinct $y, z \in R_i(x)$. Let $\alpha \in (0,1)$, and define $w = \alpha y + (1-\alpha)z$. Assume without loss of generality that $y < z$. If $w < \tilde{x}^i$, then $y < w < \tilde{x}^i$, and then single-peakedness implies wP_iy. By transitivity, wP_ix. If $w > \tilde{x}^i$, then $\tilde{x}^i < w < z$, and then single-peakedness implies wP_iz. By transitivity, wP_ix. If $w = \tilde{x}^i$, then wP_iy, and transitivity implies wP_ix. Thus, each (P_i, R_i) is strictly convex. I'll leave the other direction to you. ∎

Note that, if $X \subseteq \mathbb{R}$ is convex and (P_i, R_i) is strictly quasi-concave, then the dual pair has a utility representation. An implication of Propositions 1.20 and 2.1 is a characterization of single-peakedness in terms of utility representations.

Corollary 2.1 *Let $X \subseteq \mathbb{R}$ be convex. The profile $((P_1, R_1), \ldots, (P_n, R_n))$ is single-peaked with respect to $\leq$ if and only if, for all $i \in N$, (P_i, R_i) has a utility representation u_i satisfying (i) u_i is strictly quasi-concave, and (ii) u_i has a maximizer.*

I follow Austen-Smith and Banks in defining a weakening of the usual single-peakedness condition. We say a profile $((P_1, R_1), \ldots, (P_n, R_n))$ is *weakly single-peaked* if there exists a weak linear ordering, say $\preceq$, of X such that, for each $i \in N$, there exists $\tilde{x}^i \in X$ satisfying the following conditions:

- for all $y \in X$, $\tilde{x}^i R_i y$;
- for all $y, z \in X$, if $z \prec y \prec \tilde{x}^i$, then yR_iz;

- for all $y, z \in X$, if $\tilde{x}^i \prec y \prec z$, then yR_iz.

In this case, we say $((P_1, R_1), \ldots, (P_n, R_n))$ is *weakly single-peaked with respect to* $\preceq$. Let

$$\mathbf{W} \;=\; \{((P_1,R_1),\ldots,(P_n,R_n)) \mid ((P_1,R_1),\ldots,(P_n,R_n)) \text{ is weakly single-peaked }\}.$$

We refer to the assumption that $PR(\Theta) = \mathbf{W}$ as *weakly single-peaked domain.*

Thus, if we move to the left or right from $\tilde{x}_i$, alternatives grow weakly worse for the individual. Obviously, weak single-peakedness allows indifferences that are ruled out by single-peakedness. As the next variant of Proposition 2.1 shows, it amounts to dropping the "strict" from strict convexity.

Proposition 2.2 *Let $X \subseteq \mathbb{R}$ be convex. The profile $((P_1, R_1), \ldots, (P_n, R_n))$ of weak orders is weakly single-peaked with respect to $\leq$ if and only if, for all $i \in N$, (i) (P_i, R_i) is convex, and (ii) R_i has a maximal element, i.e., there exists $\tilde{x}^i \in X$ such that, for all $y \in X$, we have $\tilde{x}^i R_i y$.*

For a final example of an environment with single-peaked preferences, suppose, for example, that a society must decide on some level of a public good to be provided, assumed to be a quantity in $[0, \overline{x}]$. Individuals also receive utility from "income," which we allow to be positive or negative, for simplicity. Suppose that i's endowment of money is $\overline{y}_i$, and that the cost of x units of public good is $C(x)$ (where $C(\cdot)$ is strictly convex and continuous) to be divided equally among the individuals. Thus, an alternative is $(x, y_1, \ldots, y_n)$ such that $x \geq 0$ and $y_i = \overline{y}_i - C(x)/n$ for each i. Suppose that each individual i has preferences over $(x, y_1, \ldots, y_n)$ vectors with a utility representation u_i satisfying

$$u_i(x, y_1, y_2, \ldots, y_n) \;=\; v_i(x) + y_i,$$

where v_i is continuous and strictly concave. Since a public good level determines incomes uniquely, we can reduce the set of alternatives to $X = [0, \overline{x}]$. Then each i has "induced preferences" on X with utility representation u_i^* such that

$$u_i^*(x) \;=\; v_i(x) - \frac{C(x)}{n} + \overline{y}_i,$$

which is continuous and strictly concave. Therefore, u_i^* has a unique maximizer and is strictly quasi-concave, which implies that each (P_i, R_i) is strictly convex. By Proposition 2.1, therefore, induced preferences are single-peaked with respect to $\leq$.

Example 3: The Multi-dimensional Spatial Model. Now suppose a society must choose from a set of public policy vectors or from a set of feasible allocations of public goods. In that case, if there are d policy dimensions or public goods, then we have $X \subseteq \mathbb{R}^d$. An alternative is $x = (x_1, \ldots, x_d)$, where x_k is the location of public policy on the kth dimension or the amount of kth public good provided. Usually, we either suppose there are no restrictions on the set of alternatives, i.e., $X = \mathbb{R}^d$ or $X = \mathbb{R}^d_+$, or we suppose there is some convex, continuous technology for the production of public goods or for the determination of public policy, implying that X is convex, closed, and possibly compact. Typically, we assume that X has nonempty interior in $\mathbb{R}^d$, implying that X is infinite, in which case the specification of X suggests extra structure that we are often willing to impose in applications.

A classic special case of the multidimensional spatial model has played an important role in the theoretical political science literature. Given $X \subseteq \mathbb{R}^d$, recall that (P_i, R_i) is Euclidean if i has a unique maximal alternative in X, the individual's "ideal point," denoted $\tilde{x}_i$, and, for all $x, y \in X$, xR_iy if and only if $||x - \tilde{x}_i|| \leq ||y - \tilde{x}_i||$. In words, R_i is Euclidean if i's indifference curves are concentric circles (more generally, spheres or hyperspheres), with alternatives closer to i's ideal point being better for i. Let

$$\mathbf{E} \;\; = \;\; \{((P_1, R_1), \ldots, (P_n, R_n)) \mid \forall i \in N : (P_i, R_i) \text{ is Euclidean } \}.$$

We refer to the assumption that $PR(\Theta) = \mathbf{E}$ as *Euclidean domain.* Almost all of our results hold under much more general assumptions about individual preferences.

A minimal condition is continuity of preferences, i.e., each (P_i, R_i) is continuous: for every $x \in X$, the sets $R_i(x)$ and $R_i^{-1}(x)$ are closed, or equivalently, $P^{-1}(x)$ and $P(x)$ are open. This assumption, which formalizes the idea that alternatives that are very close to each other should seem similar to the individual, is a fundamental one in the analysis of infinite models. Note that, if X is assumed to be only closed or compact, then X could be finite. In that case, individual preferences are automatically continuous. Indeed, many of the results below assume only compactness of X and continuity of individual preferences, and they therefore apply to all finite models: given any model with a finite number of alternatives, $x_1, \ldots, x_k$, and arbitrary individual preferences over these alternatives, we can imbed $x_1, \ldots, x_k$ in $\mathbb{R}^d$ to get a finite set $X \subseteq \mathbb{R}^d$ and define preferences over X as originally specified — this gives us a special case of the multi-dimensional spatial model with a compact set of alternatives and continuous individual preferences!

Another, often reasonable, restriction is convexity of preferences, i.e., each (P_i, R_i) is convex: for all $x \in X$, $R_i(x)$ is convex. This is a formalization of the idea that, if we start from one alternative and move in the direction of a preferred alternative, then that change will be an improvement. This assumption does not in itself rule out the possibility that an alternative, x, contains private good components, e.g., we might have $x = (x_1, \ldots, x_n, x_{n+1})$, where, for $i = 1, \ldots, n$, x_i is an amount of a private good consumed by individual i, and x_{n+1} is the amount of a public good consumed by all. Thus, these assumptions are quite general.

A strengthening of convexity is that, for each individual i, (P_i, R_i) is strictly convex: for all $x \in X$, all $y, z \in R_i(x)$ with $y \neq z$, and all $\alpha \in (0, 1)$, we have $\alpha y + (1 - \alpha)z \in P_i(x)$. This assumption rules out the above-mentioned possibility of private good components of x. In the above example, individual i might be indifferent between alternatives y and z, where y and z differ only in the private good consumption levels of individual $j \neq i$, but then strict convexity would require $w = (1/2)y + (1/2)zP_iy$. The problem is that w also differs from y only in j's private good consumption, so i should be indifferent between all three bundles.

We say the profile $((P_1, R_1), \ldots, (P_n, R_n))$ satisfies *limited shared weak preferences (LSWP)* if, for all $x \in X$ and all $G \subseteq N$,

$$|\bigcap_{i \in G} R_i(x)| \geq 2 \quad \Rightarrow \quad \bigcap_{i \in G} R_i(x) \subseteq \operatorname{clos} \bigcap_{i \in G} P_i(x).$$

In other words, if all members of a group weakly prefer some alternative $y \neq x$ to x, then there are alternatives arbitrarily close to y that all members of a group strictly prefer to x. Note that LSWP rules out "locally maximal" elements of (P_i, R_i), unless they are global, and in that case there can be at most one. The condition is clearly implied by strict convexity. (Why?) To see that it is strictly weaker than strict convexity, note that it is satisfied whenever linear domain holds, as it can in the discrete model. Unlike strict convexity, LSWP is consistent with the existence of private goods and is, in fact, satisfied in a very large class of such models. We will see that many of our results for the multidimensional model may use this weak assumption rather than strict convexity, and so they actually carry over to the discrete choice model.

I have suggested two interpretations of the multidimensional spatial model. One is that dimensions represent different aspects of public policy: for example, some dimensions might indicate public policy on economic issues, such as the level of central bank independence or the restrictiveness of regulations, while others indicate how liberal or conservative public policy is on various social issues, such as freedom of speech, abortion rights, etc. In this case, we typically assume that individual preferences are continuous and convex (possibly strictly), and that each individual has a unique maximal alternative, an "ideal point," denoted $\tilde{x}^i$, reflecting that individual's ideal combination of positions on all policy issues.

The other is that dimensions represent levels of different public goods. In this case, a common type of preference restriction is that each R_i is *monotonic,* i.e., for all $x, y \in X$, if $x \gg y$ (so x is greater than y in every component), then xP_iy. This, of course, merely formalizes the idea that public goods are indeed "goods," or at least not bads, because more of all of them is unambiguously good for the individuals.

Something for you to think about: monotonicity seems like a reasonable assumption, but in the one-dimensional public good problem with single-peaked preferences, above, it is violated. How do you reconcile these observations?

A strengthening of monotonicity is the restriction that, for each individual i, (P_i, R_i) is *strictly monotonic,* i.e., for all $x, y \in X$, if $x > y$ (so $x \geq y$, greater in at least one component), then xP_iy. Like strict convexity, this restriction also rules out private good components. (Do you see why?)

There is actually a close connection between these two interpretations. Suppose that the group has a fixed amount m of money to purchase ℓ public goods at fixed, positive prices p. All vectors in $\mathbb{R}^{\ell}_{+}$ of provision levels are conceivable, and each individual has continuous, strictly convex, strictly monotonic preferences over $\mathbb{R}^{\ell}_{+}$. Because of the budget constraint, however, the set of alternatives is the simplex $X = \{x \in \mathbb{R}^{\ell}_{+} \mid p \cdot x \leq m\}$ in $\mathbb{R}^{\ell}_{+}$. Indeed, by monotonicity, all vectors x such that $p \cdot x < m$ are Pareto dominated, so it is customary to treat the set of alternatives as the outer face of this simplex, where the budget is exhausted: $X = \{x \in \mathbb{R}^{\ell}_{+} \mid p \cdot x = m\}$ in $\mathbb{R}^{\ell}_{+}$. Individual preferences will be given by indifference curves on this face: they are continuous and strictly convex, as before. An important feature of preferences induced on this simplex is that each individual will have an ideal point $\tilde{x}_i$, a vector of provision levels strictly preferred to every other. Thus, we are back to the model of public policy described above.

Example 4: Private Good Exchange Economies. A group N of individuals is endowed with amounts of k resources that must be allocated. Let $w \in \mathbb{R}^{k}_{+}$ denote the social endowment. Thus, $w = (w_1, \ldots, w_k)$, where w_j is the total endowment of the jth resource. In some examples of private good allocation environments, individual property rights over these resources may be defined, in which case w_j is the total amount of privately and publicly owned units of the jth resource. An alternative $x \in \mathbb{R}^{nk}_{+}$ is an allocation of resources to the individuals, i.e., $x = (x^1, \ldots, x^n)$, where $x^i \in \mathbb{R}^{k}_{+}$ is a vector indicating i's consumption levels. The set of alternatives is then $X = \{x \in \mathbb{R}^{nk}_{+} \mid \sum_{i=1}^{n} x^i = w\}$.

Because the set of alternatives is a subset of multidimensional Euclidean space, private good exchange economies are related to the spatial model. In fact, the familiar restriction of continuous preferences is always imposed, and convexity and monotonicity are typically assumed. Because the set of feasible allocations in an exchange economy has empty interior, however, some of our results for the multidimensional spatial model — even though they are very general with respect to preferences — do not specialize to private good exchange economies.

A fundamental preference restriction, not seen in the spatial model, is that an individual i's preferences depend only on i's consumption: for all $x, y \in X$, if $x^i = y^i$, then xI_iy. This restriction, which can only be formulated when some form of private consumption is assumed, is always assumed in private good exchange environments. It leads to a convexity condition between convexity and strict convexity. We say (P_i, R_i) is *strictly convex in own consumption* if, for all $x \in X$, all $y, z \in R(x)$ with $y^i \neq z^i$, and for all $\alpha \in (0, 1)$, we have $w = \alpha y + (1-\alpha)zP_ix$. The crucial assumption here is, of course, that y and z give i distinct consumption bundles, so that the mixture w allows i a non-trivial improvement.

Exchange environments also suggest a monotonicity assumption between monotonicity and strict monotonicity. We say (P_i, R_i) is *strictly monotonic in own consumption* if, for all $x, y \in X$, $x^i > y^i$ implies xP_iy.

Perhaps the simplest example of a private good exchange economy is the situation where n individuals must allocate a single fixed resource, often called a "dollar," a "cake," or a "pie." Thus, the set of alternatives is just the outer face of a simplex in $\mathbb{R}^n$. Letting $w > 0$ denote the total amount available, an alternative is $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$, where $\sum_{i=1}^n x_i = w$ and x_i denotes i's non-negative consumption of the resource. Here, monotonicity and the assumption of no externalities immediately imply the following restriction on preferences: xP_iy if and only if $x_i > y_i$. I will adhere to the least culinary of the standard terminologies by referring to such a problem as a *divide the dollar environment.*

2.3 Preference Aggregation Rules

An Assortment of PARs

There are numerous ways of aggregating preferences of individuals. Here are some examples, defined, for simplicity, in terms of strict preference.

simple majority, F_{SM}

$$xP_{SM}(\theta)y \text{ if and only if } p(x, y|\theta) > n/2.$$

relative majority, F_{RM}

$$xP_{RM}(\theta)y \text{ if and only if } p(x, y|\theta) > p(y, x|\theta).$$

simple Pareto, F_{SP}

$$xP_{SP}(\theta)y \text{ if and only if } p(x, y|\theta) = n.$$

relative Pareto, F_{RP}

$$xP_{RP}(\theta)y \text{ if and only if } r(x, y|\theta) = n \text{ and } p(x, y|\theta) > 0.$$

Here I have defined strict social preference. Note: In doing so, I must be careful to ensure that $P_F(\theta)$ is asymmetric in every state. (You can check that this holds.)

Weak social preference is obtained in the usual way. For example, $xR_{SM}(\theta)y$ if and only if not $yP_{SM}(\theta)x$, which holds if and only if not $p(y, x|\theta) > n/2$, which is equivalent to $r(x, y) \geq n/2$. Weak social preferences for other PARs are derived similarly — I leave that to you. Note that, as long as $P_F(\theta)$, defined above, is asymmetric, weak social preference will be complete, as required.

See Figure 2.2 for an examples of each of these PARs when X is finite.

Figure 2.2: Social preferences, finite X

Note the nestings: $P_{SM}(\theta) \subseteq P_{RM}(\theta)$ and $P_{SP}(\theta) \subseteq P_{RP}(\theta)$. Under linear domain, $P_{SM}(\theta) = P_{RM}(\theta)$ and $P_{SP}(\theta) = P_{RP}(\theta)$.

What do social preferences corresponding to these PARs look like when X is infinite? See Figure 2.3 for upper sections of strict and weak social preference for the simple majority and simple Pareto PARs.

Figure 2.3: Social preferences, infinite X

The following class of PARs generalizes F_{SM} and F_{SP} by establishing a numerical quota, q, needed for a strict social preference. Let q be an integer, and, in order to ensure asymmetry of strict social preference, assume $q > n/2$.

q-rule, F_q

$$xP_q(\theta)y \text{ if and only if } p(x, y|\theta) \geq q.$$

Simple majority is the case $q = \frac{n+1}{2}$ when n is odd, or $q = \frac{n}{2} + 1$ when n is even. Generally, simple majority is $q = \lceil \frac{n+1}{2} \rceil$, where $\lceil \cdot \rceil$ denotes "smallest integer greater than or equal to." And simple Pareto is the case $q = n$.

The next rule generalizes simple Pareto by designating an arbitrary group $G \subseteq N$ that can impose a common strict preference among its members on society. Assume $G \neq \emptyset$.

G-rule, F_G

$$xP_G(\theta)y \text{ if and only if } G \subseteq P(x, y|\theta),$$
$$\text{i.e., } P_G(\theta) = \bigcap_{i \in G} P_i(\theta).$$

Equivalently, given $G \neq \emptyset$,

$$R_G(\theta) = \bigcup_{i \in G} R_i(\theta)$$

for all $\theta \in \Theta$. That is, x is weakly preferred to y under G-rule if some member of G weakly prefers x to y.

Simple Pareto is the special case where $G = N$. Note that the classes of G-rules and q-rules are non-nested: there are G-rules that are not q-rules (e.g., whenever G is a proper subset of N), and there q-rules that are not G-rules (e.g., simple majority).

We write F_i for the special case of G-rule with $G = \{i\}$ for some $i \in N$. In this case, i is a "dictator," for the social preference relation is simply equal to i's.

A class of PARs that generalize many of the ones previously defined is the "weighted" q-rules, where each individual i is given a weight $w_i \geq 0$ and q satisfies $q > (\sum_{i \in N} w_i)/2$. We write $w = (w_1, \ldots, w_n)$ for a vector of such weights.

weighted q-rule, $F_{w,q}$

$$xP_{w,q}(\theta)y \text{ if and only if } \sum_{i \in P(x,y|\theta)} w_i \geq q.$$

Simple majority rule, for example, is obtained by setting $w_i = 1$ for each $i \in N$ and $q = \lceil \frac{n+1}{2} \rceil$. We obtain G-rule by setting $w_i = 1$ for each $i \in G$, $w_i = 0$ otherwise, and $q = |G|$. To obtain "weighted majority rule," where a group can impose its preferences on society if and only if it has more than half of the total weight, define

$$q = \min\{\sum_{i \in G} w_i \mid G \subseteq N, \sum_{i \in G} w_i > (\sum_{i \in N} w_i)/2\}.$$

In fact, any q strictly between this number and $(\sum_{i \in N} w_i)/2$ would work as well.

Here is a PAR defined using the idea of transitive closure of weak preference, which we saw in the analysis of the weak top cycle. At this point, we define the transitive closure PAR using simple majority preference. Later, we will define the weak top cycle derived from general PARs. Note that it is now much more convenient to define social preferences in terms of weak preference.

simple majority transitive closure, F_T

$$xR_T(\theta)y \text{ if and only if } xT_{R_{SM}(\theta)}y.$$

Thus, x is weakly socially preferred to y here if we can get from x to y in a finite number of weak simple majority preference steps.

Another PAR is defined using the idea of covering from our analysis of the weak uncovered set. At this point, we define covering using simple majority preferences. Later, we will define the weak uncovered set derived from general PARs.

simple majority covering, F_C

$$xP_C(\theta)y \;\text{ if and only if }\; R_{SM}(x|\theta) \subseteq P_{SM}(y|\theta).$$

Thus, $P_C(\theta)$ is just $C_{R_{SM}(\theta)}$. It follows that $xR_C(\theta)y$ if and only if there exists $z \in X$ such that $xR_{SM}(\theta)zR_{SM}(\theta)y$. When $xP_C(\theta)y$, we say that x "covers" y.

We could also define the covering PAR that generates the strong uncovered set, i.e., $xP_{SM}(\theta)y$ and $P_{SM}(x|\theta) \subseteq R_{SM}(y|\theta)$, but we'll just stick with the above.

A yet different type of PAR is defined for finite X. It was invented by Jean-Charles de Borda and bears his name. First, let

$$\begin{aligned} B_i^x(\theta) &= |\{y \in X \mid xP_i(\theta)y\}| - |\{y \in X \mid yP_i(\theta)x\}| \\ B^x(\theta) &= \sum_{i=1}^{n} B_i^x(\theta). \end{aligned}$$

In words, $B_i^x(\theta)$ registers the net number of strict preferences for x over other alternatives in i's preference ordering, and $B^x(\theta)$ sums these across individuals.

Borda rule, F_B

$$xP_B(\theta)y \;\text{ if and only if }\; B^x(\theta) > B^y(\theta).$$ [3]

One interpretation of Borda rule is that it uses the number of alternatives between, say, x and y in an individual's ranking to measure "intensity" of preference. That is okay, *if that is how you define intensity.* Since we define preferences as simply collections of binary comparisons, however, there is no guarantee that this measure will correspond to the ordinary language meaning.

Decisive and Blocking Coalitions

Given a PAR F, we say G is *decisive* if, for all distinct $x, y \in X$ and all $\theta \in \Theta$,

$$G \subseteq P(x, y|\theta) \;\;\Rightarrow\;\; xP_F(\theta)y.$$

When the members of G strictly prefer any x to any y, then x is strictly socially preferred to y. Let $\mathcal{D}(F)$ denote the collection of decisive coalitions of F.

Note that we must have $G \cap G' \neq \emptyset$ for all $G, G' \in \mathcal{D}(F)$, for we assume the existence of a free pair: if $G, G' \in \mathcal{D}(F)$ are disjoint, then there exists $\theta \in \Theta$ such that $G \subseteq P(x, y|\theta)$ and $G' \subseteq P(y, x|\theta)$, but then $xP_F(\theta)y$ and $yP_F(\theta)x$, contradicting asymmetry.

We say G is *blocking* if, for all distinct $x, y \in X$ and all $\theta \in \Theta$,

$$G \subseteq P(x, y|\theta) \;\;\Rightarrow\;\; xR_F(\theta)y.$$

Thus, a blocking coalition has less power than a decisive coalition, in that it can enforce only a weak social preference. Let $\mathcal{B}(F)$ denote the collection of blocking coalitions.

Clearly, $\mathcal{D}(F) \subseteq \mathcal{B}(F)$. We next develop some slightly deeper (but not much!) results.

Let $\mathcal{G}$ denote a collection of groups. We say $\mathcal{G}$ is ...

- *proper* if, for all $G, G' \in \mathcal{G}$, we have $G \cap G' \neq \emptyset$,
- *strong* if, for all $G \subseteq N$, there exists $G' \in \mathcal{G}$ such that $G' \subseteq G$ or $G' \subseteq N \setminus G$,
- *monotonic* if, for all $G \in \mathcal{G}$ and all $G' \subseteq N$, $G \subseteq G'$ implies $G' \in \mathcal{G}$.

Note that, for proper $\mathcal{G}$, we must have $G \neq \emptyset$ for all $G \in \mathcal{G}$. We do not rule out the possibility, however, that $\mathcal{G} = \emptyset$. Note also that, if $\mathcal{G}$ is monotonic, then the following condition is equivalent to $\mathcal{G}$ being strong: for all $G \subseteq N$, $G \notin \mathcal{G}$ implies $N \setminus G \in \mathcal{G}$.

We say F is *strong* if $\mathcal{D}(F)$ is strong.

Proposition 2.3

1. *$\mathcal{D}(F)$ and $\mathcal{B}(F)$ are monotonic.*
2. *For all $G \in \mathcal{D}(F)$ and all $G' \in \mathcal{B}(F)$, $G \cap G' \neq \emptyset$.*
3. *F is strong if and only if, for all $x, y \in X$ and all $\theta \in \Theta$, $xR_F(\theta)y$ implies $R(x, y|\theta) \in \mathcal{D}(F)$.*
4. *If F is strong, then $\mathcal{D}(F) = \mathcal{B}(F)$.*

Proof: To prove part 2, let $G \in \mathcal{D}(F)$ and $G' \in \mathcal{B}(F)$, and suppose $G \cap G' = \emptyset$. Let $\{x, y\}$ be a free pair, and take any $\theta \in \Theta$ such that $G \subseteq P(x, y|\theta)$ and $G' \subseteq P(y, x|\theta)$. Then $xP_F(\theta)y$ and $yR_F(\theta)x$, a contradiction. To prove part 3, assume F is strong. Pick $x, y \in X$ and $\theta \in \Theta$ such that $xR_F(\theta)y$. Then $P(y, x|\theta) \notin \mathcal{D}(F)$. Since $\mathcal{D}(F)$ is strong, $R(x, y|\theta) \in \mathcal{D}(F)$, proving one direction. Now assume F is not strong, so there exists $G \subseteq N$ such that $G \notin \mathcal{D}(F)$ and $N \setminus G \notin \mathcal{D}(F)$. Let $\{x, y\}$ be any free pair, and let $\theta \in \Theta$ satisfy $P(x, y|\theta) = G$ and $P(y, x|\theta) = N \setminus G$. By completeness, either $xR_F(\theta)y$ and $R(x, y|\theta) \notin \mathcal{D}(F)$, or $yR_F(\theta)x$ and $R(y, x|\theta) \notin \mathcal{D}(F)$, proving the converse. For part 4, take any $G \in \mathcal{B}(F)$. Let $\{x, y\}$ be a free pair, and let $\theta \in \Theta$ satisfy $P(x, y|\theta) = G$ and $P(y, x|\theta) = N \setminus G$. Then $xR_F(\theta)y$, which, by part 3, implies $G \in \mathcal{D}(F)$. ∎

Note that part 2 of Proposition 2.3 is actually stronger than the condition that $\mathcal{D}(F)$ is proper. (Why?) Is the converse of part 4 true?

You should try to figure out the decisive and blocking coalitions for the PARs defined above. (Borda rule and the weak covering PAR are a little tricky ...)

The condition that $\mathcal{D}(F)$ is strong is fairly restrictive. The prototypical example of such a PAR is simple majority rule with n odd. You can check, however, that weighted majority rule is strong if and only if there is no partition I, J of N such that $\sum_{i \in I} w_i = \sum_{i \in J} w_i = (\sum_{i \in N} w_i)/2$. In one sense, this is true for "most" assignments of weights.

For technical reasons, we define weaker notions of decisive and blocking coalitions. We say G is *semi-decisive for x over y* if, for all $\theta \in \Theta$,

$$[G = P(x, y|\theta) \text{ and } N \setminus G = P(y, x|\theta)] \quad \Rightarrow \quad xP_F(\theta)y.$$

The group G is simply *semi-decisive* if it semi-decisive for every x over every y. Thus, a semi-decisive group is weaker than a decisive one in that it can enforce a strict social preference only when opposed by all non-members.

We say G is *semi-blocking for x over y* if, for all $\theta \in \Theta$,

$$[G = P(x, y|\theta) \text{ and } N \setminus G = P(y, x|\theta)] \quad \Rightarrow \quad xR_F(\theta)y.$$

The group G is simply *semi-blocking* if it semi-blocking for every x over every y.

Clearly, if a group is semi-decisive for some x over some y, then it is also semi-blocking for x over y.

Conditions on PARs

Next, I define several conditions on PARs.

Pareto For all $\theta \in \Theta$, $P_N(\theta) \subseteq P_F(\theta)$

In words, if every individual prefers some x to some y, then x is strictly socially preferred to y. Equivalently, N is decisive.

weak Pareto For all $\theta \in \Theta$, $P_N(\theta) \subseteq R_F(\theta)$

In words, if every individual prefers some x to some y, then x is socially weakly preferred to y. Equivalently, N is blocking.

independence of irrelevant alternatives (IIA) For all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$[P(x, y|\theta) = P(x, y|\theta') \text{ and } R(x, y|\theta) = R(x, y|\theta') \text{ and } xP_F(\theta)y] \quad \Rightarrow \quad xP_F(\theta')y.$$

An equivalent definition is this: for all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$[P(x,y|\theta) = P(x,y|\theta') \text{ and } R(x,y|\theta) = R(x,y|\theta') \text{ and } xR_F(\theta)y] \quad \Rightarrow \quad xR_F(\theta')y.$$

Can you see why?

For yet another equivalent definition, we might say F satisfies IIA if, for all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$[PR(\theta)|_{\{x,y\}} = PR(\theta')|_{\{x,y\}}] \quad \Rightarrow \quad [R_F(\theta)|_{\{x,y\}} = R_F(\theta')|_{\{x,y\}}],$$

where $R_i|_Y = R_i \cap (Y \times Y)$ is the "restriction" of R_i to Y, and $(R_1, \ldots, R_n)|_Y = (R_1|_Y, \ldots, R_n|_Y)$.

Thus, if individual preferences between two alternatives, x and y, are the same in two states, then the social preference between them should be the same as well. That is, individual preferences over other alternatives are "irrelevant" in the determination of social preferences over x and y.

The motivation for this condition is straightforward if we view a social preference between alternatives x and y as indicating the social choice from the pair $\{x, y\}$. In that case, if literally no other alternatives are feasible, then it is plausible that they would not affect the process through which social decisions are ultimately determined. That would be the case, for example, if those social decisions are the result of a non-cooperative game played by the individuals. From this perspective, IIA is an "implementation" constraint.

The next lemma is pretty trivial (right?), but we will use it in our analysis of collective rationality.

Lemma 2.1 *Assume F satisfies IIA. Let $\{x, y\}$ be any free pair of alternatives, and let $G \subseteq N$ be any group. Either G is semi-blocking for x over y, or $N \setminus G$ is semi-decisive for y over x.*

The next three conditions are as defined in Austen-Smith and Banks.

neutrality For all $x, y, w, z \in X$ and all $\theta, \theta' \in \Theta$,

$$[P(x,y|\theta) = P(w,z|\theta') \text{ and } R(x,y|\theta) = R(w,z|\theta') \text{ and } xP_F(\theta)y] \quad \Rightarrow \quad wP_F(\theta')z.$$

This condition strengthens IIA and formalizes the idea that no alternatives are "special." If individual preferences over x and y in one state are the same as individual preferences over w and z in another, social preferences should be too (with w playing the same role as x, z playing the same role as y).

monotonicity For all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$[P(x,y|\theta) \subseteq P(x,y|\theta') \text{ and } R(x,y|\theta) \subseteq R(x,y|\theta') \text{ and } xP_F(\theta)y] \quad \Rightarrow \quad xP_F(\theta')y.$$

This condition strengthens IIA by adding the idea that increased support for some x over some y should preserve a social preference for x over y.

anonymity For all $x, y \in X$ and all $\theta, \theta' \in \Theta$

$$[p(x,y|\theta) = p(x,y|\theta') \text{ and } r(x,y|\theta) = r(x,y|\theta') \text{ and } xP_F(\theta)y] \quad \Rightarrow \quad xP_F(\theta')y.$$

This condition formalizes the idea that no individuals are special: the social preference between x and y depends only on the numbers of individuals who prefer (strictly and weakly) x over y, and not on their identities. Note that it also strengthens IIA.

In all of the above conditions save Pareto, a social preference in one state, θ, may have implications for social preferences in other states. For that reason, they are referred to as "inter-profile" (or in our framework, "interstate") conditions.

All of the above PARs satisfy Pareto. All but F_T, F_C, and F_B satisfy IIA (Can you see why?) All but F_T, F_C, and F_B satisfy neutrality and monotonicity. Most (which?) are anonymous.

We will use three conditions that differ from neutrality, monotonicity, and anonymity in that they are expressed in terms of the decisive coalitions of a PAR. We have already defined an aggregation rule F to be strong if $\mathcal{D}(F)$ is strong. We now define two more conditions.

decisiveness For all $x, y \in X$ and all $\theta \in \Theta$,

$$xP_F(\theta)y \quad \Rightarrow \quad P(x,y|\theta) \in \mathcal{D}(F).$$

In words, a strict social preference is only possible if the members of a decisive coalition all prefer one alternative to another. This is satisfied by F_{SM}, F_{SP}, F_q, F_G, and $F_{w,q}$, for example, but not the other PARs we've seen.

weak decisiveness For all $x, y \in X$ and all $\theta \in \Theta$,

$$xP_F(\theta)y \quad \Rightarrow \quad R(x,y|\theta) \in \mathcal{D}(F).$$

This condition is clearly weaker than decisiveness. Note that, under unrestricted or linear domain, it is implied by the conjunction of neutrality and monotonicity. (Why?) See Austen-Smith and Banks for a large class of PARs, called "voting rules," that satisfy weak decisiveness but not decisiveness.

You can check that, under linear domain, F strong implies F is decisive. Note that, by part 3 of Proposition 2.3, F strong implies F is weakly decisive.

2.4 Simple PARs

A PAR F is *simple* if there exists a proper collection $\mathcal{G}$ of groups such that

$$xP_F(\theta)y \text{ if and only if } \exists G \in \mathcal{G} : xP_G(\theta)y,$$

or equivalently,

$$P_F(\theta) \;=\; \bigcup_{G\in\mathcal{G}} \bigcap_{i\in G} P_i(\theta),$$

for all $\theta \in \Theta$. In words, x is strictly socially preferred to y if and only if all the members of some group in $\mathcal{G}$ prefer x to y. In this case, we denote the PAR by $F_{\mathcal{G}}$, and we call $\mathcal{G}$ a *representation* of the PAR.

Equivalently, by DeMorgan's law, F is simple if there exists a proper $\mathcal{G}$ such that

$$R_F(\theta) \;=\; \bigcap_{G\in\mathcal{G}} \bigcup_{i\in G} R_i(\theta)$$

for all $\theta \in \Theta$. That is, x is weakly socially preferred to y if and only if, in every $G \in \mathcal{G}$, there is some member of G who weakly prefers x to y.

Let's check that $F_{\mathcal{G}}$ is well-defined, i.e., that $P_{\mathcal{G}}(\theta)$ is asymmetric and $R_{\mathcal{G}}(\theta)$ is complete. Suppose $xP_{\mathcal{G}}(\theta)y$, so there exists $G \in \mathcal{G}$ such that, for all $i \in G$, $xP_i(\theta)y$. Suppose $yP_{\mathcal{G}}(\theta)x$ as well, so there exists $G' \in \mathcal{G}$ such that, for all $i \in G'$, $yP_i(\theta)x$. Since $\mathcal{G}$ is proper, there exists $i \in G \cap G'$, but then $xP_i(\theta)y$ and $yP_i(\theta)x$, contradicting asymmetry of $P_i(\theta)$. Thus, $P_{\mathcal{G}}(\theta)$ is asymmetric and completeness of $R_{\mathcal{G}}(\theta)$ follows.

Associated with every PAR F is a related simple PAR $F_{\mathcal{G}}$, defined by $\mathcal{G} = \mathcal{D}(F)$. Thus, $xP_{\mathcal{D}(F)}(\theta)y$ if and only if the members of some group that is decisive for F all prefer x to y. So, for relative majority rule F_{RM}, we have $F_{\mathcal{D}(F_{RM})} = F_{SM}$, and for relative Pareto, we have $F_{\mathcal{D}(F_{RP})} = F_{SP}$. For simplicity, we write $C_{\mathcal{D}(F)}(\theta)$ for the core of $F_{\mathcal{D}(F)}$ at θ. Note that, for all PARs F and all $\theta \in \Theta$, we must have $C_F(\theta) \subseteq C_{\mathcal{D}(F)}(\theta)$. (Right?)

Characterization of Simple PARs

Note that simple PARs satisfy IIA, neutrality, and monotonicity. A simple PAR satisfies Pareto if and only if it has a nonempty representation $\mathcal{G}$. You can check that a simple PAR satisfies anonymity if and only if it is a q-rule for some q.

The same PAR may have several representations. For example, q-rule may be obtained as a special case by defining

$$\mathcal{G} \;=\; \{G \subseteq N \mid |G| \geq q\} \quad \text{or} \quad \mathcal{G} \;=\; \{G \subseteq N \mid |G| = q\}.$$

Every representation $\mathcal{G}$ determines a unique maximal representation, $\mathcal{G}\uparrow$, defined as follows:

$$\mathcal{G}\uparrow \quad = \quad \{G \subseteq N \mid \exists G' \in \mathcal{G} \text{ such that } G' \subseteq G\}.$$

You can check that $F_{\mathcal{G}} = F_{\mathcal{G}\uparrow}$. Also, note that $\mathcal{G}$ is monotonic if and only if $\mathcal{G} = \mathcal{G}\uparrow$.

The PAR q-rule is the simple PAR with $\mathcal{G}\uparrow = \{G \subseteq N \mid |G| \geq q\}$; G-rule is the simple PAR with $\mathcal{G}\uparrow = \{G' \subseteq N \mid G \subseteq G'\}$; and $F_{w,q}$ is the simple PAR with $\mathcal{G}\uparrow = \{G \subseteq N \mid \sum_{i \in G} w_i \geq q\}$.

Similarly, every representation $\mathcal{G}$ determines a unique minimal representation, $\mathcal{G}\downarrow$, defined as follows: $G \in \mathcal{G}\downarrow$ if and only if $G \in \mathcal{G}$ and there is no proper subset $G' \subsetneqq G$ such that $G' \in \mathcal{G}$. Again, $F_{\mathcal{G}} = F_{\mathcal{G}\downarrow}$.

Given a collection $\mathcal{G}$, let

$$\mathcal{G}^* \quad = \quad \{G \subseteq N \mid \forall G' \in \mathcal{G} : G \cap G' \neq \emptyset\}$$

be the collection of all groups having nonempty intersection with all groups in $\mathcal{G}$. Thus, if $N = \{1, 2, 3, 4\}$, then we might have the following.

$\mathcal{G}$	$\mathcal{G}^*$	$\mathcal{G}^{**}$
123	$12, 13, 14$	123
124	$23, 24, 34$	124
134	$123, 124$	134
234	$134, 234$	234
1234	1234	1234

The next lemma shows that the above duality holds quite generally. In fact, it holds whenever $\mathcal{G} = \mathcal{G}\uparrow$.

Lemma 2.2 *Let $\mathcal{G}$ be a collection of groups. Then*

1. $\mathcal{G}^{**} = \mathcal{G}\uparrow$,
2. $\mathcal{G}^* = (\mathcal{G}\uparrow)^* = (\mathcal{G}^*)\uparrow$,
3. *$\mathcal{G}$ is proper if and only if $\mathcal{G}^*$ is strong,*
4. *$\mathcal{G}$ is strong if and only if $\mathcal{G}^*$ is proper,*
5. *if $\{B_i \mid i \in N\}$ is a collection of binary relations on X, then*

$$\bigcap_{G \in \mathcal{G}} \bigcup_{i \in G} B_i = \bigcup_{G \in \mathcal{G}^*} \bigcap_{i \in G} B_i \quad \textit{and} \quad \bigcup_{G \in \mathcal{G}} \bigcap_{i \in G} B_i = \bigcap_{G \in \mathcal{G}^*} \bigcup_{i \in G} B_i.$$

Proof: I'll leave the proof of part 1 to you. Note the consequence that $(\mathcal{G}^{**})^* = (\mathcal{G}\uparrow)^*$. Now apply part 1 to $\mathcal{G}^*$ to get $(\mathcal{G}^*)^{**} = (\mathcal{G}^*)\uparrow$. Therefore,

$$(\mathcal{G}\uparrow)^* \;=\; (\mathcal{G}^*)\uparrow,$$

and it is clear that

$$(\mathcal{G}\uparrow)^* \subseteq \mathcal{G}^* \subseteq (\mathcal{G}^*)\uparrow,$$

which proves part 2. For part 3, suppose $\mathcal{G}$ is proper. If $\mathcal{G}^*$ is not strong, then there exists $G \subseteq N$ such that $G' \in \mathcal{G}^*$ for no $G' \subseteq G$ and no $G' \subseteq N \setminus G$. In particular, $G \notin \mathcal{G}^*$ and $N \setminus G \notin \mathcal{G}^*$. The former implies there exists $G' \in \mathcal{G}$ such that $G' \subseteq N \setminus G$, and the latter implies there exists $G'' \in \mathcal{G}$ such that $G'' \subseteq G$. But then $G' \cap G'' = \emptyset$, a contradiction. Therefore, $\mathcal{G}^*$ is strong. I leave the other direction to you. For part 4, apply part 3 to $\mathcal{G}^*$. Then $\mathcal{G}^*$ is proper if and only if $\mathcal{G}^{**} = \mathcal{G}\uparrow$ is strong, which holds if and only if $\mathcal{G}$ is strong. Finally, I'll prove one direction of the first half of part 5. Take any $(x, y) \in X \times X$, and suppose that, for every $G \in \mathcal{G}$, there is some $i \in G$ such that xB_iy. Letting $G = \{i \in N \mid xB_iy\}$, we have $G \cap G' \neq \emptyset$ for every $G' \in \mathcal{G}$, so $G \in \mathcal{G}^*$, as required. ∎

The following corollary, which gives us alternative formulations of strict and weak social preferences, is immediate.

Corollary 2.2 *Let F be a simple PAR with representation $\mathcal{G}$. Then*

$$P_F(\theta) \;=\; \bigcap_{G \in \mathcal{G}^*} \bigcup_{i \in G} P_i(\theta) \quad \text{and} \quad R_F(\theta) \;=\; \bigcup_{G \in \mathcal{G}^*} \bigcap_{i \in G} R_i(\theta),$$

for all $\theta \in \Theta$.

The connection between decisive and blocking coalitions is quite tight for simple PARs. Recall part 2 of Proposition 2.3, which essentially says

$$\mathcal{D}(F) \subseteq \mathcal{B}(F)^* \text{ and } \mathcal{B}(F) \subseteq \mathcal{D}(F)^*.$$

The next proposition strengthens this conclusion by establishing a duality between decisive and blocking coalitions.

Proposition 2.4 *Let F be a simple PAR with representation $\mathcal{G}$. Then*

1. $\mathcal{D}(F) \;=\; \mathcal{G}\uparrow = \mathcal{B}(F)^*$,
2. $\mathcal{B}(F) \;=\; \mathcal{G}^* \;=\; \mathcal{D}(F)^*$,

Proof: I first consider $\mathcal{D}(F) = \mathcal{G}\uparrow$. One inclusion, that $\mathcal{G}\uparrow \subseteq \mathcal{D}(F)$, is clear. For the opposite inclusion, take $G \subseteq N$ and suppose there is no $G' \in \mathcal{G}$ such that $G' \subseteq G$. Letting $\{x, y\}$ be a free pair, take any state $\theta \in \Theta$ such that $P(x, y|\theta) = G$ and $P(y, x|\theta) = N \setminus G$. Then $yR_F(\theta)x$ (why?), and it follows that $G \notin \mathcal{D}(F)$. Therefore, $\mathcal{D}(F) = \mathcal{G}\uparrow$. I'll let you prove that $\mathcal{B}(F) = \mathcal{G}^*$. Now note that

$$\mathcal{D}(F) = \mathcal{G}\uparrow = \mathcal{G}\uparrow^{**} = (\mathcal{G}\uparrow^*)^* = (\mathcal{G}^*)^* = \mathcal{B}(F)^*,$$

where we use parts 1 and 2 of Lemma 2.2. Finally, $\mathcal{B}(F) = \mathcal{D}(F)^*$ follows from $\mathcal{D}(F) = \mathcal{B}(F)^*$, from part 1 of Lemma 2.2, and from monotonicity of $\mathcal{B}(F)$. ∎

An immediate consequence of Lemma 2.2 and Proposition 2.4 is that $\mathcal{B}(F)$ is strong. Furthermore, the complement of any not-decisive group is blocking, and the complement of any not-blocking group is decisive. Thus, for every G, either $G \in \mathcal{D}(F)$ or $N \setminus G \in \mathcal{B}(F)$. I'll let you prove the former claim, stated in the following corollary.

Corollary 2.3 *Let F be a simple PAR. Then*

1. $\mathcal{B}(F) = \{G \mid N \setminus G \notin \mathcal{D}(F)\}$,
2. $\mathcal{D}(F) = \{G \mid N \setminus G \notin \mathcal{B}(F)\}$.

We can now define strict and weak social preference in terms of decisive and blocking coalitions. Note that, in contrast to the general definition of blocking coalition, for a simple PAR, a common *weak* preference among the members of some blocking group implies a social weak preference. Thus, for simple PARs, blocking groups have considerably more power than in general.

Proposition 2.5 *Let F be a simple PAR. Then*

$$P_F(\theta) = \bigcup_{G \in \mathcal{D}(F)} \bigcap_{i \in G} P_i(\theta) \text{ and } R_F(\theta) = \bigcup_{G \in \mathcal{B}(F)} \bigcap_{i \in G} R_i(\theta),$$

for all $\theta \in \Theta$.

Proof: Let $\mathcal{G}$ be a representation of F. Because $\mathcal{D}(F) = \mathcal{G}\uparrow$, we have

$$P_F(\theta) = \bigcup_{G \in \mathcal{G}\uparrow} \bigcap_{i \in G} P_i(\theta) = \bigcup_{G \in \mathcal{D}(F)} \bigcap_{i \in G} P_i(\theta).$$

And, by Corollary 2.2 and Proposition 2.4, we have

$$R_F(\theta) = \bigcup_{G \in \mathcal{G}^*} \bigcap_{i \in G} R_i(\theta) = \bigcup_{G \in \mathcal{B}(F)} \bigcap_{i \in G} R_i(\theta),$$

as required. ∎

I'll let you prove that a PAR is simple if and only if it satisfies decisiveness. Thus, every simple PAR is weakly decisive.

Continuity of Simple PARs

Though not an issue in finite choice environments, when X is infinite we have seen that existence of maximal elements depends crucially on continuity properties of a preference relation. Thus, assuming continuous domain, we would like to know conditions on PARs that ensure continuity of social preferences.

It is straightforward to verify that, assuming each $R_i(\theta)$ is continuous, the majority weak preference relation $R_{SM}(\theta)$ will also be continuous. The same holds true for simple Pareto. The next proposition shows that all simple preference aggregation rules preserve continuity properties of individual preferences.

Proposition 2.6 *Assume $X \subseteq \mathbb{R}^d$. Let F be a simple PAR, and let $\theta \in \Theta$ be any state.*

1. *If each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous, then $(P_F(\theta), R_F(\theta))$ is upper semicontinuous.*
2. *If each $(P_i(\theta), R_i(\theta))$ is lower semicontinuous, then $(P_F(\theta), R_F(\theta))$ is lower semicontinuous.*
3. *Assume X is connected. If each $(P_i(\theta), R_i(\theta))$ is continuous, then $R_F(\theta)$ is closed and $P_F(\theta)$ is open.*

Proof: I'll prove part 1. Suppose that, for all $i \in N$ and all $x \in X$, $R_i(x|\theta) = \{y \in X \mid yR_i(\theta)x\}$ is closed. By Proposition 2.5, we have

$$R_F(x|\theta) \;=\; \bigcup_{G \in \mathcal{B}(F)} \bigcap_{i \in G} R_i(x|\theta).$$

Since intersections of closed sets are closed, and since finite unions of closed sets are closed, it follows that $R_F(x|\theta)$ is closed. Thus, $R_F(\theta)$ is upper semicontinuous. The proof of part 2 is similar, and part 3 follows because each $R_i(\theta)$ is closed. ∎

It is clear that continuity of social preferences does not carry over to weakly decisive PARs generally, for F_{RM} and F_{RP} can easily generate discontinuities. See Figure 2.4.

Figure 2.4: Discontinuous relative majority preferences

The following corollaries are an immediate consequence of Proposition 2.6 and our results on nonemptiness of choice sets. The first shows that, under weak background assumptions, nonemptiness of the core of a simple PAR follows under the weak transitivity property of Condition F or, therefore, under P-acyclicity.

Corollary 2.4 *Assume $X \subseteq \mathbb{R}^d$ is compact. Let F be a simple PAR, and let $\theta \in \Theta$ be any state such that each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous. If $(P_F(\theta), R_F(\theta))$ satisfies Condition F, then $C_F(\theta) \neq \emptyset$.*

A different continuity issue relates to the properties of the mapping $R_F \colon \Theta \to \mathcal{R}_c$. We have given the space of complete, closed preference relations a notion of convergence, and we can define a notion on Θ in a straightforward manner. Given two states $\theta, \theta' \in \Theta$ with $PR(\theta), PR(\theta') \in \mathcal{R}_c^N$, define the distance between the states as

$$\rho(\theta, \theta') \quad = \quad \sum_{i \in N} \rho(R_i(\theta), R_i(\theta')),$$

where we abuse notation slightly by using ρ for this metric. That is, we simply sum the distance between an individual's preferences in the two states.

A set $\Theta' \subseteq \Theta$ is *open* if, for every $\theta \in \Theta'$, there exists $\epsilon > 0$ such that the open ball

$$B_\epsilon(\theta) \quad = \quad \{\theta' \in \Theta \mid \rho(\theta, \theta') < \epsilon\}$$

is contained in Θ'. A set $\Theta' \subseteq \Theta$ is *closed* if its complement is open.

We say a sequence $\{\theta^m\}$ converges to θ if $\rho(\theta^m, \theta) \to 0$. Equivalently, $\theta^m \to \theta$ if, for all $i \in N$, $\rho(R_i(\theta^m), R_i(\theta)) \to 0$. Clearly, $\Theta' \subseteq \Theta$ is closed if, for every sequence $\{\theta_m\}$ in Θ' converging to some $\theta \in \Theta$, we have $\theta \in \Theta'$.

Assuming $PR(\Theta) \subseteq \mathcal{R}_c^N$ and $R_F(\Theta) \subseteq \mathcal{R}_c$, we say the mapping R_F is *continuous at* θ if, for every sequence $\{\theta^m\}$ such that $\theta^m \to \theta$, we have $R_F(\theta^m) \to R_F(\theta)$. The next proposition establishes fairly general conditions under which continuity holds for all simple PARs.

Proposition 2.7 *Assume $X \subseteq \mathbb{R}^d$ is compact, $PR(\Theta) \subseteq \mathcal{R}_c^N$, and $R_F(\Theta) \subseteq \mathcal{R}_c$. Let F be a simple PAR, and let $\theta \in \Theta$ be such that $PR(\theta)$ satisfies LSWP. Then $R_F \colon \Theta \to \mathcal{R}_c$ is continuous at θ.*

Proof: Let $\theta^m \to \theta$. For each m, let $R^m = R_F(\theta^m)$, and let $R = R_F(\theta)$. To prove that $\{R^m\}$ converges to R, first take any sequence $\{(x^m, y^m)\}$ converging to (x, y) such that $x^m R^m y^m$ for all m. For each m, there exists $G^m \in \mathcal{B}(F)$ such that $x^m R_i(\theta^m) y^m$ for all $i \in G^m$. Because N is finite, there exists $G \in \mathcal{G}$ such that $G = G^m$ for infinitely many

m, and then $xR_i(\theta)y$ for all $i \in G$ follows from Lemma 1.3. Thus, xRy. Now take any $x, y \in X$ with xRy, so there exists $G \in \mathcal{B}(F)$ such that $xR_i(\theta)y$ for all $i \in G$. If $x = y$, then, by reflexivity, $xR^m y$ for all m. Suppose $x \neq y$. Then, by LSWP, for each k there exists $x_k \in P_G(y|\theta)$ such that $||x_k - x|| < 1/k$. (Why?) Furthermore, there exists $m_k > k$ such that $x_k \in P_G(y|\theta^{m_k})$, for otherwise we would have $yR_i(\theta^m)x_k$ for some $i \in G$ and infinitely many m, implying $yR_i(\theta)x_k$ (why?), a contradiction. Thus, we have a sequence $\{(x_k, y)\}$ and a subsequence $\{R^{m_k}\}$ such that $(x_k, y) \to (x, y)$ and $x_k R^{m_k} y$ for all k. Therefore, $R^m \to R$, by Lemma 1.4. ∎

Transitivity of Simple PARs

Another property important in establishing the existence of maximal elements is transitivity. The next results set the stage for our general analysis of collective rationality, showing that, when the domain of individual preferences is large, transitivity of strict and weak social preference corresponding to a simple PAR is very restrictive.

Proposition 2.8 *Assume $|X| \geq 3$ and $PR(\Theta) \supseteq \mathbf{L}$. Let F be a simple PAR. Then F satisfies Pareto and $P_F(\theta)$ is transitive for all $\theta \in \Theta$ if and only if $F = F_G$ for some nonempty $G \subseteq N$.*

Proof: I will prove one direction. Assume F satisfies Pareto and $P_F(\theta)$ is transitive for all $\theta \in \Theta$. Let $\mathcal{G}$ be a representation of F, nonempty by Pareto, and let G be a minimal group in $\mathcal{G}$. Suppose there exists $G' \in \mathcal{G}$ such that $G \setminus G' \neq \emptyset$. Take distinct alternatives x, y, and z and any state $\theta \in \Theta$ with preferences over these alternatives as follows.

$R_{G\cap G'}(\theta)$	$R_{G\setminus G'}(\theta)$	$R_{N\setminus G}(\theta)$
x	y	z
y	z	x
z	x	y

Because $xP_{G'}(\theta)y$ and $yP_G(\theta)z$, we have $xP_F(\theta)yP_F(\theta)z$. Note, however, that $P(x, z|\theta) \subseteq G \cap G' \subsetneq G$. By minimality of G, therefore, $P(x, z|\theta) \notin \mathcal{G}$, so not $xP_F(\theta)z$, contradicting transitivity of $P_F(\theta)$. Therefore, $G \subseteq G'$ for all $G' \in \mathcal{G}$, so $F = F_G$. That $G \neq \emptyset$ follows because $\mathcal{G}$ is proper. ∎

Examples of PARs satisfying the assumptions of Proposition **??** are F_{SP} and F_{RP}, in which case $G = N$. Note, as illustrated in Figure 2.2, that these PARs do not always generate transitive weak social preference. As the next proposition shows, adding that condition yields the existence of a "dictator."

Proposition 2.9 *Assume $|X| \geq 3$ and $PR(\Theta) \supseteq \mathbf{L}$. Let F be a simple PAR. Then F satisfies Pareto and $R_F(\theta)$ is transitive for all $\theta \in \Theta$ if and only if $F = F_i$ for some $i \in N$.*

Proof: I will prove one direction. Assume F satisfies Pareto and $R_F(\theta)$ is transitive for all $\theta \in \Theta$. Let $F = F_G$ for some nonempty $G \subseteq N$, and suppose that $i, j \in G$ for distinct i and j. Take distinct alternatives x, y, and z, and take any state $\theta \in \Theta$ with preferences over these alternatives as follows.

$R_i(\theta)$	$R_j(\theta)$	$R_{N\setminus\{i,j\}}(\theta)$
y	z	z
z	x	x
x	y	y

Because $xP_j(\theta)y$ and $yP_i(\theta)z$, we have $xR_F(\theta)yR_F(\theta)z$. But, by Pareto, $zP_F(\theta)x$, contradicting transitivity of $R_F(\theta)$. Thus, $|G| = 1$. ■

You should think about how the conclusions of Propositions 2.8 and 2.9 would change if Pareto were dropped.

Acyclicity of strict social preference is even more fundamental to nonempty social choice sets, but we will wait to take up that issue.

2.5 The Discrete Choice Model

In this section, I will examine the issue of collective rationality when individual preferences are unrestricted, as in the discrete choice model. In following sections I will extend the analysis to other collective choice environments.

We have seen the importance of continuity, transitivity, and convexity conditions in guaranteeing nonempty maximal sets. In this section, I will consider the following transitivity conditions on a PAR F.

***R*-transitivity** For all $\theta \in \Theta$, $R_F(\theta)$ is transitive.

***P*-transitivity** For all $\theta \in \Theta$, $P_F(\theta)$ is transitive.

***P*-acyclicity** For all $\theta \in \Theta$, $P_F(\theta)$ is acyclic.

Generally, F_{SP}, F_{RP}, F_G, and F_C are P-transitive. And F_T and F_B are R-transitive.

Thus, for all $\theta \in \Theta$, $R_{SP}(\theta)$, $R_{RP}(\theta)$, $R_G(\theta)$, and $R_C(\theta)$ are weak quasi-orders, while $R_T(\theta)$ and $R_B(\theta)$ are weak orders. Corresponding strict preferences are strict quasi-orders, etc.

We have seen that simple rules satisfy P-transitivity and R-transitivity only under very restrictive conditions. In fact, $R_{SM}(\theta)$, $R_{RM}(\theta)$, $R_q(\theta)$, and $R_{\mathcal{G}}$ don't even have to be weak suborders. Consider the simple example below, named "Condorcet's Paradox" after the Marquis de Condorcet, a scientist and contemporary of Borda's who is generally credited with initiating the formal analysis of voting.

Condorcet's Paradox. Let $n = 3$, $X = \{x, y, z\}$, and consider a state with individual preferences as follows.

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$
x	y	z
y	z	x
z	x	y

Clearly, these individual preferences generate a majority-rule strict preference cycle,

$$xP_{SM}(\theta)yP_{SM}(\theta)zP_{SM}(\theta)x,$$

and, as a consequence, the majority core is empty.

The idea of the paradox can be extended to public good environments. In Figure 2.5, for example, there are three voters with Euclidean preferences and a majority preference cycle through x, y, and z. In fact, you can check that the presence of such cycles is critical in this example, because the majority core is empty.

Figure 2.5: Majority preference cycle

There is a classical theorem, due to McGarvey, to the effect that, when X is finite, strict majority preferences can be arbitrarily ugly, subject to the constraint of asymmetry. In other words, majority preference aggregation does not entail any restrictions on social preferences other than asymmetry.

Proposition 2.10 (McGarvey) *Given a finite set X and an arbitrary asymmetric relation P on X, there exists a set N of individuals and a profile $((P_1, R_1), \ldots, (P_n, R_n)) \in \mathbf{L}$ of preferences, say at state θ, such that $P_{SM}(\theta) = P$.*

I'll let you think about the proof. Note that, in McGarvey's Theorem, the set of individuals is taken as a variable. That means different P's may require different numbers of individuals. The theorem is stated only for majority rule. I do not think it has been generalized to arbitrary PARs. (Indeed, it would not be true for every PAR, so the question is: for which PARs does a McGarvey result hold?) The assumption of finiteness is important in the theorem – there is presently no such characterization of majority preference on infinite sets.

The next proposition shows that the logic of Condorcet's Paradox is actually quite general. It differs from McGarvey's Theorem in that it establishes the possibility of cycles for most every size of the set of individuals. Moreover, the cycle established in the proposition is a nasty one that runs through the entire set of alternatives. In particular, the majority core is empty.

Proposition 2.11 *Assume that $n \geq 3$, that $|X| \geq 3$, that $n = 4 \Rightarrow |X| \geq 4$, and that* $\mathbf{R}(\Theta) \supseteq \mathbf{L}$. *There exists $\theta \in \Theta$ such that $P_{SM}(\theta) = P_{RM}(\theta)$ violates acyclicity. Furthermore, the cycle is exhaustive: for all $x, y \in X$, there exist $k, l \leq 4$ and $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l \in X$ such that*

$$xP_{SM}(\theta)x_1P_{SM}(\theta)x_2P_{SM}(\theta)\cdots x_k = yP_{SM}(\theta)y_1P_{SM}(\theta)\cdots y_l = x.$$

Proof: I'll prove the result for the $n \neq 4$ case, leaving the $n = 4$ case to you. Divide N into disjoint groups G_1, G_2, and G_3 such that, for all $j \in \{1, 2, 3\}$, $|N \backslash G_j| > n/2$. (We can do this because $n = 3$ or $n \geq 5$.) Let w and z be distinct alternatives. By assumption, there is some state $\theta \in \Theta$ with preferences as follows.

$R_{G_1}(\theta)$	$R_{G_2}(\theta)$	$R_{G_3}(\theta)$
w	z	$X\backslash\{w, z\}$
z	$X\backslash\{w, z\}$	w
$X\backslash\{w, z\}$	w	z

In the above profile, $X \setminus \{w, z\}$ indicates where the alternatives other than w and z are ranked. Any weak linear order within that subset will do. You can check that the claim of the proposition holds at this state. ∎

Transitivity of Weak Social Preference

Though the weaker conditions are also of interest, I turn first to the possibility of transitivity of social weak preference. The reasons for this are threefold: first, we know that transitivity is sufficient for the existence of maximal elements in finite sets (and, with compactness and continuity, in infinite sets); second, there is some philosophical interest in the possibility that groups of people, when acting collectively, might exhibit the same rationality that individuals do (which would lend some credence to the idea of a "group consciousness"); third, this is historically the first and most important approach to the problem.

Our first lemma derives an implication of R-transitivity without the use of any ancillary conditions on the PAR F.

Lemma 2.3 *Assume that $|X| \geq 3$, that $PR(\Theta) \supseteq$* $\mathbf{L}$, *and that F satisfies R-transitivity. If G and G' are semi-blocking, then, for all distinct $x, y \in X$, $N \backslash (G \cap G')$ is not semi-decisive for x over y.*

Proof: Suppose G and G' are semi-blocking, and take any two alternatives $x, y \in X$. Let z be any distinct alternative, and let $\theta \in \Theta$ be a state with individual preferences over these alternatives as follows.

$R_{G\setminus G'}(\theta)$	$R_{G\cap G'}(\theta)$	$R_{G'\setminus G}(\theta)$	$R_{N\setminus(G\cup G')}$
x	y	z	x
y	z	x	z
z	x	y	y

Since G is semi-blocking, we have $yR_F(\theta)z$. Since G' is semi-blocking, we have $zR_F(\theta)x$. By R-transitivity, $yR_F(\theta)x$. Thus, $N \setminus (G \cap G')$ is not semi-decisive for x over y. ∎

Adding the assumption that F satisfies IIA, then the conclusion of Lemma 2.3 can be strengthened to: $G \cap G'$ is semi-blocking. (Why?)

The next lemma uses weak Pareto and IIA to show that, if a coalition is semi-blocking in one instance (i.e., for one alternative over another), then it is semi-blocking generally.

Lemma 2.4 *Assume that $|X| \geq 3$, that $PR(\Theta) \supseteq \mathbf{L}$, and that F satisfies weak Pareto, IIA, and R-transitivity. If G is semi-blocking for some x over some $y \neq x$, then it is semi-blocking.*

Proof: Let G be semi-blocking for some x over some $y \neq x$. Let z be any distinct alternative, and consider a state $\theta \in \Theta$ with preferences over x, y, and z as follows.

$R_G(\theta)$	$R_{N\setminus G}(\theta)$
x	y
y	z
z	x

Since G is semi-blocking, we have $xR_F(\theta)y$. By weak Pareto, we have $yR_F(\theta)z$. By R-transitivity, $xR_F(\theta)z$. By IIA, G is semi-blocking for x over z. Since z was arbitrary, this shows that G is semi-blocking for x over every $z \neq x$. A similar argument shows that G is semi-blocking for every $w \neq y$ over y. (You should try to prove this step.) Of course, x and y could be any alternatives such that G is semi-blocking for one over the other in some state, so this argument can be repeated for any such pair of distinct alternatives. Now take *any two* alternatives $s, t \in X$. Assume without loss of generality that $x \neq s$, and take any w distinct from x and s. Then G is semi-blocking for x over w. Repeating the above argument for x and w, G is semi-blocking for s over w. If $w = t$, then we are done. If not, then, again by the above argument, G is semi-blocking for s over t. Therefore, G is semi-blocking. ∎

Our main result on R-transitive PARs, next, combines the above observations to deduce the existence of an individual with veto power.

weak dictatorship There exists $i \in N$ such that, for all $\theta \in \Theta$, $P_i(\theta) \subseteq R_F(\theta)$.

Equivalently, there is some $i \in N$ such that, for all $\theta \in \Theta$, $P_F(\theta) \subseteq R_i(\theta)$. The condition means that some singleton group is blocking, so there is some individual whose strict preference prohibits a strict social preference in the opposite direction. We call this individual a "weak dictator," or a "vetoer," or a "blocker." Note that there may be several weak dictators. If a PAR has a weak dictator, we say it is *weakly dictatorial.*

Proposition 2.12 *Assume* $|X| \geq 3$ *and either* $PR(\Theta) = \mathbf{U}$ *or* $PR(\Theta) = \mathbf{L}$. *If* F *satisfies weak Pareto, IIA, and R-transitivity, then* F *is weakly dictatorial.*

Proof: By weak Pareto, N is semi-blocking. Let G be a minimal semi-blocking group. If $|G| > 1$, then take any $i \in G$. Taking distinct $x, y \in X$, a free pair by linear domain, Lemma 2.1 implies that either $G \setminus \{i\}$ is semi-blocking for x over y or $(N \setminus G) \cup \{i\}$ is semi-blocking for y over x. If $G \setminus \{i\}$ is semi-blocking for x over y, then, by Lemma 2.4, it is semi-blocking, contradicting minimality of G. Likewise, if $(N \setminus G) \cup \{i\}$ is semi-blocking for y over x, then it is semi-blocking. But then, by Lemma 2.3, it follows that $\{i\} = G \cap [(N \setminus G) \cup \{i\}]$ is semi-blocking, again contradicting minimality of G. Therefore, $|G| = 0$ or $|G| = 1$. In the first case, $R_F(\theta) = X \times X$ for all $\theta \in \Theta$ (why?), and every individual is a weak dictator. Consider the second case, where $G = \{i\}$ for some $i \in N$. Take distinct $x, y, z \in X$ and a state $\theta \in \Theta$ with individual preferences over these alternatives as follows.

$R_i(\theta)$	$R_{N\setminus\{i\}}(\theta)$
x	z
z	$[x\ y]$
y	

(Here "$[x\ y]$" indicates that individual preferences between x and y are arbitrary in this state.) Since $\{i\}$ is semi-blocking, we have $xR_F(\theta)z$. By weak Pareto, we have $zR_F(\theta)y$. By R-transitivity, $xR_F(\theta)y$. Because x and y are arbitrary here, IIA implies that $\{i\}$ is blocking, as desired. ∎

As consequence of Proposition 2.12, merely strengthening weak Pareto to Pareto, we deduce a striking conclusion.

dictatorship There exists $i \in N$ such that, for all $\theta \in \Theta$, $P_i(\theta) \subseteq P_F(\theta)$.

The condition means that some singleton group is decisive. Thus, power is concentrated in a single individual, in the sense that all strict preferences of the individual are imposed on society. We call this individual a *dictator* for F. If a PAR has a dictator, we say it is *dictatorial.*

Proposition 2.13 (Arrow) *Assume $|X| \geq 3$ and either $PR(\Theta) = \mathbf{U}$ or $PR(\Theta) = \mathbf{L}$. If F satisfies Pareto, IIA, and R-transitivity, then F is dictatorial.*

Proof: Let i be a weak dictator. Take any distinct alternatives $x, y, z \in X$ and state $\theta \in \Theta$ with individual preferences over these alternatives as follows.

$R_i(\theta)$	$R_{N\setminus\{i\}}(\theta)$
x	z
z	$[x\ y]$
y	

Since i is a weak dictator, we have $xR_F(\theta)z$. By Pareto, we have $zP_F(\theta)y$. By R-transitivity, $xP_F(\theta)y$. By IIA, i is a dictator. ∎

You should consider the robustness of this "impossibility theorem" to relaxations of various assumptions. If we relax R-transitivity to P-transitivity, for example, the Pareto rules, F_{SP} and F_{RP}, satisfy Arrow's assumptions. What if $|X| = 2$? What if we drop Pareto? What if we drop IIA?

As Proposition 2.9 shows, the converse of Arrow's theorem does hold for simple PARs. It also holds when $PR(\Theta) = \mathbf{L}$. The converse does not hold when $PR(\Theta) = \mathbf{U}$, as you should check: if F is dictatorial, it does not necessarily satisfy IIA or R-transitivity. (Though dictatorship does imply Pareto.)

Can you see where the proof of Proposition 2.13 uses $PR(\Theta) = \mathbf{U}$ or $PR(\Theta) = \mathbf{L}$ instead of merely $PR(\Theta) \supseteq \mathbf{L}$? Does the conclusion of the proposition hold if we assume merely the latter inclusion? What if we assume $PR(\Theta) = \mathbf{U}$ instead?

Transitivity of Strict Social Preference

As a consequence of Arrow's theorem, assuming IIA and the minimal condition of Pareto, we cannot expect collective preferences to exhibit the same regularity properties that we expect of individual preferences — except in groups with highly asymmetric power structures.

Weakening R-transitivity to P-transitivity, we know that there do exist nondictatorial PARs satisfying the remainder of Arrow's conditions: F_{SP} and F_{RP}, for example. In fact, given any G with $1 < |G| \leq n$, F_G satisfies Pareto, IIA, and P-transitivity and is nondictatorial. We will see, however, that this gives us a nearly complete characterization of such PARs!

Lemma 2.5 *Assume that $|X| \geq 3$, that $PR(\Theta) \supseteq \mathbf{L}$, and that F satisfies P-transitivity. If G and G' are semi-decisive, then, for all distinct $x, y \in X$, $N \setminus (G \cap G')$ is not semi-blocking for x over y.*

Proof: Suppose G and G' are semi-decisive, and take any two alternatives $x, y \in X$. Let z be any distinct alternative, and let $\theta \in \Theta$ be a state with individual preferences over these alternatives as follows.

$R_{G \setminus G'}(\theta)$	$R_{G \cap G'}(\theta)$	$R_{G' \setminus G}(\theta)$	$R_{N \setminus (G \cup G')}$
x	y	z	x
y	z	x	z
z	x	y	y

Since G is semi-decisive, we have $yP_F(\theta)z$. Since G' is semi-decisive, we have $zP_F(\theta)x$. By P-transitivity, $yP_F(\theta)x$. Thus, $N \setminus (G \cap G')$ is not semi-blocking for x over y. ∎

Adding the assumption that F satisfies IIA, then the conclusion of Lemma 2.5 can be strengthened to: $G \cap G'$ is semi-decisive. (Why?)

Lemma 2.6 *Assume that $|X| \geq 3$, that $PR(\Theta) \supseteq \mathbf{L}$, and that F satisfies Pareto, IIA, and P-transitivity. If G is semi-decisive for some x over some $y \neq x$, then it is semi-decisive.*

Proof: Let G be semi-decisive for some x over some $y \neq x$. Let z be any distinct alternative, and consider a state $\theta \in \Theta$ with preferences over x, y, and z as follows.

$R_G(\theta)$	$R_{N \setminus G}(\theta)$
x	y
y	z
z	x

Since G is semi-decisive, we have $xP_F(\theta)y$. By Pareto, we have $yP_F(\theta)z$. By P-transitivity, $xP_F(\theta)z$. By IIA, G is semi-decisive for x over z. Since z was arbitrary, this shows that G is semi-decisive for x over every $z \neq x$. A similar argument shows that G is semi-decisive for every $w \neq y$ over y. (You should try to prove this step.) Of course, x and y could be any alternatives such that G is semi-decisive for one over the other in some state, so this argument can be repeated for any such pair of distinct alternatives. Now take *any two* alternatives $s, t \in X$. Assume without loss of generality that $x \neq s$, and take any w distinct from x and s. Then G is semi-decisive for x over w. Repeating the above argument for x and w, G is semi-decisive for s over w. If $w = t$, then we are done. If not, then, again by the above argument, G is semi-decisive for s over t. Therefore, G is semi-decisive. ∎

We need one more condition before we can state the main result on P-transitive social preferences.

oligarchy There is some $G \subseteq N$ such that, for all $\theta \in \Theta$,

$$\bigcap_{i \in G} P_i(\theta) \subseteq P_F(\theta) \subseteq \bigcap_{i \in G} R_i(\theta)$$

Note that the second inclusion is equivalent to: for all $i \in G$, $P_F(\theta) \subseteq R_i(\theta)$. Thus, each $i \in G$ is a weak dictator, or equivalently, $\{i\}$ is a blocking coalition for each $i \in G$.

In words, the oligarchy condition means that power is concentrated in the members of a group, G, in the sense that the members of G can impose a common strict preference on society; moreover, each member of G has individual power, in the form of a veto over strict social preferences, i.e., if $i \in G$ strictly prefers x to y, then society cannot have the opposite preference. We say G is an *oligarchy,* and we refer to a member of G as an *oligarch.* If F has an oligarchy, we call it *oligarchical.*

Clearly, if i is a dictator, then $\{i\}$ is an oligarchy, and vice versa.

Proposition 2.14 (Gibbard) *Assume $|X| \geq 3$ and either $PR(\Theta) = \mathbf{U}$ or $PR(\Theta) = \mathbf{L}$. If F satisfies Pareto, IIA, and P-transitivity, then F is oligarchical.*

Proof: By Pareto, N is semi-decisive. Let G be a minimal semi-decisive group, and note that G is nonempty. (Why?) To see that G is decisive, take distinct $x, y, z \in X$ and a state $\theta \in \Theta$ with individual preferences over these alternatives as follows.

$R_G(\theta)$	$R_{N\setminus G}(\theta)$
x	z
z	$[x\ y]$
y	

(Here, $[x\ y]$ indicates that individual preferences between x and y are arbitrary.) Because G is semi-decisive, we have $xP_F(\theta)z$. By Pareto, we have $zP_F(\theta)y$. By P-transitivity, we have $xP_F(\theta)y$. Because x and y are arbitrary here, IIA implies that G is decisive. Thus, $\bigcap_{i\in G} P_i(\theta) \subseteq P_F(\theta)$ for all θ. I claim that $P_F(\theta) \subseteq \bigcap_{i\in G} R_i(\theta)$ for all θ as well. Take any distinct $x, y \in X$, a free pair by linear domain, and any $i \in G$. If $\{i\}$ is not semi-blocking for x over y, then, by Lemma 2.1, $N \setminus \{i\}$ is semi-decisive for y over x. By Lemma 2.6, $N \setminus \{i\}$ is semi-decisive. And by Lemma 2.5, it follows that $G \setminus \{i\} = G \cap [N \setminus \{i\}]$ is semi-decisive, contradicting minimality of G. Therefore, $\{i\}$ is semi-blocking. To see that $\{i\}$ is blocking, take distinct $x, y, z \in X$ and a state $\theta \in \Theta$ with individual preferences over these alternatives as follows.

$R_i(\theta)$	$R_{N\setminus\{i\}}(\theta)$
x	z
z	$[x\ y]$
y	

By Pareto, we have $zP_F(\theta)y$. If $yP_F(\theta)x$, then, by P-transitivity, we have $zP_F(\theta)x$. But, because $\{i\}$ is semi-blocking, we have $xR_F(\theta)z$, a contradiction. Therefore, $xR_F(\theta)y$. Because x and y are arbitrary, IIA implies that $\{i\}$ is blocking, as desired. ■

Since F_{SP} and F_{RP} satisfy all of Gibbard's conditions, they are oligarchical. Can you see which G is the oligarchy?

You should check the robustness of Gibbard's theorem to relaxations of the assumptions. What if $|X| = 2$? What if we drop Pareto or IIA? You can check that, again, the result does not hold if we assume merely $PR(\Theta) \supseteq \mathbf{L}$.

Note that it is not generally true that oligarchical PARs satisfy IIA or P-transitivity. (Why?)

Acyclicity of Strict Social Preference

What if we drop any of Gibbard's conditions? I'll let you think about Pareto and IIA. Suppose we weaken P-transitivity to P-acyclicity. Are there non-oligarchical PARs that satisfy Pareto, IIA, and this condition? The next example shows there are.

The following PAR, call it F_{SC}, is based on the voting rule of the pre-1965 UN Security Council. Let $n = 11$, with the first five individuals corresponding to the permanent members of the Security Council. Define

$$xP_{SC}(\theta)y \;\Leftrightarrow\; [\{1,2,3,4,5\} \subseteq R(x,y|\theta) \text{ and } p(x,y|\theta) \geq 7].$$

This clearly satisfies Pareto and IIA. Is it oligarchical? If a group were an oligarchy, it would necessary include $\{1,2,3,4,5\}$, because each permanent member can veto strict social preference. But $\{1,2,3,4,5\}$ is not decisive: an oligarchy would have to contain some $i > 5$. But no such i has a veto, so there is no oligarchy. Note that F_{SC} is P-acyclic. (Can you see why?)

Because F_{SC} is not oligarchical yet satisfies Pareto and IIA, it must not satisfy P-transitivity generally. I'll let you construct an example in which strict social preference for this PAR violates transitivity.

While this does give us an example of a non-oligarchical PAR satisfying Pareto, IIA, and P-acyclicity, it is apparent that F_{SC} does concentrate some power in a relatively small group. We will see this observation formalized and generalized shortly.

We first need another condition on PARs, one that is satisfied in most any democratic societies, though not by F_{SC}.

virtual unanimity For all $x, y \in X$ and all $\theta \in \Theta$, $p(x,y|\theta) \geq n-1$ implies $xP_F(\theta)y$.

In words, if all individuals or all but one prefer x to y strictly, then so does society. Equivalently, $G \in \mathcal{D}(F)$ for every group G with $|G| \geq n-1$. Note that virtual unanimity implies Pareto.

The next proposition, proved by Tom Schwartz, indicates that strict social preference cycles can be a deep problem, even if we do not assume IIA. It assumes, however, a large enough number of alternatives. (Compare this result to Austen-Smith and Banks's Lemma 2.3.)

Proposition 2.15 (Schwartz) *Assume $|X| \geq n$ and $PR(\Theta) \supseteq \mathbf{L}$. If F satisfies virtual unanimity, then it violates P-acyclicity.*

Proof: Since $|X| \geq n$, we can take n distinct alternatives and label them $x_1, x_2, \ldots, x_n$. Take $\theta \in \Theta$ such that individual preferences restricted to $\{x_1, x_2, \ldots, x_n\}$ are as follows.

$R_1(\theta)$	$R_2(\theta)$	$\cdots$	$R_i(\theta)$	$\cdots$	$R_n(\theta)$
x_2	x_3		x_{i+1}		x_1
x_3	x_4		x_{i+2}		x_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_n	x_1		x_{i-1}		x_{n-1}
x_1	x_2		x_i		x_n.

(This construction is called a "Latin Square".) Note that $G \setminus \{i\} = P(x_i, x_{i+1}|\theta)$ (letting $x_{n+1} = x_1$), for all i, so $p(x_i, x_{i+1}|\theta) = n - 1$. By virtual unanimity, we have

$$x_1 P_F(\theta) x_2 P_F(\theta) x_3 \cdots P_F(\theta) x_n P_F(\theta) x_1,$$

contradicting P-acyclicity. ∎

Note that Borda rule never produces strict social preference cycles, so it cannot satisfy virtual unanimity when $|X| \geq n$. I'll let you verify this. Next is another condition on PARs.

collegiality $\bigcap \mathcal{D}(F) \neq \emptyset$.

Roughly, this means there is some individual who is in every decisive coalition. If there is such an individual, we say F is *collegial*, and we call $\bigcap \mathcal{D}(F)$ the *collegium* of F.

Note, however, the logical convention that the intersection of the *empty* family of coalitions is N itself: let $\mathcal{D}(F) = \emptyset$; take any $i \in N$, and note that, for all $G \in \mathcal{D}(F)$, $i \in G$; therefore, $i \in \bigcap \mathcal{D}(F)$. Of course, the universal generalization here is vacuously satisfied. Technically, then, collegiality means that either there are no decisive coalitions or there are, and some individual is in all of them. The minimal condition of Pareto is equivalent to $\mathcal{D}(F) \neq \emptyset$, in which case collegiality reduces to the more intuitive idea.

Example: F_{SC} is collegial, and the collegium is the permanent members.

How does a collegium compare to an oligarchy?

1. If G is an oligarchy, then it is a collegium.

2. G may be a collegium yet give no member a veto: F_B when $|X| \geq n$, because then $\mathcal{D}(F_B) = \{N\}$.

3. G may be a collegium yet not be decisive: F_{SC}.

4. G may be a collegium yet give no member a veto *and* not be decisive: let $N = \{1, 2, 3, 4\}$, pick two distinct alternatives a and b, and define F by

$$xP_F(\theta)y \quad \Leftrightarrow \quad \begin{cases} xP_1(\theta) \cap P_2(\theta) \cap (P_3(\theta) \cup P_4(\theta))y \\ \text{or } x = aP_1(\theta) \cap P_3(\theta) \cap P_4(\theta)b = y \\ \text{or } x = bP_2(\theta) \cap P_3(\theta) \cap P_4(\theta)a = y. \end{cases}$$

   Here, $\mathcal{D}(F) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$, so $\bigcap \mathcal{D}(F) = \{1, 2\}$. This group is not decisive. Moreover, assuming linear domain, neither 1 nor 2 has a veto. Also, F is P-acyclic and satisfies IIA.

Note that the examples of F_B and F, in (4) above, violate neutrality. As the next proposition shows, if a collegial PAR satisfies weak decisiveness, then, indeed, every member of the collegium will have a veto.

Proposition 2.16 *Assume F satisfies weak decisiveness. If $i \in \bigcap \mathcal{D}(F)$, then, for all $\theta \in \Theta$, $P_i(\theta) \subseteq R_F(\theta)$.*

Proof: Take any $i \in \bigcap \mathcal{D}(F)$ and any $\theta \in \Theta$ such that $xP_i(\theta)y$. If $yP_F(\theta)x$, then, by weak decisiveness, $R(y, x|\theta) \in \mathcal{D}(F)$, contradicting $i \in \bigcap \mathcal{D}(F)$. Therefore, $xR_F(\theta)y$. ∎

The next result, due to Don Brown, gives us a restriction implied by P-acyclicity and is in the same line of results by Arrow and Gibbard. It is quite different, however, in that it does not impose IIA but does impose a restriction on the number of alternatives.

Proposition 2.17 (Brown) *Assume $|X| \geq n$ and $PR(\Theta) \supseteq \mathbf{L}$. If F is P-acyclic, then it is collegial.*

Brown's result is actually equivalent to Proposition 2.15 and follows from the earlier result by observing that F is not collegial if and only if it satisfies virtual unanimity.

Combining Propositions 2.16 and 2.17, we have the following result: If $|X| \geq n$, if all profiles of weak linear orders are possible, and if F satisfies Pareto, monotonicity, neutrality, and P-acyclicity, then some individual has a veto. (This is essentially Austen-Smith and Banks's Theorem 2.5.)

Because these results assume $|X| \geq n$, they give us only a limited characterization of P-acyclic PAR's.

Given a PAR F, define the *Nakamura number* of F, denoted $\mathcal{N}(F)$, as

$$\mathcal{N}(F) \quad = \quad \min\left\{|\mathcal{G}| \mid \mathcal{G} \subseteq \mathcal{D}(F) \text{ and } \bigcap \mathcal{G} = \emptyset\right\}$$

if $\bigcap \mathcal{D}(F) = \emptyset$ (i.e., F is non-collegial); otherwise (i.e., F is collegial), set $\mathcal{N}(F) = |2^X| > |X|$. (We can actually do this even when X is infinite, in which case $|\cdot|$ denotes the "cardinality" of a set.)

In words, $\mathcal{N}(F)$ is the size of the smallest collection of decisive groups having empty intersection.

Proposition 2.18 *For every PAR F, $\mathcal{N}(F) \geq 3$. If F is non-collegial, then $\mathcal{N}(F) \leq n$.*

Proof: If F is collegial, then $\mathcal{N}(F) = |2^X| \geq 3$, since $|X| \geq 2$. Otherwise, take any $G, G' \in \mathcal{D}(F)$. Since $\mathcal{D}(F)$ is proper, $G \cap G' \neq \emptyset$. Therefore, $\mathcal{N}(F) \geq 3$. If F is non-collegial, then, for each $i \in N$, there exists $G^i \in \mathcal{D}(F)$ with $i \notin G^i$. So $\bigcap\{G^i \mid i \in N\} = \emptyset$. Therefore, $\mathcal{N}(F) \leq |\{G^i \mid i \in N\}| \leq n$. ∎

The next result, due to Nakamura, shows us how the concept of Nakamura number figures in a more general analysis of P-acyclicity. Note that it does not use any restriction on the number of alternatives.

Proposition 2.19 (Nakamura) *Assume $PR(\Theta) \supseteq \mathbf{L}$. If F is P-acyclic, then $\mathcal{N}(F) > |X|$.*

Proof: Suppose $|X| \geq \mathcal{N}(F)$, which implies that F is non-collegial. Let $m = \mathcal{N}(F)$, and let $\{G^1, G^2, \ldots, G^m\} \subseteq \mathcal{D}(F)$ satisfy $\bigcap_{j=1}^m G^j = \emptyset$. Define

$$\begin{aligned}
H^1 &= N \backslash G^1 \\
H^2 &= G^1 \backslash G^2 \\
H^3 &= (G^1 \cap G^2) \backslash G^3 \\
H^4 &= (G^1 \cap G^2 \cap G^3) \backslash G^4 \\
&\vdots \\
H^m &= (G^1 \cap G^2 \cap \cdots \cap G^{m-1}) \backslash G^m.
\end{aligned}$$

Note that H^j may be empty. See Figure 2.6 for an example of this construction.

These sets satisfies the following three properties:

Figure 2.6: The construction of the H^j's

1. $N = \bigcup_{j=1}^{m} H^j$ (i is in first H^j such that $i \notin G^j$)
2. $H^j \cap H^k = \emptyset$ for all distinct j and k
3. $G^j \subseteq N \backslash H^j$.

Since $|X| \geq \mathcal{N}(F) = m$, we may take distinct alternatives $x_1, x_2, \ldots, x_m$ and a state $\theta \in \Theta$ such that individual preferences restricted to $\{x_1, x_2, \ldots, x_m\}$ are as follows.

$R_{H^1}(\theta)$	$R_{H^2}(\theta)$	$\cdots$	$R_{H^j}(\theta)$	$\cdots$	$R_{H^m}(\theta)$
x_2	x_3		x_{j+1}		x_1
x_3	x_4		x_{j+2}		x_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_m	x_1		x_{j-1}		x_{m-1}
x_1	x_2		x_j		x_m.

Since $G^j \subseteq N \backslash H^j$, each $N \backslash H^j$ is decisive. Therefore,

$$x_1 P_F(\theta) x_2 P_F(\theta) x_3 \cdots P_F(\theta) x_m P_F(\theta) x_1,$$

contradicting P-acyclicity. ∎

Note that this result generalizes Brown's theorem: Assume $|X| \geq n$, all profiles of weak linear orders are possible, but F is non-collegial. Then $\mathcal{N}(F) \leq n \leq |X|$, and Nakamura's theorem implies F is not P-acyclic.

The next proposition helps clarify the implications of Nakamura's theorem by calculating the Nakamura numbers of various PARs.

Proposition 2.20

1. *If F is non-collegial and strong, then $\mathcal{N}(F) = 3$.*
2. *If $q < n$, then $\mathcal{N}(F) = \lceil \frac{n}{n-q} \rceil$.*
3. *If $n = 4$, then $\mathcal{N}(F_{SM}) = 4$. If $n \geq 3$, $n \neq 4$, then $\mathcal{N}(F_{SM}) = 3$.*

Proof: 1. Since F is non-collegial, there exist distinct minimal decisive groups G and G'. Since $\mathcal{D}(F)$ is proper, $G \cap G' \neq \emptyset$. By minimality, $G \cap G' \notin \mathcal{D}(F)$. Since $\mathcal{D}(F)$ is strong, $N \backslash (G \cap G') \in \mathcal{D}(F)$. Clearly, $G \cap G' \cap (N \backslash (G \cap G')) = \emptyset$, so $\mathcal{N}(F) \leq 3$. By Proposition

2.18, $\mathcal{N}(F) = 3$. 2. Let $\mathcal{N}(F_q) = m$, and take distinct $G_1, G_2, \ldots, G_m \in \mathcal{D}(F_q)$ with $\bigcap_{j=1}^m G_j = \emptyset$. By DeMorgan's Law, $N = \bigcup_{j=1}^m (N \backslash G_j)$. This implies

$$n \;=\; |\bigcup_{j=1}^{m} (N\backslash G_j)| \;\leq\; \sum_{j=1}^{m} |N\backslash G_j|.$$

Since $G_j \in \mathcal{D}(F_q)$, $|G_j| \geq q$, so $|N\backslash G_j| \leq n - q$. Thus,

$$n \;\leq\; \sum_{j=1}^{m} |N\backslash G_j| \;\leq\; m(n-q),$$

implying

$$\mathcal{N}(F_q) \;=\; m \;\geq\; \frac{n}{n-q}.$$

Since $\mathcal{N}(F)$ is an integer, $\mathcal{N}(F) \geq \lceil \frac{n}{n-q} \rceil$. Set $t = \lceil \frac{n}{n-q} \rceil$. For the opposite inequality, I will find distinct $G_1, G_2, \ldots, G_t \in \mathcal{D}(F_q)$ with $\bigcap_{j=1}^t G_j = \emptyset$. Since $t - 1 < \frac{n}{n-q}$, i.e., $(t-1)(n-q) < n$, we can construct $t-1$ pairwise disjoint groups of size $n-q$, call them $H_1, H_2, \ldots, H_{t-1}$, with some individuals leftover.

$$\underbrace{1, 2, \ldots, n-q}_{H_1}, \underbrace{n-q+1, \ldots, 2(n-q)}_{H_2}, \ldots, \underbrace{(t-2)(n-q)+1, \ldots, (t-1)(n-q)}_{H_{t-1}},$$
$$\underbrace{(t-1)(n-q)+1, \ldots, n}_{\text{leftover}}$$

Since $t(n-q) = \left\lceil \frac{n}{n-q} \right\rceil (n-q) \geq \left(\frac{n}{n-q}\right)(n-q) = n$, the leftover group, call it H_t, contains no more than $n-q$ members. Thus, for each $j = 1, \ldots, t$, $G_j = N\backslash H_j$ has at least q members, so $G_j \in \mathcal{D}(F_q)$. Also, $\bigcap_{j=1}^t G_j = \emptyset$, as required. 3. Let's use part 2 for the special case of simple majority rule: $q = \lceil \frac{n+1}{2} \rceil$. If n is odd, then

$$\left\lceil \frac{n}{n-q} \right\rceil \;=\; \left\lceil \frac{n}{n - \frac{n+1}{2}} \right\rceil \;=\; \left\lceil \frac{2n}{n-1} \right\rceil \;\leq\; 3$$

if and only if $\frac{2n}{n-1} \leq 3$, i.e., $n \geq 3$. With our earlier proposition, this means that $\mathcal{N}(F_{SM}) = 3$, whenever n is odd and ≥ 3. If n is even, then

$$\left\lceil \frac{n}{n-q} \right\rceil \;=\; \left\lceil \frac{n}{n - \frac{n}{2} - 1} \right\rceil \;=\; \left\lceil \frac{2n}{n-2} \right\rceil \;\leq\; 3$$

if and only if $\frac{2n}{n-2} \leq 3$, i.e., $n \geq 6$. You can check that $\mathcal{N}(F_{SM}) = 4$ when $n = 4$, proving the claim. ∎

These results yield the following corollary.

Corollary 2.5 *Assume $PR(\Theta) \supseteq \mathbf{L}$.*

1. *If F is P-acyclic and strong, then $|X| = 2$.*
2. *If $P_q \subseteq P_F$ and F is P-acyclic, then $|X| < \lceil \frac{n}{n-q} \rceil$.*
3. *If $n \neq 4$, $n \geq 3$, $P_{SM} \subseteq P_F$, and F is P-acyclic, then $|X| = 2$.*
4. *If $n = 4$, $P_{SM} \subseteq P_F$, and F is P-acyclic, then $|X| \leq 3$.*

Nakamura also proved the following sufficient condition for P-acyclicity.

Proposition 2.21 (Nakamura) *Let F be simple. If $\mathcal{N}(F) > |X|$, then F is P-acyclic.*

Proof: Suppose $\mathcal{N}(F) > |X|$ but F is not P-acyclic. Then there is some θ and $x_1, x_2, \ldots, x_k$ such that

$$x_1 P_F(\theta) x_2 P_F(\theta) \cdots x_{k-1} P_F(\theta) x_k = x_1.$$

Without loss of generality, we may assume $x_1, x_2, \ldots, x_{k-1}$ are distinct (why?), so $|X| \geq k-1$. Since F is simple,

$$\{P(x_1, x_2|\theta), P(x_2, x_3|\theta), \ldots, P(x_{k-1}, x_k|\theta), P(x_k, x_1|\theta)\} \subseteq \mathcal{D}(F).$$

Since $\mathcal{N}(F) > |X| \geq k-1$, we must have $\bigcap_{h=1}^{k-1} P(x_h, x_{h+1}|\theta) \neq \emptyset$. But then there is some $i \in N$ with $x_1 P_i(\theta) x_2 P_i(\theta) \cdots x_{k-1} P_i(\theta) x_k = x_1$, a contradiction. ∎

As a corollary, we get the following characterization of P-acyclicity for simple PARs, complementing Propositions 2.8 and 2.9 on transitivity of strict and weak social preference for simple PARs. It differs from these earlier propositions in that the characterization places a joint restriction on F and the number of alternatives.

Corollary 2.6 *Assume $PR(\Theta) \supseteq \mathbf{L}$. Let F be a simple PAR. Then F is P-acyclic if and only if $\mathcal{N}(F) > |X|$.*

2.6 The One-Dimensional Spatial Model

We've examined the possibility of collective rationality under broad domain restrictions (e.g., linear domain or unrestricted domain), and we've discovered rather severe implications for the distribution of power among individuals (though less so for the case of P-acyclicity).

We now investigate the domain restriction of single-peakedness in the one-dimensional spatial model, where it is quite intuitive. It turns out that single-peakedness is enough to get excellent transitivity properties for many PARs. The next proposition imposes weak decisiveness to get transitivity of strict social preferences and the stronger condition that $\mathcal{D}(F)$ is strong to get transitivity of weak social preferences. Note that the conditions of the proposition are stated for individual preferences at a single state, rather than on the domain of possible preferences. An implication is that, under single-peaked domain, these transitivity properties hold at every state.

Proposition 2.22 *Assume $PR(\theta)$ is single-peaked.*

1. *If F is weakly decisive, then $P_F(\theta)$ is transitive.*
2. *If F is strong, then $R_F(\theta)$ is transitive.*

Proof: Let $PR(\theta)$ be single-peaked with respect to $\prec$, and take any $x, y, z \in X$ such that $xP_F(\theta)yP_F(\theta)z$. If $x = y$ or $y = z$, then we have $xP_F(\theta)z$. Otherwise, one of the following cases must obtain:

1. $x \prec y \prec z$	3. $y \prec z \prec x$	5. $z \prec x \prec y$
2. $y \prec x \prec z$	4. $z \prec y \prec x$	6. $x \prec z \prec y$.

Case 1: Since $xP_F(\theta)y$ and F is weakly decisive, we have $R(x, y|\theta) \in \mathcal{D}(F)$. Since individual preferences are single-peaked, $R(x, y|\theta) \subseteq P(x, z|\theta)$ (right?), so $P(x, z|\theta) \in \mathcal{D}(F)$, which implies $xP_F(\theta)z$. Case 2: Since $yP_F(\theta)z$ and F is weakly decisive, we have $R(y, z|\theta) \in \mathcal{D}(F)$. By single-peakedness, $R(y, z|\theta) \subseteq P(x, z|\theta)$, so $xP_F(\theta)z$. Case 3: Since $xP_F(\theta)y$ and F is weakly decisive, $R(x, y|\theta) \in \mathcal{D}(F)$. By single-peakedness, $R(x, y|\theta) \subseteq P(z, y|\theta)$, implying $zP_F(\theta)y$, a contradiction. Thus, this case cannot occur. Case 4 is symmetric to 1, case 5 is symmetric to 2, and case 6 is symmetric to 3. With part 3 of Proposition 2.3, the proof of part 2 proceeds in the same way. ∎

As the following example illustrates, the assumption of single-peakedness in part 1 of Proposition 2.22 cannot be dropped, not even weakened to weak single-peakedness, even for majority rule with n odd. Let $n = 3$, $X = \{x, y, z\}$, and consider a state θ with preferences as follows.

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$
x	y	z
yz	x	y
	z	x

Then $yP_{SM}(\theta)xP_{SM}(\theta)z$ but $zR_{SM}(\theta)y$, violating P-transitivity.

The next example shows that the conclusion of part 1 of Proposition 2.22 cannot be strengthened to R-transitivity, and that the assumption that F is strong in part 2 cannot be weakened to weak decisiveness.

$R_1(\theta)$	$R_2(\theta)$
y	z
x	y
z	x

This profile is single-peaked with respect to $x \prec y \prec z$, but we have $xR_{SM}(\theta)zR_{SM}(\theta)y$ and $yP_{SM}(\theta)x$, violating R-transitivity.

Note that $n = 2$ is even in the previous example. Otherwise, if n were odd, then the second part of Proposition 2.22 would ensure R-transitivity.

Transitivity of strict social preferences is more than we need for nonemptiness of choice sets. Austen-Smith and Banks show that weak single-peakedness is actually sufficient for P-acyclicity of social preferences.

Proposition 2.23 *Assume $PR(\theta)$ is weakly single-peaked. If F is simple, then $P_F(\theta)$ is acyclic.*

Proof: Let $PR(\theta)$ be weakly single-peaked with respect to $\preceq$. Take any finite number of alternatives, $x_1, \ldots, x_k$, and suppose that $x_1 P_F(\theta) x_2 \cdots P_F(\theta) x_k$. Assume, without loss of generality, that $x_k \prec x_{k-1}$. I claim that, for all $h, j = 1, \ldots, k$, if $x_h \prec x_j \preceq x_{h+1}$, then $x_h \prec x_{j+1}$. Otherwise, we have $x_{j+1} \preceq x_h \prec x_j \preceq x_{h+1}$, and, by weak single-peakedness, $P(x_{h+1}, x_h|\theta) \subseteq R(x_j, x_{j+1}|\theta)$ (why?), implying $P(x_{h+1}, x_h|\theta) \cap P(x_{j+1}, x_j|\theta) = \emptyset$. Since F is simple, however, it must be that $P(x_{h+1}, x_h|\theta)$ and $P(x_{j+1}, x_j|\theta)$ are decisive, a contradiction. The result now follows because $x_k \prec x_1$ (why?), which implies $x_k \neq x_1$. ∎

We have seen that the conclusion of Proposition 2.23 cannot be strengthened to P-transitivity or R-transitivity. One way it differs from our earlier propositions is that it assumes F is simple, rather than just weakly decisive. To see that we cannot assume just weak decisiveness, or even that F is strong, let $n = 3$, $X = \{x, y, z\}$, and define F as follows. For all $v, w \in X$, say $vP_F(\theta)w$ if and only if any of three conditions holds:

- $vP_1(\theta)w$ and $vR_2(\theta)w$,
- $vP_3(\theta)w$ and $vR_1(\theta)w$,
- $vP_2(\theta)w$ and $vP_3(\theta)w$.

This PAR is weakly decisive. In fact, F, so-defined, is neutral and monotonic, and $\mathcal{D}(F) = \mathcal{D}(F_{SM})$ is strong. But it is not decisive. Now consider the following profile.

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$
x	z	y
yz	xy	z
		x

This profile is weakly single-peaked with respect to $x \prec y \prec z$. Note, however, that $xP_F(\theta)yP_F(\theta)zP_F(\theta)x$, violating P-acyclicity.

Thus, when X is finite and F is weakly decisive, single-peakedness implies that the core, $C_F(\theta)$, is nonempty. When X is finite and F is simple, weak single-peakedness implies that the core is nonempty. When X is infinite, nonemptiness of the core is not so immediate: we have not established upper semicontinuity of the social preference relation, nor have we imposed compactness on the set of alternatives. (In fact, we have not imposed any topological structure on the set of alternatives.)

The next task is to derive a sharp characterization of the core under single-peakedness for a large class of PARs. As a consequence, we will obtain an existence result for infinite sets of alternatives, i.e., the usual conditions of compactness and continuity are unneeded. We first note that such a characterization for simple PARs carries over immediately to all weakly decisive PARs.

Recall that, for all PARs F and all $\theta \in \Theta$, we must have $C_F(\theta) \subseteq C_{\mathcal{D}(F)}(\theta)$. For single-peaked profiles, the converse holds for all weakly decisive PARs.

Proposition 2.24 *Assume $PR(\theta)$ is single-peaked. If F is weakly decisive, then $C_F(\theta) = C_{\mathcal{D}(F)}(\theta)$.*

Proof: Let $PR(\theta)$ be single-peaked with respect to $\preceq$. Take $x \in C_{\mathcal{D}(F)}(\theta)$ and suppose $x \notin C_F(\theta)$, i.e., there is some $y \in X$ such that $yP_F(\theta)x$. Assume without loss of generality that $x \prec y$. By single-peakedness, $x \prec \tilde{x}^i$ for all $i \in R(y,x|\theta)$. Let $z = \min\{\tilde{x}^i \mid i \in R(y,x|\theta)\}$, where the min operation refers to $\preceq$. By single-peakedness, $R(y,x|\theta) \subseteq P(z,x|\theta)$. By weak decisiveness, $R(y,x|\theta) \in \mathcal{D}(F)$, so $zP_{\mathcal{D}(F)}(\theta)x$, contradicting $x \in C_{\mathcal{D}(F)}(\theta)$. ∎

Given a state θ and preference profile $PR(\theta)$ single-peaked with respect to $\preceq$, define

$$\begin{aligned} N^+_\preceq(x|\theta) &= \{i \in N \mid x \prec \tilde{x}^i\} \\ N^-_\preceq(x|\theta) &= \{i \in N \mid \tilde{x}^i \prec x\}. \end{aligned}$$

Define the groups

$$\begin{aligned} G_1 &= \{i \in N \mid N^+_\preceq(\tilde{x}^i|\theta) \notin \mathcal{D}(F)\} \\ G_2 &= \{i \in N \mid N^-_\preceq(\tilde{x}^i|\theta) \notin \mathcal{D}(F)\}. \end{aligned}$$

Note that $G_1 \neq \emptyset$, because, letting i have the "greatest" ideal point according to the ordering $\prec$, we have $\tilde{x}_i \in G_1$. Similarly, $G_2 \neq \emptyset$. Thus, we can take

$$i_1 \in \arg\min\{\tilde{x}^i \mid i \in G_1\} \;\text{ and }\; i_2 \in \arg\max\{\tilde{x}^i \mid i \in G_2\},$$

where the minimum and maximum are with respect to $\prec$. Note that, if $\tilde{x}^{i_1} \prec \tilde{x}^i$, then $i \in G_1$; similarly, if $\tilde{x}^i \prec \tilde{x}^{i_2}$, then $i \in G_2$.

Furthermore, if F satisfies Pareto, then G_1 and G_2 are decisive. This is true of G_1, for example, if $G_1 = N$. If $G_1 \subsetneqq N$, then let i be a solution to $\max_{i \in N \setminus G_1} \tilde{x}^i$, and note that $G_1 = N^+_{\prec}(\tilde{x}^i|\theta) \in \mathcal{D}(F)$. (Do you see why?) The same argument establishes that $G_2 \in \mathcal{D}(F)$. Thus, because $\mathcal{D}(F)$ is proper, we have $G_1 \cap G_2 \neq \emptyset$, i.e., $\tilde{x}^{i_1} \preceq \tilde{x}^{i_2}$.

We say x is an *F-median with respect to $\preceq$ at θ* if

$$N^+_{\prec}(x|\theta) \notin \mathcal{D}(F) \text{ and } N^-_{\prec}(x|\theta) \notin \mathcal{D}(F).$$

The next proposition establishes that the set of F-medians is independent of the particular ordering $\preceq$ in the definition of single-peakedness.

Proposition 2.25 *Assume $PR(\theta)$ is single-peaked with respect to $\preceq$ and with respect to $\preceq'$. Then x is an F-median with respect to $\preceq$ at θ if and only if it is an F-median with respect to $\preceq'$ at θ.*

Proof: Suppose x is an F-median with respect to $\preceq$ but not an F-median with respect to $\preceq'$. Without loss of generality, suppose $N^+_{\preceq'}(x|\theta) \in \mathcal{D}(F)$, and take $i \in \arg\min\{\tilde{x}^k \mid k \in N^+_{\preceq'}(x|\theta)\}$, where the minimum is with respect to $\preceq'$. By single-peakedness with respect to $\preceq'$, $\tilde{x}^i P_k(\theta) x$ for all $k \in N^+_{\preceq'}(x|\theta)$. Since x is an F-median with respect to $\preceq$, we cannot have $N^+_{\preceq'}(x|\theta) \subseteq N^+_{\preceq}(x|\theta)$ or $N^+_{\preceq'}(x|\theta) \subseteq N^-_{\preceq}(x|\theta)$. Thus, we may assume that $\tilde{x}^j \preceq x \prec \tilde{x}^i$ for some $j \in N^+_{\preceq'}(x|\theta)$. But then single-peakedness with respect to $\preceq$ implies $x P_j(\theta) \tilde{x}^i$, a contradiction. ∎

Thus, we use the term "F-median at θ" and write simply $M_F(\theta)$ for the set of F-medians at θ.

Proposition 2.26 *Assume $PR(\theta)$ is single-peaked. If F satisfies Pareto, then*

$$M_F(\theta) = \{x \in X \mid \tilde{x}^{i_1} \preceq x \preceq \tilde{x}^{i_2}\} \neq \emptyset.$$

Proof: If $x \prec \tilde{x}^{i_1}$, then $G_1 \subseteq N^+_{\prec}(x|\theta) \in \mathcal{D}(F)$, so $x \notin M_F(\theta)$. Similarly, $\tilde{x}^{i_2} \prec x$ implies $x \notin M_F(\theta)$. Now take x such that $\tilde{x}^{i_1} \preceq x \preceq \tilde{x}^{i_2}$. By the first relation, $N^+_{\prec}(x|\theta) \notin \mathcal{D}(F)$, and, by the second relation, $N^-_{\prec}(x|\theta) \notin \mathcal{D}(F)$. Thus, $x \in M_F(\theta)$. This establishes the equality in the proposition. That $M_F(\theta) \neq \emptyset$ follows because $\mathcal{D}(F)$ is proper: there exists $i \in G_1 \cap G_2$, and then $\tilde{x}^i \in M_F(\theta)$. ∎

Now that we have characterized the F-medians, we can give a characterization of the core when individual preferences are single-peaked. With Proposition 2.26, the following proposition establishes the nonemptiness of the core, given any weakly decisive PAR, even for infinite X.

Proposition 2.27 *Assume $PR(\theta)$ is single-peaked.*

1. *For every PAR F, $C_F(\theta) \subseteq M_F(\theta)$.*
2. *If F is weakly decisive, then $M_F(\theta) \subseteq C_F(\theta)$.*

Proof: For part 1, take $x \in C_F(\theta)$ and suppose $x \notin M_F(\theta)$. Without loss of generality, say $N^+_{\prec}(x|\theta) \in \mathcal{D}(F)$, and let $\tilde{x}^i$ be the minimum ideal point above x according to $\prec$. Since $PR(\theta)$ is single-peaked, $N^+_{\prec}(x|\theta) = P(\tilde{x}^i, x|\theta)$, so $\tilde{x}^i P_F(\theta)x$, contradicting $x \in C_F(\theta)$. For part 2, take $x \in M_F(\theta)$, and suppose there is some $y \in X$ such that $yP_F(\theta)x$. Without loss of generality, suppose $x \prec y$. Since $PR(\theta)$ is single-peaked, $R(y, x|\theta) \subseteq N^+_{\prec}(x|\theta)$. Since F is weakly decisive, $yP_F(\theta)x$ implies $R(y, x|\theta) \in \mathcal{D}(F)$. But then $N^+(x|\theta) \in \mathcal{D}(F)$, contradicting $x \in M_F(\theta)$. ■

Let's examine the idea of F-median in some special cases. If F is a q-rule, then $x \in M_F(\theta)$ if and only if $|N^+_{\prec}(x|\theta)| < q$ and $|N^-_{\prec}(x|\theta)| < q$. Ordering the ideal points $\tilde{x}^1 \preceq \tilde{x}^2 \preceq \cdots \preceq \tilde{x}^n$, the medians are as in Figure 2.7.

Figure 2.7: F-medians for q-rules

If $q = \lceil \frac{n+1}{2} \rceil$, then we have two cases, depending whether n is even or odd, as in Figure 2.8. When n is odd, there is only one majority rule core point: it is the median of the distribution of ideal points.

Figure 2.8: F-medians for majority rule

This result provides the logic for the famous "Median Voter Theorem," generally attributed to Harold Hotelling, Anthony Downs, or Duncan Black. Suppose two candidates compete for the votes of an odd number of voters by simultaneously choosing platforms in a one-dimensional space. If the voters' preferences are single-peaked, then the unique Nash equilibrium is for both candidates to locate at the median of the voters' ideal points.

The uniqueness of the majority rule core when n is odd can actually be obtained quite generally, as the next proposition shows. In fact, as long as the collection of decisive groups is strong, the core must consist of the ideal point of some voter.

Proposition 2.28 *Assume $PR(\theta)$ is single-peaked. If F is strong, then $C_F(\theta) = \{\tilde{x}^i\}$ for some $i \in N$. Furthermore, for all $x \in X \setminus \{\tilde{x}^i\}$, we have $\tilde{x}^i P_F(\theta) x$.*

Proof: Note that F is weakly decisive, so $C_F(\theta) = M_F(\theta)$. Furthermore, because $\mathcal{D}(F)$ is strong, we have $\tilde{x}^{i_1} = \tilde{x}^{i_2}$ and $M_F(\theta) = \{\tilde{x}^{i_1}\}$. That $\tilde{x}^{i_1}$ is strictly socially preferred to every other alternative follows from Proposition 2.22. (Do you see why?) ∎

Note that, when F is strong and we assume single-peaked domain, the social weak preference $R_F(\theta)$ is complete and transitive, i.e., a weak order. Moreover, by the previous proposition, there is a unique alternative at the top of that ordering, which we may take to be the ideal point of individual i_1. But we have not explored the nature of the social ordering. Is it single-peaked? (You can check that it is.) Is it equal to the ordering of i_1, i.e., do we have $R_F(\theta) = R_{i_1}(\theta)$? (Well?)

The definition of F-median can be extended to weakly single-peaked profiles, in which case we may consider whether Proposition 2.27 extends as well. The next example, due to Austen-Smith and Banks, shows that it does not, even when individuals have unique ideal points (so the definition of F-median is as before) and F is strong. Let $n = 3$ and $X = \{x, y, z\}$, and consider the following profile.

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$
y	x	z
z	yz	y
x		x

This profile is weakly single-peaked with respect to $x \prec y \prec z$, and the unique majority rule median is y, but $C_{SM}(\theta) = \{y, z\}$. Thus, core points need not be medians.

For another example, go back to our example of a weakly single-peaked profile, a PAR with F strong, and a strict social preference cycle: there, y is the unique median, but $xP_F(\theta)y$. Thus, medians need not be core points.

A more general construction of the set of F-medians must allow for "intervals" of ideal points, which must have well-defined suprema and infima. To deal with this, suppose $X \subseteq \mathbb{R}$ and $PR(\theta)$ is weakly single-peaked with respect to $\leq$, so that each $M(R_i(\theta))$ is a nonempty interval. Given $G \subseteq N$, let

$$\tilde{X}_G(\theta) \;=\; \bigcup_{i \in G} M(R_i(\theta)),$$

and define

$$\overline{x}(\theta) \;=\; \min_{G \in \mathcal{D}(F)} \sup \tilde{X}_G(\theta) \;\text{ and }\; \underline{x}(\theta) \;=\; \max_{G \in \mathcal{D}(F)} \inf \tilde{X}_G(\theta),$$

where we use the convention that $\sup Y = \infty$ if Y is unbounded above and $\inf Y = -\infty$ if Y is unbounded below. Then we define the F-medians, still denoted $M_F(\theta)$, to be the interval from $\underline{x}(\theta)$ to $\overline{x}(\theta)$, with endpoints included depending on whether the supremum and infimum are achieved in the above. Of course, if each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous, then each $M(R_i(\theta))$ is closed (right?), and then $M_F(\theta) = [\underline{x}(\theta), \overline{x}(\theta)]$.[4] You should check that, when $PR(\theta)$ is single-peaked, this definition agrees with our earlier one.

The next result extends one direction of Proposition 2.27 to the weakly single-peaked case. Note that it gives us nonemptiness of the core. I leave the proof to you.

Proposition 2.29 *Assume that $X \subseteq \mathbb{R}$, that $PR(\theta)$ is weakly single-peaked with respect to $\leq$, and that each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous. If F is simple, then $M_F(\theta) \subseteq C_F(\theta)$.*

That the opposite inclusion does not hold, even when F is simple and strong, follows from the above example for simple majority rule from Austen-Smith and Banks. To see that the stated inclusion does not hold when F is just weakly decisive, or even when F is strong, go back to our example of a weakly single-peaked profile and a strict social preference cycle.

There is a simple procedure for finding the F-medians, and therefore a subset of core points, for a weakly single-peaked profile. You can check that, under the assumptions of Proposition 2.29, $x \in M_F(\theta)$ if there is a selection $(\tilde{x}^1, \ldots, \tilde{x}^n)$ such that $\tilde{x}^i \in M(R_i(\theta))$ for all $i \in N$ and x is a median with respect to $(\tilde{x}^1, \ldots, \tilde{x}^n)$.

See Sections 4.5 and 4.6 in Austen-Smith and Banks for a different domain restriction, "order restricted preferences," that can be helpful in analyzing two-dimensional problems.

2.7 The Multi-dimensional Spatial Model

We now impose the assumption that X is a subset of d-dimensional Euclidean space. In addition, we often impose further structure on the set of alternatives and on individual preferences. If $d = 1$, if X is convex, and if preferences are strictly convex, then we are back to the one-dimensional model with single-peaked preferences. If X is finite, as is allowed in some results, then we essentially have the discrete choice model. Our main interest here, however, will be on the multi-dimensional model, with $d \geq 2$. In fact, some of the results in this section assume that X has nonempty interior in $\mathbb{R}^d$, ruling out finite models and private good exchange economies.

[4] Unless one of these end points is ∞ or $-\infty$. Those are not actually points in $\mathbb{R}$, so they are not actually alternatives.

Arrow's Theorem

We first consider the issue of social rationality, where we impose IIA and allow individual preferences to vary. Unlike the discrete choice model, however, we can no longer use the assumptions of linear or unrestricted domain...

[More to do here.]

Characterization of the Core

While the core is nonempty and coincides with the F-medians in the one-dimensional spatial model, the core can be empty in multi-dimensional collective choice problems — even when the usual regularity conditions hold.

To see this, refer to Figure 2.5, where we have three individuals with Euclidean preferences over a two-dimensional space. If x is not on the contract curve for individuals 1 and 2, there is an alternative, y, that both individuals strictly prefer, i.e., $yP_{SM}(\theta)x$. By that logic, a core alternative would have to be on all three contract curves, which is impossible. Thus, the simple majority core is empty. Note that this logic applies whenever the three ideal points form a triangle — no matter how close they are to collinear. Thus, we see that the simple majority core is "fragile" as well.

Before continuing with the analysis of this issue, we note a useful connection between the core of a PAR F and the core of the associated simple PAR $F_{\mathcal{D}(F)}$. We know that $C_F(\theta) \subseteq C_{\mathcal{D}(F)}(\theta)$ generally. The next proposition establishes the other direction for all weakly decisive PARs under weak conditions on individual preferences.

Proposition 2.30 *Assume $X \subseteq \mathbb{R}^d$ and $PR(\theta)$ satisfies LSWP. If F is weakly decisive, then $C_F(\theta) = C_{\mathcal{D}(F)}(\theta)$.*

Proof: Take $x \in C_{\mathcal{D}(F)}(\theta)$ and suppose $x \notin C_F(\theta)$, i.e., there is some $y \in X$ such that $yP_F(\theta)x$. By weak decisiveness, $G = R(y, x|\theta) \in \mathcal{D}(F)$. By LSWP, there exists $z \in P_G(x|\theta)$. Thus, $G \subseteq P(z, x|\theta) \in \mathcal{D}(F)$, so $zP_{\mathcal{D}(F)}(\theta)x$, contradicting $x \in C_{\mathcal{D}(F)}(\theta)$. ∎

So the cores of weakly decisive PARs are determined by associated simple PARs. This result justifies the terminology "majority core" (without the "simple" or "relative" modifier), which I may use later. Thus, any characterization for the core of a simple rule will apply to all weakly decisive PARs with the same collection of decisive groups.

Furthermore, Proposition 1.23 immediately yields the following result.

Proposition 2.31 *Assume $X \subseteq \mathbb{R}^d$ and $(P_F(\theta), R_F(\theta))$ satisfies thin indifference. If $C_F(\theta) \neq \emptyset$, then there exists $x \in X$ such that $C_F(\theta) = \{x\}$. Furthermore, for all $y \in X \backslash \{x\}$, we have $xP_F(\theta)y$.*

Under fairly standard assumptions in the multi-dimensional spatial model, social preferences satisfy thin indifference. Because that is a nice condition to have, the next proposition is an important (though straightforward) one.

Proposition 2.32 *Assume $X \subseteq \mathbb{R}^d$ and $PR(\theta)$ satisfies LSWP. If F is strong, then $(P_F(\theta), R_F(\theta))$ satisfies thin indifference.*

Proof: Take $y \in R_F(x|\theta)$, so $G = R(y, x|\theta) \in \mathcal{D}(F)$, for otherwise, since $\mathcal{D}(F)$ is strong, we would have $P(x, y|\theta) \in \mathcal{D}(F)$, a contradiction. Suppose $y \neq x$ and take any open set $Y \subseteq X$ such that $y \in Y$. By LSWP, there exists $z \in P_G(x|\theta) \cap Y$. Thus, $G \subseteq P(z, x|\theta) \in \mathcal{D}(F)$, implying $z \in P_F(x|\theta)$. Thus, $y \in \text{clos} P_F(x|\theta)$. ∎

A straightforward implication of Propositions 2.31 and 2.32 is that, if LSWP is satisfied, if F is strong, and if the core is nonempty, then it must be a singleton.

Note that the conclusion of Proposition 2.31 is not that $C_F(\theta)$ consists of the ideal point of some individual. In fact, that conclusion does not hold generally under the assumptions of the corollary. Consider simple majority rule in the environment of Figure 2.9.

Figure 2.9: The core is not an ideal point

If we add the assumption that each individual's preferences has a differentiable utility representation, then we can deduce the stronger result.

Proposition 2.33 *Assume that $X \subseteq \mathbb{R}^d$ and that each $(P_i(\theta), R_i(\theta))$ has a differentiable utility representation $u_i \colon X \to \mathbb{R}$ such that $\nabla u_i(x) = 0$ only if $x \in M(R_i(\theta))$. If F is strong and $C_F(\theta) \neq \emptyset$, then $C_F(\theta) \cap intX = \bigcup_{i \in N} M(R_i(\theta))$.*

I'll leave the proof to you. To see that the restrictions on individual preferences are needed for the latter result, consider the example in the following figure, where x^* is a relative majority core point but is no one's ideal point.

The following proposition may also be of interest.

Figure 2.10: Core still not an ideal point

Proposition 2.34 *Assume $X \subseteq \mathbb{R}^d$. If $(R_{\mathcal{D}(F)}(\theta), P_{\mathcal{D}(F)}(\theta))$ satisfies thin indifference, then so does $(P_F(\theta), R_F(\theta))$.*

Proof: Take any $x \in X$ and $y \in R_F(x|\theta)$. If $y = x$, then clearly $y \in \{x\} \cup \mathrm{clos}P_F(x|\theta)$. Otherwise, take any open set Y containing y. Note that $yR_F(\theta)x$ implies $yR_{\mathcal{D}(F)}(\theta)x$. Since $(R_{\mathcal{D}(F)}(\theta), P_{\mathcal{D}(F)}(\theta))$ satisfies thin indifference, there exists $z \in Y$ such that $zP_{\mathcal{D}(F)}(\theta)x$. This implies $zP_F(\theta)x$, as required. ■

[Do medians in all directions here.]

Nonemptiness of the Core

Recall that, when all profiles of weak linear orders are possible, $\mathcal{N}(F) > |X|$ is necessary for P-acyclicity of social preferences. The next result shows that, with a sufficiently rich domain $d+1$ plays the role of the cardinality of $|X|$: the condition $\mathcal{N}(F) > d+1$ is necessary for nonemptiness of the core to hold generally. Note that the result does not impose any restrictions on the PAR F.

Proposition 2.35 (Schofield; Strnad) *Assume that $X \subseteq \mathbb{R}^d$ has nonempty interior and $PR(\Theta) \supseteq \mathbf{E}$. If $C_F(\theta) \neq \emptyset$ for all $\theta \in \Theta$, then $\mathcal{N}(F) > d+1$.*

Proof: I will be somewhat less formal in this proof. Suppose $d+1 \geq \mathcal{N}(F)$. Let $m = \mathcal{N}(F)$, and take $G_1, G_2, \ldots, G_m \in \mathcal{D}(F)$ such that $\bigcap_{j=1}^m G_j = \emptyset$. Since X contains an open set and $d+1 \geq m$, we can find m distinct points, $z_1, z_2, \ldots, z_m \in X$ that are pointwise equidistant from each other.

Figure 2.11: Equidistant points for $d = 3$

.

This is depicted in Figure 2.11 for $d = 3$. We construct a profile of Euclidean preferences as follows: for each $i \in N$, let $\gamma(i) = \{j \mid i \notin G_j\}$, and define

$$\tilde{x}^i = \frac{1}{|\gamma(i)|} \sum_{j \in \gamma(i)} z_j.$$

Note that $\bigcap_{j=1}^m G_j = \emptyset$ implies $|\gamma(i)| \geq 1$ for all i, so $\tilde{x}^i$ is well-defined. By assumption, there exists $\theta \in \Theta$ such that each $(P_i(\theta), R_i(\theta))$ is Euclidean with ideal point $\tilde{x}^i$. Note that $\tilde{x}^i$ lies on the face opposite z_j if and only if $i \in G_j$. Claim: $C_F(\theta) = \emptyset$. Why? Take any $x \in X$. There is some face, say that opposite z_j, that x is *not* on. Let y be the point on that face closest to x. Since preferences are Euclidean, we have $yP_i(\theta)x$ for all $i \in G_j$. Since $G_j \in \mathcal{D}(F)$, $yP_F(\theta)x$. ∎

We will now use previous results, and one new one, to extend this parallel: in many environments, $\mathcal{N}(F) > d+1$ is sufficient for nonemptiness of the core of a simple PAR.

Recall, from Proposition 1.27, that compactness of X, upper semi-continuity of (P, R), and Condition F imply $M(P) \neq \emptyset$. Recall, from Proposition 1.28, that convexity of X and upper semi-continuity and semi-convexity of (P, R) imply Condition F.

Applying these results to social preferences, $C_F(\theta) \neq \emptyset$ follows if:

- X is compact and convex,
- $(P_F(\theta), R_F(\theta))$ is upper semicontinuous and semi-convex.

We already have a result, Proposition 2.6, that gives a weak sufficient condition on individual preferences for upper semi-continuity of social preferences: if F is simple, then we need only assume that each $R_i(\theta)$ is upper semicontinuous.

What about semi-convexity? That is addressed next.

Proposition 2.36 *Assume that $X \subseteq \mathbb{R}^d$ is convex and that, for all $i \in N$, $(P_i(\theta), R_i(\theta))$ is convex. Let F be a simple PAR. If $d+1 < \mathcal{N}(F)$, then $(R_F(\theta), P_F(\theta))$ is semi-convex.*

Proof: By Caratheodory's Theorem, if $Y \subseteq \mathbb{R}^d$ and $y \in \text{conv}Y$, then we can find $y^0, y^1, \ldots,$ $y^d \in Y$ such that $y \in \text{conv}\{y^0, y^1, \ldots, y^d\}$. Take any $x \in X$ and suppose that $x \in \text{conv}P_F(x|\theta)$. By Caratheodory's Theorem, there exist $y^0, y^1, \ldots, y^d \in P_F(x|\theta)$ such that $x \in \text{conv}\{y^0, y^1, \ldots, y^d\}$. Since F is simple, for each $j = 0, 1, \ldots, d$, we have $G^j = P(y^j, x|\theta) \in \mathcal{D}(F)$. Since $d+1 < \mathcal{N}(F)$, we must have $\bigcap_{j=1}^d G^j \neq \emptyset$, i.e., there exists i such that $y^j P_i(\theta)x$ for all $j = 0, 1, \ldots, d$. But then $x \in \text{conv}P_i(x|\theta)$, contradicting convexity of $(R_i(\theta), P_i(\theta))$. ∎

Combining these observations, we have the following result on nonemptiness of the core of a simple rule.

Proposition 2.37 *Assume that $X \subseteq \mathbb{R}^d$ is compact and convex and that, for all $i \in N$, $(P_i(\theta), R_i(\theta))$ is upper semicontinuous and convex. Let F be a simple PAR. If $\mathcal{N}(F) > d+1$, then $C_F(\theta) \neq \emptyset$.*

Using Proposition 2.30, this gives us the following corollary for all weakly decisive PARs.

Corollary 2.7 *Assume that $X \subseteq \mathbb{R}^d$ is compact and convex and that, for all $i \in N$, $(P_i(\theta), R_i(\theta))$ is upper semicontinuous and strictly convex. Let F be weakly decisive. If $\mathcal{N}(F) > d+1$, then $C_F(\theta) \neq \emptyset$.*

To see the role of semi-convexity in Proposition 2.37, go back to one of our earlier examples: three voters with Euclidean preferences over a two-dimensional space, ideal points arranged in a triangle. We can make this space compact and convex. Clearly, individual preferences are continuous and strictly convex. If simple majority preferences were semi-convex, Proposition 2.37 would yield nonemptiness of the core, which clearly does not hold. Indeed, as we have noted, simple majority preferences are not semi-convex in this example. If $d+1$ were less than $\mathcal{N}(F)$, Proposition 2.36 would yield semi-convexity, but you can see that condition does not hold: $d+1 = 3 = \mathcal{N}(F)$.

As an application of Proposition 2.37, add a fourth individual. Then the Nakamura number of simple majority rule is $\mathcal{N}(F) = 4 > 3 = d+1$, and the result tells us that the majority core is nonempty — regardless of the preferences of the individuals. If we add another dimension as well, however, the antecedent condition of the proposition is violated.

Symmetry Conditions for Majority Core Points

Note that the condition $\mathcal{N}(F) > d+1$ in Proposition 2.35 is necessary for nonemptiness of the core as θ varies across Θ, generating all profiles of Euclidean preferences. It is possible, however, that $d+1 \geq \mathcal{N}(F)$ and $C_F(\theta) \neq \emptyset$ in particular states (though there will be other states in which the core is empty). This is pictured below in two examples: $n = 3$, which we have seen before, and $n = 5$.

Figure 2.12: Nonempty core with $d+1 \geq \mathcal{N}(F)$.

Our next goal is to understand, given *any* state, the conditions under which the majority core is nonempty. We skip some interesting topics, like "medians in all directions" and the separating hyperplane characterization of core points. (See Austen-Smith and Banks's Theorem 5.6).

Note that, in the last examples, while the majority core was nonempty, it exhibited two properties: first, the core point was the ideal point of one individual; second, individual gradients at the core point were matched against each other in a very precise way. We will see that these observations generalize.

Given a state θ in which each $(P_i(\theta), R_i(\theta))$ has a differentiable utility representation u_i, given a group $G \subseteq N$, and given an alternative $x \in \mathrm{int}X$, we say that *radial symmetry* holds at x if, for all $p \in \mathbb{R}^d$,

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(x) = \alpha p\}| \quad = \quad |\{i \in N \mid \exists \alpha > 0 : \nabla u_i(x) = -\alpha p\}|.$$

This is pictured below.

Figure 2.13: Radial symmetry

The next proposition, due to Charlie Plott, uses these concepts to deduce a strong necessary condition on majority core points.

Proposition 2.38 (Plott) *Assume $X \subseteq \mathbb{R}^d$ and:*

1. *n is odd;*
2. *each $(P_i(\theta), R_i(\theta))$ is represented by some differentiable u_i;*
3. *$x^* \in \mathit{int}X$;*
4. *for all $i, j \in N$, $\nabla u_i(x^*) = \nabla u_j(x^*) = 0$ implies $i = j$ (no shared critical points at x^*).*

If $x^* \in C_{SM}(\theta)$, then (i) there is some $i \in N$ with $\nabla u_i(x^*) = 0$, and (ii) radial symmetry holds at x^*.

Proof: To establish (i), suppose $x^* \in C_{SM}(\theta)$ but $\nabla u_i(x^*) \neq 0$ for all $i \in N$. Then there is at least one $p \neq 0$ such that $\nabla u_i(x^*) \cdot p \neq 0$ for all i. (Why?) Since n is odd, either

$$\{i \in N \mid \nabla u_i(x^*) \cdot p > 0\} \quad \text{or} \quad \{i \in N \mid \nabla u_i(x^*) \cdot p < 0\}$$

is a majority. Without loss of generality, suppose the former. Since $x^* \in \mathrm{int}X$, we can pick $\epsilon > 0$ small enough that $x^* + \epsilon p \in X$. Furthermore, $u_i(x^* + \epsilon p) > u_i(x^*)$ for all i in the first set above. (For this, recall that $\nabla u_i(x^*) \cdot p$ is the derivative of u_i in the direction p:

$$\nabla u_i(x^*) \cdot p \quad = \quad \lim_{\epsilon \to 0} \frac{u_i(x^* + \epsilon p) - u_i(x^*)}{\epsilon} > 0.)$$

So for small enough $\epsilon > 0$, this is true for a majority of individuals, so $(x^* + \epsilon p)P_{SM}(\theta)x^*$, a contradiction. Therefore, $\nabla u_i(x^*) = 0$ for some $i \in N$. (Note that, for this conclusion, we used only that $\mathcal{D}(F)$ was strong.) Now suppose that (ii) is false, i.e., there is some $p \in \mathbb{R}^d$ such that

$$|\{j \neq i \mid \exists \alpha > 0 : \nabla u_j(x^*) = \alpha p\}| \quad \neq \quad |\{j \neq i \mid \exists \alpha > 0 : \nabla u_j(x^*) = -\alpha p\}|.$$

Call the first group G_1 and the second G_2. Without loss of generality, suppose $|G_1| > |G_2|$. If $G_1 \cup G_2 \cup \{i\} = N$, then define the vector $q = 0$. Otherwise, $d \geq 2$ and we can define q as a non-zero vector such that

$$\{j \in N \mid \nabla u_j(x^*) \cdot q = 0\} \quad = \quad G_1 \cup G_2 \cup \{i\}.$$

(Why?) See the figure below.

Figure 2.14: A helpful picture

Therefore,

$$\{j \in N \mid \nabla u_j(x^*) \cdot q > 0\} \cup \{j \in N \mid \nabla u_j(x^*) \cdot q < 0\} \quad \subseteq \quad N \backslash (G_1 \cup G_2 \cup \{i\}).$$

Call the first group on the lefthand side H_1 and the second H_2, and without loss of generality suppose $|H_1| \geq |H_2|$. Note that $|G_1| + |H_1| \geq \frac{n+1}{2}$ (why?), so $G_1 \cup H_1$ is a majority. If we can find some $y \in X$ such that $u_j(y) > u_j(x^*)$ for all $j \in G_1 \cup H_1$, we have $yP_{SM}(\theta)x^*$, a contradiction. First, pick $\delta > 0$ small enough that

$$\nabla u_j(x^*) \cdot (q + \delta p) > 0$$

for all $j \in H_1$. Also, note that

$$\nabla u_j(x^*) \cdot (q + \delta p) \; = \; \delta \nabla u_j(x^*) \cdot p \; > \; 0$$

for all $j \in G_1$. Now we can use an argument similar to that used before. Picking $\epsilon > 0$ small enough, we have $x^* + \epsilon(q + \delta p) \in X$ and

$$u_j(x^* + \epsilon(q + \delta p)) \quad > \quad u_j(x^*)$$

for all $j \in G_1 \cup H_1$. Setting $y = x^* + \epsilon(q + \delta p)$, we have $yP_{SM}(\theta)x^*$, a contradiction, and the proof is complete. ∎

Does Plott's theorem need n odd? Yes. See Figure 2.15.

Note that, in Figure 2.15(a), x^* satisfies (i) but violates (ii). In Figure 2.15(b), x^* satisfies (ii) but violates (i). Is it possible to find an example in which x^* violates both (i) and (ii)? As the following proposition makes clear, no.

Figure 2.15: n even

Proposition 2.39 *Assume $X \subseteq \mathbb{R}^d$ and:*

1. *each $(P_i(\theta), R_i(\theta))$ is represented by some differentiable u_i;*
2. *$x^* \in intX$;*
3. *for all $i, j \in N$, $\nabla u_i(x^*) = \nabla u_j(x^*) = 0$ implies $i = j$.*

If $x^* \in C_{SM}(\theta)$, then either (i) there is some $i \in N$ with $\nabla u_i(x^*) = 0$, or (ii) both $\nabla u_i(x^*) \neq 0$ for all $i \in N$ and radial symmetry holds at x^*.

Proof: I will sketch the proof. Let $x^* \in C_{SM}(\theta)$, and suppose neither (i) nor (ii) hold. Add another individual, say $n+1$, giving us an odd number, with ideal point at x^*. Thus, $\nabla u_{n+1}(x^*) = 0$. Since (i) does not hold, no other individual has a zero gradient at x^*. By Proposition 2.38, then, if x^* is a majority core point for this augmented society, radial symmetry must hold. Since (ii) does not hold, x^* is not a majority core point, i.e., there is a majority, say G, of individuals who strictly prefer some $y \in X$ to x^*. By construction, $G \subseteq N$. Returning to the original model, we have $yP_{SM}(\theta)x^*$, a contradiction. ∎

Does Plott's theorem need no shared critical points at x^*? Yes. See Figure 2.16 for an example.

Figure 2.16: Shared critical points at x^*

Can you see how you could extend Plott's theorem to account for shared critical points? (We would need to define a weaker type of radial symmetry.)

Though the first part of Plott's theorem generalizes to all strong PARs, the second part clearly does not: consider F_i, the PAR defined as the preference relation of individual i. Indeed, the second part does not generalize even to non-dictatorial PARs with $\mathcal{D}(F)$ strong. Let $n = 5$, and let $\mathcal{D}(F)$ consist of every group containing individual 1 and some other $i \neq 1$ and the group $\{2, 3, 4, 5\}$. Consider preferences as in Figure 2.17.

Figure 2.17: Non-dictatorial PAR, $\mathcal{D}(F)$ strong

Note that, under the conditions of Plott's theorem, x^* must be a critical point of some individual's utility function. To see that x^* need not be anyone's ideal point, consider the example of Figure 2.10, following Proposition 2.33. Note that, as that proposition implies, some individual has non-convex preferences in the example.

At any rate, we have a strong necessary condition for x to be a majority rule core point when n is odd. The next result addresses sufficiency. We say u_i is *pseudo-concave* if it is differentiable and, for all $x, y \in \text{int}X$, $u_i(y) > u_i(x)$ implies $\nabla u_i(x) \cdot (y - x) > 0$. Note that pseudo-concavity implies differentiability and quasi-concavity (why?), but not the converse. (Think about $u_i(x) = x^3$.)

Proposition 2.40 (Plott) *Assume $X \subseteq \mathbb{R}^d$ and each $(P_i(\theta), R_i(\theta))$ has a pseudo-concave utility representation u_i. Let $x \in \text{int}X$. If radial symmetry holds at x, then $x \in C_{SM}(\theta)$.*

Proof: Suppose radial symmetry holds at x but $yP_{SM}(\theta)x$ for some y. Then, by pseudo-concavity,

$$P(y, x|\theta) \quad \subseteq \quad \{i \in N \mid \nabla u_i(x) \cdot (y - x) > 0\}.$$

The latter group is a majority. But then $\{i \in N \mid \nabla u_i(x) \cdot (y - x) < 0\}$ is a minority, contradicting radial symmetry. ∎

Thus, radial symmetry essentially characterizes the majority rule core points. Note that, unlike Proposition 2.38, Proposition 2.40 does not assume n is odd. How does this sufficiency result generalize beyond majority rule? I'll let you think about it.

Majority Core Usually Empty

Plott's Theorem tells us that, when n is odd (etc.), it is very difficult to find majority rule core points. Indeed, not only are core points rare, but, even when they exist, they are "fragile." That is, a minor perturbation of individual utility functions can remove an alternative from the core. We will prove formally that, for our metric on Θ, the set of states in which the majority core is empty is "generic," defined as follows. A set $\Theta' \subseteq \Theta$ is *dense* if, for every $\theta \in \Theta$ and every $\epsilon > 0$, $B_\epsilon(\theta) \cap \Theta' \neq \emptyset$. In other words, clos$\Theta' = \Theta$. Then we say Θ' is *generic* if it contains an open, dense subset of Θ.

Equivalently, Θ' is generic if its complement $\Theta \setminus \Theta'$ is contained in a closed set with empty interior, i.e., clos$(\Theta \setminus \Theta')$ contains no open set.

We first need the following result on upper hemicontinuity of the core. The result assumes continuity of the mapping R_F from states to social weak preference relations. Note that, from Proposition 2.7, we have conditions under which this is satisfied by all simple PARs.

Proposition 2.41 *Assume $X \subseteq \mathbb{R}^d$ is compact, $PR(\Theta) \subseteq \mathcal{R}_c^N$, and $R_F(\Theta) \subseteq \mathcal{R}_c$. Let $R_F \colon \Theta \to \mathcal{R}_c$ be continuous at θ. Then $C_F(\theta)$ is closed and C_F is upper hemicontinuous at θ.*

Proof: Take any open set U with $C_F(\theta) \subseteq U$, and note that $C_F(\theta) = M(R_F(\theta))$. We know from Proposition 1.49 that M is upper hemicontinuous on $\mathcal{R}_c$, so there exists an open set W containing $R_F(\theta)$ such that, for all $R \in W$, $M(R) \subseteq U$. Assuming R_F is continuous at θ, there is an open set V containing θ such that, for all $\theta' \in V$, $R_F(\theta') \in W$. Thus, $C_F(\theta') \subseteq U$ for all $\theta' \in V$, as required. I'll let you prove that $C_F(\theta)$ is closed. ■

The next proposition establishes weak conditions under which the set of states for which the core is nonempty is closed in the metric ρ. Given a PAR F, let $\Theta_F \subseteq \Theta$ be the subset of states for which $C_F(\theta) \neq \emptyset$.

Proposition 2.42 *Assume $X \subseteq \mathbb{R}^d$ is compact, $PR(\Theta) \subseteq \mathcal{R}_c^N$, and $R_F(\Theta) \subseteq \mathcal{R}_c$. Let $R_F \colon \Theta \to \mathcal{R}_c$ be continuous. Then Θ_F is closed.*

Proof: Consider a sequence $\{\theta^m\}$ of states and a state θ such that $\theta^m \in \Theta_F$ for all m and $\theta^m \to \theta$. For each m, let x^m be an element of the core of the mth state, i.e., $x^m \in C_F(\theta^m)$. Since X is compact, the sequence $\{x^m\}$ has a subsequence that converges to some $x \in X$. Without loss of generality, index this subsequence by m. By Proposition 2.41, $x \in C_F(\theta)$, so $\theta \in \Theta_F$. ■

Of course, I eventually want to argue that the set $\Theta \setminus \Theta_F$ is generic. Since Θ_F is closed, this amounts to showing that Θ_F contains no open set. The next lemma, on nowhere denseness of unions of closed sets, will be helpful. It is taken from McKelvey (1979).

Lemma 2.7 *Let $X \subseteq \mathbb{R}^d$ be endowed with the relative topology, and let $Y \subseteq X$. If $Y = \bigcup_{i=1}^n Y_i$, where each $Y_i \subseteq X$ is closed and has empty interior, then Y is closed and has empty interior.*

In contrast to Proposition 2.42, the next result will depend on our specification of the domain of preferences. Different authors have considered different restrictions. We will use a subset of those in Plott's theorem. We say u_i is *strictly pseudo-concave* if it is differentiable and, for all $x, y \in \text{int}X$ with $x \neq y$, $u_i(y) \geq u_i(x)$ implies $\nabla u_i(x) \cdot (y - x) > 0$. Note that, if (P_i, R_i) has a strictly pseudo-concave representation, then it is strictly convex (but not conversely).

Let $\mathcal{R}_{sp} \subseteq \mathcal{R}_c^N$ denote the set of preference profiles such that each (P_i, R_i) is representable by some strictly pseudo-concave $u_i \colon \mathbb{R}^d \to \mathbb{R}$, with no shared critical points.[5] Let $\Theta_{SM}^\circ \subseteq \mathcal{R}_c^N$

[5] Did you notice that I'm assuming u_i is defined over all of $\mathbb{R}^d$? Can you see where I use this in the next proof? This is cheating a little, but what the heck.

denote the set of profiles with a majority core point interior to X. For simplicity, I will show that Θ°_{SM}, rather than Θ_{SM}, contains no open sets.

Proposition 2.43 *Assume n is odd and $X \subseteq \mathbb{R}^d$ is compact and convex with $d \geq 2$, $PR(\Theta) = \mathcal{R}^N_{sp}$, and $R_{SM}(\Theta) \subseteq \mathcal{R}_c$. Then $\Theta \setminus \Theta^\circ_{SM}$ is dense in Θ with respect to the Hausdorff metric.*

Proof: Take any $\theta \in \Theta^\circ_{SM}$, where u_i is a strictly pseudo-concave representation of $(P_i(\theta), R_i(\theta))$. Thus, $C_{SM}(\theta)$ contains an element interior to X, say x^*. Note that, with our assumptions that n is odd and that each u_i is strictly pseudo-concave, Proposition 2.33 implies that $C_{SM}(\theta) = \{\tilde{x}^i\}$ for some $i \in N$. Thus, x^* is the unique core point and $x^* = \tilde{x}^i$. Furthermore, radial symmetry must hold at x^*. Let $C_{jh} \subseteq X$ be the alternatives that are "Pareto optimal for j and h," i.e.,

$$C_{jh} \quad = \quad \{x \in X \mid \forall y \in X : xR_j(\theta)y \text{ or } xR_h(\theta)y\},$$

and note that the set of interior points at which radial symmetry holds is contained in $C = \bigcup_{j,h \neq i} C_{jh}$. Furthermore, each C_{jh} is closed and, since $d \geq 2$, is nowhere dense. (Why?) By Lemma 2.7, C is nowhere dense. For arbitrary integer m, we can therefore find an interior point z^m within $1/m$ of x^* such that $z^m \notin C$. Define the utility function $u_i^m(x) = u_i(x - z^m + x^*)$. This is a new strictly pseudo-concave representation such that $\nabla u_i^m(z^m) = 0$, i.e., the unique maximizer of u_i^m is z^m. Let $\theta^m \in \Theta$ be a state such that u_i^m is a representation of $R_i(\theta^m)$ and the preferences of other individuals are unchanged. The weak preference $R_i(\theta^m)$ can be made arbitrarily close to $R_i(\theta)$ in by a suitable choice of m. (Check this.) I claim that, for high enough m, $C_{SM}(\theta^m) \cap \text{int}X = \emptyset$. To see this, suppose there exists $x^m \in C_{SM}(\theta^m)$ for infinitely many m. By Proposition 2.38, we have $x^m = \tilde{x}^j$ for some $j \in N$. By construction, the radial symmetry condition does not hold at z^m, so we must have $j \neq i$. Since N is finite, we may assume, going to a subsequence if needed, that $x^m = \tilde{x}^j$ for all m. By Proposition 2.41, C_{SM} has closed graph, so $\tilde{x}^j \in C_{SM}(\theta)$. But this contradicts our assumption that $C_{SM}(\theta) = \{x^*\}$. Thus, we can perturb θ to obtain an empty core, as required. ■

2.8 The Weak Top Cycle

Recall the definition of the weak top cycle set: given a dual pair (P, R), it is $TOP(R) = M(T_R)$. That is, it is the maximal elements of the transitive closure of weak preference. We have already seen this idea in the context of collective choice with the definition of the PAR F_T, the transitive closure of weak simple majority preference. The core of this PAR is $C_T(\theta) = TOP(R_{SM}(\theta))$, where we use the social weak preference $R_{SM}(\theta)$ for R.

We can apply this definition more generally to any PAR. Given a PAR F, define the *weak top cycle of F at θ* by $WTOP_F(\theta) = TOP(R_F(\theta))$.

Properties of the Weak Top Cycle

Recall some properties of the weak top cycle applied to social preferences, where $W_F(\theta)$ is the union of $P_F(\theta)$-dominant $R_F(\theta)$-cycles.

- $WTOP_F(\theta) = C_F(\theta) \cup W_F(\theta)$.
- The weak top cycle is externally $P_F(\theta)$-stable and, if $Y \subseteq X$ is externally $P_F(\theta)$-stable, then $WTOP_F(\theta) \subseteq Y$.
- Assume $C_F(\theta) \neq \emptyset$. If $(P_F(\theta), R_F(\theta))$ satisfies thin indifference, then $WTOP_F(\theta) = C_F(\theta)$.
- If X is compact and $R_F(\theta)$ is upper semicontinuous, then $WTOP_F(\theta) \neq \emptyset$.

The last item listed above is significant: whereas we have seen that $C_{SM}(\theta)$ is typically empty, the weak top cycle for majority rule will be nonempty under very general conditions. Indeed, this is true for all simple rules.

Proposition 2.44 *Assume $X \subseteq \mathbb{R}^d$ is compact and each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous. Let F be a simple PAR. Then $WTOP_F(\theta) \neq \emptyset$.*

The second-to-last item above relies on the property of thin indifference of social preferences, a sufficient condition for which was presented in Proposition 2.32. This yields the following result.

Proposition 2.45 *Assume $X \subseteq \mathbb{R}^d$ and $PR(\theta)$ satisfies LSWP. If F is strong and $C_F(\theta) \neq \emptyset$, then $WTOP_F(\theta) = C_F(\theta)$.*

As a consequence, we have conditions under which the weak top cycle and core (when nonempty) coincide. In particular, the conditions are fulfilled, under convexity assumptions, by the simple majority PAR when n is odd.

The above result on nonemptiness of the weak top cycle is stated for simple PARs, which have the required continuity properties. But relative majority rule and relative Pareto are obvious examples of PARs that do not generally generate continuous social preferences. Such PARs are captured through the following proposition. It shows that, under fairly weak conditions on individual preferences, weak decisiveness is enough.

Proposition 2.46 *Assume that $X \subseteq \mathbb{R}^d$, that each $(P_i(\theta), R_i(\theta))$ is continuous, and that $PR(\theta)$ satisfies LSWP. Let F be weakly decisive. Then $WTOP_F(\theta) \neq \emptyset$.*

Proof: From Proposition 2.6, it follows that $(P_{\mathcal{D}(F)}(\theta), R_{\mathcal{D}(F)}(\theta))$ is continuous, and then Proposition 1.40 implies that the corresponding strong top cycle is nonempty, i.e.,

$$TOP(P_{\mathcal{D}(F)}(\theta)) \neq \emptyset.$$

I claim that this strong top cycle is a subset of the weak top cycle of F at θ, i.e.,

$$TOP(P_{\mathcal{D}(F)}(\theta)) \subseteq TOP(R_F(\theta)).$$

The claim is clearly true, by Proposition 2.30, if $C_{\mathcal{D}(F)}(\theta) \neq \emptyset$, so suppose $C_{\mathcal{D}(F)}(\theta) = \emptyset$. Take any $x \in TOP(P_{\mathcal{D}(F)}(\theta))$. Proposition 1.34 implies that x is an element of some undominated $P_{\mathcal{D}(F)}(\theta)$-cycle, say Z, and by definition there must exist $y, z \in Z$ such that $xP_{\mathcal{D}(F)}(\theta)yP_{\mathcal{D}(F)}(\theta)z$. To see that $x \in TOP(R_F(\theta))$, take any $w \in X$. If $w \in Z$, then, because $x \in Z$, we have $xT_{P_{\mathcal{D}(F)}(\theta)}w$, which implies $xT_{R_F(\theta)}w$. Suppose $w \notin Z$, so by definition we have $yR_{\mathcal{D}(F)}(\theta)w$. Since $(P_{\mathcal{D}(F)}(\theta), R_{\mathcal{D}(F)}(\theta))$ is continuous, $P^{-1}_{\mathcal{D}(F)}(x|\theta) \cap P_{\mathcal{D}(F)}(z|\theta)$ is an open set containing y. By LSWP, there exists $y' \in P^{-1}_{\mathcal{D}(F)}(x|\theta) \cap P_{\mathcal{D}(F)}(z|\theta)$ such that $R(y, w|\theta) = P(y', w|\theta)$ and $P(w, y|\theta) = P(w, y'|\theta)$. (Do you see why?) We then have $y'R_F(\theta)w$: otherwise we would have $wP_F(\theta)y'$, and then weak decisiveness would imply that $P(w, y'|\theta) = R(w, y'|\theta) \in \mathcal{D}(F)$, but then we would have $wP_{\mathcal{D}(F)}(\theta)y'$, implying $w \in Z$, a contradiction. Thus, $xP_{\mathcal{D}(F)}(\theta)y'R_F(\theta)w$, which implies $xT_{R_F(\theta)}w$, as required. ∎

We can say more if we add the assumption that F is strong. In fact, it is enough if social preferences are upper semicontinuous and satisfy thin indifference: then the weak top cycles of F and $F_{\mathcal{D}(F)}$ differ only at points of closure. Note that the proposition actually gives more: the weak top cycles of these PARs essentially coincide with the strong top cycles.

Proposition 2.47 *Assume $X \subseteq \mathbb{R}^d$. If $(P_{\mathcal{D}(F)}(\theta), R_{\mathcal{D}(F)}(\theta))$ is upper semicontinuous and satisfies thin indifference, then*

$$TOP(P_{\mathcal{D}(F)}(\theta)) \subseteq TOP(P_F(\theta)) \subseteq WTOP_F(\theta) \subseteq WTOP_{\mathcal{D}(F)}(\theta) \subseteq \mathit{clos}TOP(P_{\mathcal{D}(F)}(\theta)).$$

Proof: The result follows easily from Proposition 2.31 when $C_{\mathcal{D}(F)}(\theta) \neq \emptyset$. Otherwise, recall that, by Propositions 1.38 and 1.39, we have

$$D(T_{P_{\mathcal{D}(F)}(\theta)}) = TOP(P_{\mathcal{D}(F)}(\theta)) \subseteq WTOP_{\mathcal{D}(F)}(\theta) \subseteq \text{clos}TOP(P_{\mathcal{D}(F)}(\theta)).$$

Since

$$D(T_{P_{\mathcal{D}(F)}(\theta)}) \subseteq D(T_{P_F(\theta)}) \subseteq TOP(P_F(\theta)) \subseteq WTOP_F(\theta) \subseteq WTOP_{\mathcal{D}(F)}(\theta),$$

the desired inclusions follow. ∎

The weak top cycle will be nonempty even when the core is empty, presenting an alternative approach to placing bounds on collective choices. As we will see, however, the bounds imposed by weak top cycle are typically not very restrictive: when the core is nonempty, $WTOP_F(\theta)$ can be quite large.

Extent of the Weak Top Cycle

For simplicity, we now assume $X = \mathbb{R}^d$ and consider a state θ in which individual preferences are Euclidean, i.e., for each $i \in N$, there exists $\tilde{x}^i \in X$ such that $xR_i(\theta)y$ if and only if $||x - \tilde{x}^i|| \leq ||y - \tilde{x}^i||$.

We note that, if $(P_i(\theta), R_i(\theta))$ is Euclidean, then the following statements are equivalent to $xP_i(\theta)y$:

$$\begin{aligned}
||x - \tilde{x}^i|| &< ||y - \tilde{x}^i|| \\
||x - \tilde{x}^i||^2 &< ||y - \tilde{x}^i||^2 \\
(x - \tilde{x}^i) \cdot (x - \tilde{x}^i) &< (y - \tilde{x}^i) \cdot (y - \tilde{x}^i) \\
x \cdot x - 2x \cdot \tilde{x}^i + \tilde{x}^i \cdot \tilde{x}^i &< y \cdot y - 2y \cdot \tilde{x}^i + \tilde{x}^i \cdot \tilde{x}^i \\
x \cdot x - y \cdot y &< 2\tilde{x}^i \cdot (x - y) \\
\frac{1}{2}(x + y) \cdot (x - y) &< \tilde{x}^i \cdot (x - y).
\end{aligned}$$

Geometrically, think of $x - y$ as the gradient of a linear function. The left hand side is the value of the function at $\frac{1}{2}(x + y)$. The inequality says $\tilde{x}^i$ must be on the $(x - y)$-side of the hyperplane with normal $x - y$ through $\frac{1}{2}(x + y)$, which we refer to as the "bisecting hyperplane" for x and y.

We can use this observation to derive a useful characterization of the core. We call a vector $y \in \mathbb{R}^d$ a *direction* if $||y|| = 1$. Given a direction $y \in \mathbb{R}^d$ and $c \in \mathbb{R}$, let

$$\begin{aligned}
H_{y,c} &= \{x \in \mathbb{R}^d \mid x \cdot y = c\} \\
H^+_{y,c} &= \{x \in \mathbb{R}^d \mid x \cdot y > c\} \\
\overline{H}^+_{y,c} &= \{x \in \mathbb{R}^d \mid x \cdot y \geq c\}
\end{aligned}$$

Define $H^-_{y,c}$ and $\overline{H}^-_{y,c}$ similarly.

Given a PAR F, we say $H_{y,c}$ is a F*-median hyperplane* at θ if

$$\{i \in N \mid \tilde{x}^i \in H^+_{y,c}\} \notin \mathcal{D}(F) \text{ and } \{i \in N \mid \tilde{x}^i \in H^-_{y,c}\} \notin \mathcal{D}(F).$$

Equivalently, when F is simple, $H_{y,c}$ is a F-median hyperplane at θ if

$$\{i \in N \mid \tilde{x}^i \in \overline{H}^+_{y,c}\} \in \mathcal{B}(F) \text{ and } \{i \in N \mid \tilde{x}^i \in \overline{H}^-_{y,c}\} \in \mathcal{B}(F).$$

Let $\mathbf{M}_F(\theta)$ denote the set of F-median hyperplanes at θ. Note that, for all $y \in \mathbb{R}^d$, there exists $c \in \mathbb{R}$ such that $H_{y,c} \in \mathbf{M}_F(\theta)$. (Why?)

When F is strong, for every direction $y \in \mathbb{R}^d$, there is exactly one F-median hyperplane with normal y. (Why?) Generally, assuming F satisfies Pareto, there will be a compact interval, $[c', c'']$, such that $H_{y,c}$ is a F-median hyperplane with normal y if and only if $c \in [c', c'']$. Denote this interval by C_y (suppressing dependence on F and θ). Note that $C_y = -C_{-y}$ (because $y \cdot x = c$ if and only if $(-y) \cdot x = -c$).

We say x^* is a *total F-median* at θ if, for all $y \in \mathbb{R}^d$, $H_{y,x^* \cdot y} \in \mathbf{M}_F(\theta)$. In other words, x^* is a total F-median at θ if and only if, for all $y \in \mathbb{R}^d$, $y \cdot x^* \in C_y$. Let $TM_F(\theta)$ denote the set of total F-medians at θ.

The next proposition is well-known and straightforward to prove.

Proposition 2.48 *Assume that $X \subseteq \mathbb{R}^d$, that each $(P_i(\theta), R_i(\theta))$ is Euclidean, and that $x^* \in intX$.*

1. *For every PAR F, $x^* \in C_F(\theta)$ implies $x^* \in TM_F(\theta)$.*
2. *If F is weakly decisive, then $x^* \in TM_F(\theta)$ implies $x^* \in C_F(\theta)$.*

Proof: For part 1, suppose $x^* \in C_F(\theta)$ but is not a total median. Then there exists $y \in \mathbb{R}^d$ such that $H_{y,x^* \cdot y} \notin \mathbf{M}_F(\theta)$. Without loss of generality, suppose the individuals with ideal points in $H^+_{y,x^* \cdot y}$ are decisive. For $\epsilon > 0$, let $z_\epsilon = x^* + \epsilon y$. Note that, for each $i \in N$, z_ϵ is closer than x^* to $\tilde{x}^i$ if and only if

$$\frac{1}{2}(z_\epsilon + x^*) \cdot (z_\epsilon - x^*) \quad < \quad \tilde{x}^i \cdot (z_\epsilon - x^*),$$

or equivalently,

$$x^* \cdot y + \frac{\epsilon}{2} y \cdot y \quad < \quad \tilde{x}^i \cdot y. \tag{2.1}$$

If i is such that $\tilde{x}^i \cdot y > x^* \cdot y$, then this inequality holds for small enough ϵ. Therefore, choosing $\epsilon > 0$ small enough that $z_\epsilon \in X$ and (2.1) holds for all i with $\tilde{x}^i \in H^+_{y,x^* \cdot y}$, we have $z_\epsilon P_F(\theta) x^*$, a contradiction. I'll leave part 2 for you. ∎

Before proceeding, I give a lemma on solutions to systems of weak inequalities: if $\{y_j \cdot x \geq c_j \mid j = 1, 2, \ldots, k\}$ is a system with no solution, and if, after deleting any one of the inequalities, the remaining ones admit a solution, then the remaining ones can be solved with equality.

Let $K = \{1, 2, \ldots, k\}$, and let J denote a subset of K. For $y_1, y_2, \ldots, y_k \in \mathbb{R}^d$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$, let $S(J) = \{x \in \mathbb{R}^d \mid \forall j \in J : y_j \cdot x \geq c_j\}$ denote the set of solutions to the J inequalities. Let $S^{=}(J) = \{x \in \mathbb{R}^d \mid \forall j \in J : y_j \cdot x = c_j\}$ denote the set of solutions which solve the inequalities with equality.

Lemma 2.8 *Let $y_1, y_2, \ldots, y_k \in \mathbb{R}^d$, and let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Assume that $S(K) = \emptyset$ and that, for all $j \in K$, $S(K \setminus \{j\}) \neq \emptyset$. Then, for all $j \in K$, $S^{=}(K \setminus \{j\}) \neq \emptyset$.*

Proof: Suppose that, for some $j \in K$, $S^{=}(K \setminus \{j\}) = \emptyset$. Note that the maximization problem

$$\max_{x \in \mathbb{R}^d} y_j \cdot x$$

$$\text{s.t. } y_\ell \cdot x \geq c_\ell \ (\ell \neq j)$$

has a solution, say $\hat{x}$. Furthermore, note that $y_j \cdot \hat{x} < c_j$. (Why?) I claim that $y_\ell \cdot \hat{x} = c_\ell$ for all $\ell \neq j$. If not, some constraint, say h, is not binding at $\hat{x}$, i.e., $y_h \cdot \hat{x} > c_h$. Then $S(K \setminus \{h\}) = \emptyset$: if $x \in S(K \setminus \{h\})$, then, for small enough $\epsilon > 0$, $x_\epsilon = (1-\epsilon)\hat{x} + \epsilon x$ solves the constraints in the above maximization problem (why?), and, since $y_j \cdot x \geq c_j > y_j \cdot \hat{x}$, we have

$$y_j \cdot x_\epsilon \ = \ (1-\epsilon) y_j \hat{x} + \epsilon y_j \cdot x \ = \ y_j \cdot \hat{x} + \epsilon (y_j \cdot x - y_j \cdot \hat{x}) \ > \ y_j \cdot \hat{x},$$

a contradiction. But $S(K \setminus \{h\}) \neq \emptyset$ by assumption, a contradiction. ∎

The next result was proved by McKelvey (1976) for the special case of simple majority rule. I give a version of it that covers all simple PARs and other PARs such as relative majority rule. Referred to as a "chaos" theorem, it shows that the weak top cycle (in fact, the strong top cycle, if you look at the proof carefully) exhausts the entire space of alternatives. The result assumes $X = \mathbb{R}^d$ and Euclidean preferences. McKelvey (1979) drops those assumptions, adding, however, the assumption that F is strong.

Proposition 2.49 (McKelvey) *Assume $X = \mathbb{R}^d$ and each $(P_i(\theta), R_i(\theta))$ is Euclidean. Let F be weakly decisive. If $C_F(\theta) = \emptyset$, then $WTOP_F(\theta) = X$.*

Proof: Assume $C_F(\theta) = \emptyset$, and take any $v, w \in X$. We will construct a sequence $t_0, t_1, \ldots,$ $t_k \in X$ such that $v R_F(\theta) t_k R_F(\theta) t_{k-1} \cdots R_F(\theta) t_0 = w$. Since the core is empty and F is weakly decisive, F must satisfy Pareto, so C_y is a compact interval for every direction y. Let $c_y = \min C_y$, and define $H_y = H_{y,c_y}$. Since the core is empty, Proposition 2.48 implies that there is no total median, so

$$\bigcap_{\substack{y \in \mathbb{R}^d \\ ||y|| = 1}} \overline{H}_y^{+} \ = \ \emptyset.$$

This follows because, for all $y \in \mathbb{R}^d$, $x \in \overline{H}_y^+ \cap \overline{H}_{-y}^+$ implies

$$\min C_y \;=\; c_y \;\leq\; x \cdot y \;\leq\; -c_{-y} \;=\; -\min C_{-y} \;=\; \max C_y.$$

By Helly's theorem (Rockafellar (1970), Corollary 21.3.2), there are $d{+}1$ vectors $y_0, y_1, \ldots, y_d$ with no common solution x to

$$x \cdot y_i \quad \geq \quad c_{y_i},$$

$i = 0, 1, \ldots, d$. Let $c_i = c_{y_i}$. Pick a *minimal* subset with this property, and, without loss of generality, assume that subset is $y_0, y_1, \ldots, y_p$. So, for all $j = 0, 1, \ldots, p$, there *is* a solution to

$$x \cdot y_i \quad \geq \quad c_i \;\; (i \neq j).$$

Dropping the jth inequality, Lemma 2.8 allows us solve the remaining p inequalities with equality. Let z_j solve them, i.e.,

$$z_j \cdot y_i \quad = \quad c_i \;\; (i \neq j), \tag{2.2}$$

and note that $z_j \cdot y_j < c_j$. Set $z = \frac{1}{p+1} \sum_{j=0}^{p} z_j$, and assume without loss of generality that $z = 0$. Then $c_i > 0$ for all $i = 0, 1, \ldots, p$, since

$$0 \;=\; (p+1) z \cdot y_i \;=\; \sum_{j=0}^{p} z_j \cdot y_i \;=\; p c_i + z_i \cdot y_i \;<\; (p+1) c_i,$$

where the third equality uses (2.2), and the inequality uses $z_i \cdot y_i < c_i$. See Figure 2.18.

Figure 2.18: What's going on?

Note that, for all $x \in \mathbb{R}^d$, there is some $i = 0, 1, \ldots, p$ such that $x \cdot y_i \leq 0$ (or else we could solve all $p+1$ inequalities). Given any t_k, we define t_{k+1} as follows: pick any y_i such that $t_k \cdot y_i < 0$ (there is at least one), and let

$$t_{k+1} \quad = \quad t_k + [c_i - 2 y_i \cdot t_k] y_i.$$

Note that the projection of t_k onto the subspace $H_{y_i,0}$ is equal to the projection of t_{k+1} onto this subspace. Denote it by s. Furthermore, you can check that

$$\begin{aligned} t_k &= s + (y_i \cdot t_k) y_i \\ t_{k+1} &= s - (y_i \cdot t_k + c_i) y_i. \end{aligned}$$

What's going on here? See Figure 2.19.

Figure 2.19: What's going on?

That $t_{k+1}P_F(\theta)t_k$ follows since, letting $c' = \frac{1}{2}(t_k + t_{k+1}) \cdot y_i$, we have

$$\begin{aligned} c' &= (t_k + (\frac{c_i}{2} - y_i \cdot t_k)y_i) \cdot y_i \\ &= \frac{c_i}{2} \\ &< c_i, \end{aligned}$$

which implies $c' \notin C_{y_i}$, and so $H_{y_i,c'} \notin \mathbf{M}_F(\theta)$. Also, by the Pythagorean theorem, we have

$$\begin{aligned} ||t_{k+1}||^2 &= (c_i - t_k \cdot y_i)^2 + ||s||^2 \\ ||t_k||^2 &= (t_k \cdot y_i)^2 + ||s||^2. \end{aligned}$$

Thus,

$$\begin{aligned} ||t_{k+1}||^2 - ||t_k||^2 &= c_i^2 - 2(t_k \cdot y_i)c_i \\ &\geq c_i^2. \end{aligned}$$

In particular, the squared norm of t_{k+1} is larger than that of t_k by a discrete amount. By taking k high enough, we may construct a sequence satisfying

$$t_k P_F(\theta) t_{k-1} \cdots P_F(\theta) t_2 P_F(\theta) t_1 P_F(\theta) t_0 = w$$

and such that $||t_k||^2$, and therefore $||t_k||$, is arbitrarily large. Making $||t_k||$ big enough, we can ensure $vR_F(\theta)t_k$ (how?), as required. ∎

The following corollary is (nearly) immediate. You should try to prove it.

Corollary 2.8 *Assume $X = \mathbb{R}^d$ and $PR(\Theta) \supseteq \mathbf{E}$. If $d + 1 \geq \mathcal{N}(F)$, then $WTOP_F$ is not upper hemicontinuous.*

What about Non-Euclidean Preferences?

McKelvey's later theorem is not proved constructively. There, he takes a strong simple rule F and an arbitrary alternative $x \in X$, and he considers the set of alternatives that are socially preferred to x directly or indirectly, in a finite sequence of social preference steps. Letting $P = P_F(\theta)$, this is the set

$$T_P(x) = \{y \in X \mid yT_Px\}.$$

Assuming individual preferences are continuous, $P_F(z|\theta)$ is open for all $z \in X$, and therefore $T_P(x)$ is open. (Why?) Suppose the boundary of $T_P(x)$ is nonempty. We can show that, for each y and z in the boundary of $T_P(x)$, it must be that $yI_F(\theta)z$. (Same question.) By a clever argument using a "preference diversity" assumption, McKelvey shows that there is some individual j such that the boundary (well, "frontier") of $T_P(x)$ lies within an indifference class of j. Finally, take $i \neq j$ and alternatives z and w in the boundary of $T_P(x)$ such that zI_iw. Then, under weak conditions, there must be another individual k such that zI_kw.

We conclude that the necessary conditions that must be met by boundary points of $T_P(x)$ are very restrictive. So restrictive that, typically, the boundary of $T_P(x)$ will be empty. This essentially means, assuming X is connected, that either the core is nonempty or the set $T_P(x)$ comprises almost all of X. So, if the core is empty, then $WTOP_F(\theta)$ is almost the entire space. In fact, McKelvey shows this for the *strong* top cycle. This is also referred to as a "chaos" result.

I will present a version of McKelvey's general chaos result that is stronger than his in the sense that we do not need F to be strong or even simple — merely weakly decisive — but weaker in that we prove the expansiveness of the weak top cycle, rather than the strong. In this latter respect, it is similar to the conjunction of Austen-Smith and Banks's Theorems 6.5 and 6.6. It differs from theirs in that they restrict attention to simple PARs (and, for simplicity, they impose $X = \mathbb{R}^d$ and convexity of individual preferences). Results on expansiveness of the weak top cycle do, however, extend to the strong top cycle when thin social indifference holds: by Proposition 2.47, the strong top cycle contains the weak within its closure. All of these results rely on a rather complicated restriction on decisive coalitions related to a certain individual, which I simplify somewhat.

Before proceeding to that result, I will establish a much weaker conclusion without these ancillary assumptions. In the following, let $T_F(\theta) = T_{R_F(\theta)}$ be the transitive closure of social weak preference at θ, and let $T_F(x|\theta)$ be the upper section of $T_F(\theta)$ at x.

Proposition 2.50 *Assume $X \subseteq \mathbb{R}^d$ and each $(P_i(\theta), R_i(\theta))$ is continuous. Let F be weakly decisive. For all $x \in X$ and all $y, z \in bdT_F(x|\theta)$, there exists $i \in N$ such that $yI_i(\theta)z$. Furthermore, if $Z \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta) \neq \emptyset$ or $(Z \cap P_i(z|\theta)) \setminus T_F(x|\theta) \neq \emptyset$ for every open set Z around z, then there exists $j \neq i$ such that $yI_j(\theta)z$.*

Proof: First suppose there is no $i \in N$ such that $yI_i(\theta)z$, i.e., $P(y,z|\theta) \cup P(z,y|\theta) = N$. By continuity of individual preferences, and since $y, z \in \mathrm{bd}T_F(x|\theta)$, there exist $y' \in T_F(x|\theta)$ and $z' \in X \setminus T_F(x|\theta)$ such that $P(y',z'|\theta) = P(y,z|\theta)$ and $P(z',y'|\theta) = P(z,y|\theta)$. I claim that $z'R_F(\theta)y'$. If not, then $y'P_F(\theta)z'$, and, by weak decisiveness, $R(y',z'|\theta) = P(y,z|\theta) \in \mathcal{D}(F)$. Now take $\hat{y} \in X \setminus T_F(x|\theta)$ and $\hat{z} \in T_F(x|\theta)$ such that $P(\hat{y},\hat{z}|\theta) = P(y,z|\theta)$ and $P(\hat{z},\hat{y}|\theta) = P(z,y|\theta)$. Then $P(\hat{y},\hat{z}|\theta) \in \mathcal{D}(F)$, which implies $\hat{y}P_F(\theta)\hat{z}$. But then $\hat{y} \in T_F(x|\theta)$, a contradiction. Therefore, there exists $i \in N$ such that $yI_i(\theta)z$. Suppose there is no $j \neq i$ such that $yI_j(\theta)z$, i.e., $P(y,z|\theta) \cup P(z,y|\theta) = N \setminus \{i\}$. By continuity, there exists an open set V around z such that $V \subseteq P_j(y|\theta)$ for all $j \in P(z,y|\theta)$ and $V \subseteq P_j^{-1}(y|\theta)$ for all $j \in P(y,z|\theta)$. If $(Z \cap P_i(z|\theta)) \setminus T_F(x|\theta) \neq \emptyset$ for every open set Z around z, then there exists $z' \in (V \cap P_i(z|\theta)) \setminus T_F(x|\theta)$. By continuity, since $z'P_i(\theta)y$ and $y \in \text{bd}T_F(x|\theta)$, there exists $y' \in T_F(x|\theta)$ such that $P(y',z'|\theta) = P(y,z'|\theta) = P(y,z|\theta)$ and $P(z',y'|\theta) = P(z',y|\theta) = P(z,y|\theta) \cup \{i\}$. See Figure 2.20.

Figure 2.20: What's going on?

I claim that $z'R_F(\theta)y'$. Otherwise, we have $y'P_F(\theta)z'$, and, by weak decisiveness, $R(y',z'|\theta) = P(y,z|\theta) \in \mathcal{D}(F)$. By continuity and $y,z \in \text{bd}T_F(x|\theta)$, there exist $\hat{y} \in X \setminus T_F(x|\theta)$ and $\hat{z} \in T_F(x|\theta)$ such that $P(\hat{y},\hat{z}|\theta) = P(y,z|\theta)$, implying $\hat{y}P_F(\theta)\hat{z}$. But then $\hat{y} \in T_F(x|\theta)$, a contradiction. Therefore, $z'R_F(\theta)y'$, as claimed. But then $z' \in T_F(x|\theta)$, a contradiction. If $Z \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta) \neq \emptyset$ for every open set Z around z, then there exists $z'' \in V \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta)$. By continuity, since $yP_i(\theta)z''$ and $y \in \text{bd}T_F(x|\theta)$, there exists $y'' \in T_F(x|\theta)$ such that $P(y'',z''|\theta) = P(y,z''|\theta) = P(y,z|\theta) \cup \{i\}$ and $P(z'',y''|\theta) = P(z,y|\theta)$. See Figure 2.20. I claim that $y''R_F(\theta)z''$. Otherwise, we have $z''P_F(\theta)y''$, and, by weak decisiveness, $R(z'',y''|\theta) = P(z,y|\theta) \in \mathcal{D}(F)$. This leads to a contradiction as above, establishing the claim. But then $y'' \in T_F(x|\theta)$, a contradiction. ∎

Thus, if $T_F(x|\theta)$ has two distinct boundary points, then some individual must be indifferent between them. And if that individual's indifference curve crosses the boundary of $T_F(x|\theta)$ at those points in a "transversal" way, then some other individual's indifference curve also must cross at those two points. This observation can have very restrictive implications when $|\text{bd}T_F(x|\theta)| \geq 2$ for some $x \in X$, depicted in Figure 2.21, so we would expect that, for all $x \in X$, $|\text{bd}T_F(x|\theta)| = 0$ or 1.

Figure 2.21: Restrictive implications

But the condition $|\text{bd}T_F(x|\theta)| = 1$ is possible, essentially, only if the core is a singleton and is externally $P_F(\theta)$-stable. (There are some other special cases where this can happen. Can you think of any?) And when X is connected, $|\text{bd}T_F(x|\theta)| = 0$ is only possible if either $\text{clos}T_F(x|\theta) = \emptyset$ or $\text{clos}\overline{T_F(x|\theta)} = \emptyset$. The former cannot be, since $x \in T_F(x|\theta)$, and the latter implies $T_F(x|\theta) = X$. Thus, as the next proposition states, the weak top cycle can be quite large.

Proposition 2.51 *Assume $X \subseteq \mathbb{R}^d$ is connected and $\text{bd}T_F(x|\theta) = \emptyset$ for all $x \in X$. Then $WTOP_F(\theta) = X$.*

The next proposition amplifies these conclusions for a special case of the one-dimensional spatial model. The result applies to simple majority rule with n even, i.e., $q = (n/2) + 1$, and it shows that the weak top cycle may be very large, even when the core is nonempty.

Proposition 2.52 *Assume $X = \mathbb{R}$, each $(P_i(\theta), R_i(\theta))$ is Euclidean, and, for all distinct $i, j \in N$, $\tilde{x}^i \neq \tilde{x}^j$. Let F_q be a quota rule with $q > (n+1)/2$. Then $WTOP_q(\theta) = \mathbb{R}$.*

Proof: By Proposition 2.51, it suffices to show that $|\mathrm{bd}T_q(x|\theta)| = 0$ for all $x \in X$. Recall that, by Proposition 2.27, we have $C_q(\theta) = [\tilde{x}^{i_1}, \tilde{x}^{i_2}]$ for some $i_1, i_2 \in N$. By the assumption that ideal points are distinct and $q > (n+1)/2$, it must be that $\tilde{x}^{i_1} < \tilde{x}^{i_2}$. Thus, for each $x \in X$, we see that $T_q(x|\theta)$ contains a non-degenerate interval $[\tilde{x}^{i_1}, \tilde{x}^{i_2}]$. There are two cases to rule out: $|\mathrm{bd}T_q(x|\theta)| = 1$ and $|\mathrm{bd}T_q(x|\theta)| \geq 2$. In the first case, let $\mathrm{bd}T_q(x|\theta) = \{y\}$, and assume without loss of generality that there exists $z \in T_q(x|\theta)$ such that $y < z$. Pick $w \in X$ high enough that $z < w$ and $P(y, w|\theta) = N$. If $w \in T_q(x|\theta)$, then there is an open set Y around y such that $Y \subseteq P_q(w|\theta) \subseteq T_q(x|\theta)$, contradicting our assumption that $y \in \mathrm{bd}T_q(x|\theta)$. Thus, $w \notin T_q(x|\theta)$. But then $\mathrm{clos}(T_q(x|\theta)) \cap [z, w] \neq \emptyset$ and $\mathrm{clos}(\overline{T_q(x|\theta)}) \cap [z, w] \neq \emptyset$. Since $[z, w]$ is connected, we must have $\mathrm{bd}T_q(x|\theta) \cap [z, w] \neq \emptyset$, implying $|\mathrm{bd}T_q(x|\theta)| > 1$, a contradiction. In the second case, take distinct $y, z \in \mathrm{bd}T_q(x|\theta)$. Assume without loss of generality that $|y - \tilde{x}^{i_1}| < |z - \tilde{x}^{i_1}|$. It follows that there is some open set Y around y and some open set Z around z such that this inequality holds for all $y' \in Y$ in place of y and all z' in place of z. Then our assumption of Euclidean preferences implies that $y' R_q(\theta) z'$ for all $y' \in Y$ and all $z' \in Z$. (Why?) Since $Z \cap T_q(x|\theta) \neq \emptyset$, we have $y \in \mathrm{int}T_q(x|\theta)$, a contradiction. ∎

The conclusion of Proposition 2.52 obviously does not carry over the the case where F is strong, in which case the core is a singleton and coincides with the weak top cycle set. That it does not extend generally to non-Euclidean preference profiles can be seen in the following example.

Figure 2.22: Bounded weak top cycle

We now proceed to a version of McKelvey's general chaos theorem.

To formalize McKelvey's main assumption on individual preferences, define the *frontier* of $Y \subseteq X$, denoted $\mathrm{fr}Y$, as

$$\mathrm{fr}Y \quad = \quad \mathrm{clos}(\mathrm{int}Y) \cap \mathrm{clos}(\mathrm{int}\overline{Y}),$$

where closures and interiors are with respect to the relative topology on X. Note that $\mathrm{fr}Y$ is a closed subset of the boundary of Y, and that $\mathrm{fr}Y = \mathrm{fr}(\mathrm{int}Y) = \mathrm{fr}(\mathrm{clos}Y)$. (Right?) Also, when X is connected, $\mathrm{fr}Y \neq \emptyset$ if and only if $\mathrm{int}Y$ and $\mathrm{int}\overline{Y}$ are nonempty. (Same question.)

We say a profile $((P_1, R_1), \ldots, (P_n, R_n))$ satisfies *preference diversity* if, for every $Y \subseteq X$, every $x \in \mathrm{fr}Y$, and all distinct $i, j \in N$, $I_i(x) \cap I_j(x)$ has empty interior in the relative topology on $\mathrm{fr}Y$. This assumption guarantees that no two voters have preferences with indifference curves that exactly coincide locally.

We're ready for McKelvey's theorem. It is similar to Proposition 2.50 in that it establishes precisely matched individual indifferences across the frontier of $T_F(x|\theta)$. One difference is that, because it relies on preference diversity, the proposition focuses on the frontier, rather than the boundary, of this set. This leads to a more important difference: McKelvey's theorem establishes that the frontier of $T_F(x|\theta)$ is entirely contained within an indifference class of a single individual, significantly pinning down the way in which those individual indifferences need to be matched.

Proposition 2.53 (McKelvey) *Assume that $X \subseteq \mathbb{R}^d$, that each $(P_i(\theta), R_i(\theta))$ is continuous, and that $PR(\theta)$ satisfies preference diversity. Let F be weakly decisive. Take any $x \in X$.*

1. *There is some $j \in N$ such that, for all $y \in \mathrm{fr}T_F(x|\theta)$, $\mathrm{fr}T_F(x|\theta) \subseteq I_j(y|\theta)$.*

Furthermore, take any $y, z \in \mathrm{fr}T_F(x|\theta)$ and any $i \neq j$ such that $yI_i(\theta)z$.

2. *If $P(y, z|\theta) \cup \{j\} \notin \mathcal{D}(F)$ and $(Z \cap P_i(z|\theta)) \setminus T_F(x|\theta) \neq \emptyset$ for every open set Z around z, then there exists $k \in N$, with $k \neq i, j$, such that $yI_k(\theta)z$.*

3. *If $P(z, y|\theta) \cup \{j\} \notin \mathcal{D}(F)$ and $Z \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta) \neq \emptyset$ for every open set Z around z, then there exists $k \in N$, with $k \neq i, j$, such that $yI_k(\theta)z$.*

Proof: From Proposition 2.50 and the fact that $\mathrm{fr}T_F(x|\theta) \subseteq \mathrm{bd}T_F(x|\theta)$, it follows that, for all $y, z \in \mathrm{fr}T_F(x|\theta)$, there exists $i \in N$ such that $yI_i(\theta)z$. Therefore, given any $y \in \mathrm{fr}T_F(x|\theta)$, we have $\mathrm{fr}T_F(x|\theta) \subseteq \bigcup_{i \in N}[I_i(y|\theta) \cap \mathrm{fr}T_F(x|\theta)]$. Since each $I_i(y|\theta) \cap \mathrm{fr}T_F(x|\theta)$ is closed in the relative topology on $\mathrm{fr}T_F(x|\theta)$, and since $\mathrm{fr}T_F(x|\theta)$ is nonempty by supposition, Lemma 2.7 implies that some $I_j(y|\theta) \cap \mathrm{fr}T_F(x|\theta)$ contains a nonempty open set in the relative topology on $\mathrm{fr}T_F(x|\theta)$. I claim that $\mathrm{fr}T_F(x|\theta) \subseteq I_j(y|\theta)$. Suppose there exists $z \in \mathrm{fr}(T_F(x|\theta)) \setminus I_j(y|\theta)$. Note that, by preference diversity, for all $i \neq j$, $I_i(z|\theta) \cap I_j(y|\theta)$ has empty interior in $\mathrm{fr}(T_F(x|\theta))$. Therefore, by Lemma 2.7, there exists $w \in I_j(y|\theta) \cap \mathrm{fr}(T_F(x|\theta))$ such that $wI_i(\theta)z$ for no $i \neq j$, a contradiction. This establishes the claim and proves part 1 of the proposition. Now take any $y, z \in \mathrm{fr}T_F(x|\theta)$ and any $i \neq j$ such that $yI_i(\theta)z$ and such that there is no $k \neq i, j$ for which $yI_k(\theta)z$. By continuity, there is an open set V around z such that $V \subseteq P_k(y|\theta)$ for all $k \in P(z, y|\theta)$ and $V \subseteq P_k^{-1}(y|\theta)$ for all $k \in P(y, z|\theta)$. Suppose that $P(y, z|\theta) \cup \{j\} \notin \mathcal{D}(F)$ and $(Z \cap P_i(z|\theta)) \setminus T_F(x|\theta) \neq \emptyset$ for every open set Z around z. Take $z' \in (V \cap P_i(z|\theta)) \setminus T_F(x|\theta)$. By continuity, since $z'P_i(\theta)y$ and $y \in \mathrm{fr}T_F(x|\theta)$, there exists $y' \in T_F(x|\theta)$ such that $R(y', z'|\theta) \subseteq P(y, z|\theta) \cup \{j\}$, which is not decisive. Since F is weakly decisive, therefore, we have $z'R_F(\theta)y'$, implying $z' \in T_F(x|\theta)$, a contradiction. This proves part 2 of the proposition. Suppose that $P(z, y|\theta) \cup \{j\} \notin \mathcal{D}(F)$ and $Z \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta) \neq \emptyset$ for every open set Z around z. Take $z'' \in V \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta)$.

By continuity, since $yP_i(\theta)z''$ and $y \in \mathrm{fr}T_F(x|\theta)$, there exists $y'' \in X \setminus T_F(x|\theta)$ such that $R(z'', y''|\theta) \subseteq P(z, y|\theta) \cup \{j\}$, which is not decisive. Since F is weakly decisive, therefore, we have $y'' R_F(\theta) z''$, implying $y'' \in T_F(x|\theta)$, a contradiction. Therefore, there exists $k \in N$, with $k \neq i, j$, such that $yI_k(\theta)z$, proving part 3 of the proposition. ∎

How restrictive are the assumptions in parts 2 and 3 of Proposition 2.53? To gauge this, suppose that the frontier of $T_F(x|\theta)$ is contained in an indifference class of individual j, and take two alternatives, y and z, in the frontier, and let i be indifferent between them. Moreover, let i's indifference curve through y and z cut the frontier in a "transversal" way, so $(Z \cap P_i(z|\theta)) \setminus T_F(x|\theta) \neq \emptyset$ and $Z \cap P_i^{-1}(z|\theta) \cap T_F(x|\theta) \neq \emptyset$ for every open set Z around z. Beyond these assumptions, the existence of another individual who is indifferent between y and z relies on the following:

$$P(y, z|\theta) \cup \{j\} \notin \mathcal{D}(F) \quad \text{or} \quad P(z, y|\theta) \cup \{j\} \notin \mathcal{D}(F).$$

Note that, if F is anonymous, then this condition is automatically satisfied: if $P(y, z|\theta) \cup \{j\}$ is decisive, then, by anonymity, $P(y, z|\theta) \cup \{i\}$ must be decisive, so $P(z, y|\theta) \cup \{j\}$ cannot be. Thus, it should be considered a weak condition.

McKelvey argues that the conclusions of Proposition 2.53 are very restrictive, especially when $d \geq 3$, so that we would expect them to hold only in the trivial case of $\mathrm{fr}T_F(x|\theta) = \emptyset$. When X is connected, this is only possible if $\mathrm{int}T_F(x|\theta) = \emptyset$ or $\mathrm{int}\overline{T_F(x|\theta)} = \emptyset$. The former is only possible if $\mathrm{int}R_F(x|\theta) = \emptyset$, which is only possible if x is in the core *and* the core has empty interior. Thus, for all $x \in X$, we may very well have $\mathrm{int}\overline{T_F(x|\theta)} = \emptyset$, i.e., $T_F(x|\theta)$ is dense in X.

The next proposition, which parallels Proposition 2.51, shows that the weak top cycle can be quite large as a consequence. It differs from the earlier result in that its main assumption concerns the frontier of $T_F(x|\theta)$ rather than the boundary. To compensate, we must add the extra assumption that the core is empty and accept a slightly weaker conclusion.

Proposition 2.54 *Assume $X \subseteq \mathbb{R}^d$ is connected, $(P_F(\theta), R_F(\theta))$ is continuous, fr$T_F(x|\theta) = \emptyset$ for all $x \in X$, and $C_F(\theta) = \emptyset$. For all $x \in X$, if $P_F^{-1}(x|\theta) \neq \emptyset$, then $x \in WTOP_F(\theta)$.*

Proof: Fix an arbitrary $x \in X$. Since X is connected, $\mathrm{fr}T_F(x|\theta) = \emptyset$ implies that either $\mathrm{int}T_F(x|\theta) = \emptyset$ or $\mathrm{int}\overline{T_F(x|\theta)} = \emptyset$. Suppose the former. Because $P_F(x|\theta)$ is open and $P_F(x|\theta) \subseteq \mathrm{int}T_F(x|\theta)$, it must be that $P_F(x|\theta) = \emptyset$, i.e., $x \in C_F(\theta)$, a contradiction. Thus, $\mathrm{int}T_F(x|\theta) = \emptyset$, or equivalently, $\mathrm{clos}(T_F(x|\theta)) = X$. To prove the proposition, assume $P_F^{-1}(x|\theta) \neq \emptyset$, and take any $y \in X$. By the above argument, $\mathrm{clos}(T_F(y|\theta)) = X$, so $P_F^{-1}(x|\theta) \cap \mathrm{clos}(T_F(y|\theta)) \neq \emptyset$, so $P_F^{-1}(x|\theta) \cap T_F(y|\theta) \neq \emptyset$. This implies that $x \in T_F(y|\theta)$, and we conclude that $x \in WTOP_F(\theta)$. ∎

Proposition 2.53 is stated differently than McKelvey's theorem, in that after part 1 he assumes that i is either "not a dummy voter" or is "as strong as j." I will argue that, if $yI_i(\theta)z$ and $yI_j(\theta)z$ as in the proposition, then it is weaker to assume

$$P(y,z|\theta) \cup \{j\} \notin \mathcal{D}(F) \quad \text{or} \quad P(z,y|\theta) \cup \{j\} \notin \mathcal{D}(F),$$

meaning my formulation of the assumptions of the proposition is somewhat weaker than his. I will use the phrase "j is not a swing voter" to denote this condition.

We say i is a *dummy voter for y over z at θ* if, for every $G \subseteq N \setminus \{i\}$ with

$$P(y,z|\theta) \setminus \{i\} \ \subseteq \ G \ \subseteq \ R(y,z|\theta),$$

we have

$$G \cup \{i\} \notin \mathcal{D}(F) \ \text{ or } \ G \in \mathcal{D}(F).$$

So i is not a dummy voter if there exists $G \subseteq N \setminus \{i\}$ with $P(y,z|\theta) \setminus \{i\} \subseteq G \subseteq R(y,z|\theta)$ such that $G \cup \{i\} \in \mathcal{D}(F)$ and $G \notin \mathcal{D}(F)$.

We say i is *as strong as j for y over z at θ* if, for every $G \subseteq N \setminus \{i\}$ with

$$P(y,z|\theta) \setminus \{i,j\} \ \subseteq \ G \ \subseteq \ R(y,z|\theta),$$

we have

$$G \cup \{j\} \notin \mathcal{D}(F) \ \text{ or } \ G \cup \{i\} \in \mathcal{D}(F).$$

So, no matter how we divide up indifferent voters, if j makes the group voting for y over z decisive, then so does i.

To compare the formulations, assume $yI_i(\theta)z$ and $yI_j(\theta)z$, and suppose i is not a dummy voter for y over z. Thus, there exists $G \subseteq N \setminus \{i\}$ with $P(y,z|\theta) \subseteq G \subseteq R(y,z|\theta)$ such that $G \cup \{i\} \in \mathcal{D}(F)$ and $G \notin \mathcal{D}(F)$. If j is contained in G, then $P(y,z|\theta) \cup \{j\} \notin \mathcal{D}(F)$. If j is not contained in G, then $G \cup \{i\} \subseteq R(y,z|\theta) \setminus \{j\} \in \mathcal{D}(F)$, and this implies $P(z,y|\theta) \cup \{j\} \notin \mathcal{D}(F)$. Thus, j is not a swing voter, as claimed.

Now suppose i is as strong as $j \neq i$ for y over z. Suppose that $xI_i(\theta)y$ and $xI_j(\theta)y$, and take G as in the definition such that $G \cup \{j\} \notin \mathcal{D}(F)$ or $G \cup \{i\} \in \mathcal{D}(F)$. In the first case, we have $P(y,z|\theta) \cup \{j\} \notin \mathcal{D}(F)$. In the second case, we have $P(z,y|\theta) \cup \{j\} \notin \mathcal{D}(F)$. Thus, again, j is not a swing voter.

Lastly, recall that McKelvey assumed F was strong and proved the expansiveness of the strong top cycle set. With strictly convex individual preferences, that would give us thin social indifference. McKelvey doesn't assume strict convexity, but his other assumption do, indeed, deliver thin social indifference. Then the conclusion from Proposition 2.54 that the weak top cycle can be quite large carries over, by Proposition 2.47, to the strong top cycle set. In fact, it carries over to the strong top cycle of any weakly decisive PAR F such that $F_{\mathcal{D}(F)}$ generates thin social indifference at θ.

In sum, we have seen that the weak top cycle set is of little use in bounding collective choices in the presence of an empty core: while nonempty quite generally, the top cycle set is much too big, typically encompassing the entire space of alternatives.

We are left in something of a quandary. The majority core is rarely nonempty in multiple dimensions. When it is, arbitrarily small perturbations of individual preferences can annihilate it. In such cases, the weak top cycle gives no indication that collective choices need have any relation to the core, i.e., predictions based on the core may fail to be robust to even the smallest misspecifications of individual preferences.

2.9 The Weak Uncovered Set

Recall the definition of the weak uncovered set: given a dual pair (P, R) it is $UC(R) = M(C_R)$, where the covering relation of R, C_R, is defined as

$$xC_Ry \quad \Leftrightarrow \quad R(x) \subseteq P(y).$$

We can apply this definition when, in particular, R and P are weak and strict social preferences. Given a collective choice environment and PAR F, define the *weak uncovered set of F at θ* by $WUC_F(\theta) = UC(R_F(\theta))$.

Properties of the Weak Uncovered Set

Recall some properties of the weak uncovered set applied to social preferences.

- $x \in WUC_F(\theta)$ if and only if, for all $y \in X$, there exists $z \in X$ such that $xR_F(\theta)zR_F(\theta)y$. (Two-step principle)
- Assume $C_F(\theta) \neq \emptyset$. If $(P_F(\theta), R_F(\theta))$ satisfies thin indifference, then $WUC_F(\theta) = C_F(\theta)$.
- If X is compact and $(P_F(\theta), R_F(\theta))$ is upper semicontinuous, then $WUC_F(\theta) \neq \emptyset$.
- $C_F(\theta) \subseteq WUC_F(\theta) \subseteq WTOP_F(\theta)$.

Thus, the weak uncovered set shares some of the properties of the weak top cycle. In particular, the fourth item listed above establishes nonemptiness of the weak uncovered set under the same minimal continuity condition used for the weak top cycle. Thus, we have the following proposition, which actually implies our nonemptiness result for the weak top cycle.

Proposition 2.55 *Assume $X \subseteq \mathbb{R}^d$ is compact and each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous. Let F be a simple PAR. Then $WUC_F(\theta) \neq \emptyset$.*

The third item above yields the following result, also paralleling the results for the weak top cycle.

Proposition 2.56 *Assume $X \subseteq \mathbb{R}^d$ and $PR(\theta)$ satisfies LSWP. If F is strong and $C_F(\theta) \neq \emptyset$, then $WUC_F(\theta) = C_F(\theta)$.*

I want to digress briefly on the conditions under which we get equivalence of the core (when nonempty) and the weak uncovered set: F is strong and LSWP is satisfied. If either of those assumptions is dropped, then the equivalence need not hold. For example, assume $n = 3$, differentiable, concave utility representations, as in Figure 2.23.

Figure 2.23: Core $\neq$ weak uncovered set

Here, $\nabla u_2(z) = 0$ for all z on the straight line segment above. The Plott conditions are satisfied at x^*, and imagine that it is the only such point. Therefore, the unique majority core alternative is x^*. But note that y is weakly majority preferred to x^*, so $y \in WUC_{SM}(\theta)$.

Why do we need F to be strong? Consider an environment with $n = 2$, $X = \mathbb{R}$, Euclidean preferences with $\tilde{x}^1 = -\epsilon$ and $\tilde{x}^2 = \epsilon$. Here, the simple majority core is $[-\epsilon, \epsilon]$, and it can be checked that the weak uncovered set is $[-2\epsilon, 2\epsilon]$. This is in stark contrast to the weak top cycle, which, by Proposition 2.52, is equal to the entire real line! Note also that, as ϵ goes to zero, the weak uncovered set collapses to $\{0\}$ in a continuous way.

Nonemptiness of the weak uncovered set for PARs such as relative majority rule is a difficult issue. The next proposition gives some sufficient conditions for nonemptiness, including a new one on social preferences. We say a dual pair (P, R) is *rich* if, for all $x, y \in X$, $R(x) = R(y)$ implies $x = y$. This would appear to be a very weak condition, but I do not know of any general sufficient conditions for it.

Proposition 2.57 *Assume that $X \subseteq \mathbb{R}^d$ is compact, that $(P_{\mathcal{D}(F)}(\theta), R_{\mathcal{D}(F)}(\theta))$ is upper semicontinuous and satisfies thin indifference, and that either $C_{\mathcal{D}(F)}(\theta) \neq \emptyset$ or $(P_{\mathcal{D}(F)}(\theta), R_{\mathcal{D}(F)}(\theta))$ is rich. Then*

$$\emptyset \neq WUC_F(\theta) \subseteq WUC_{\mathcal{D}(F)}(\theta).$$

Proof: The result is trivial, by Proposition 2.31, if $C_{\mathcal{D}(F)}(\theta) \neq \emptyset$, so suppose otherwise. Define xBy if and only if $xR_{\mathcal{D}(F)}(\theta)y$ and $P_{\mathcal{D}(F)}(x|\theta) \subseteq R_{\mathcal{D}(F)}(y|\theta)$. Note that, by thin indifference and upper semi-continuity, xBy implies $R_{\mathcal{D}(F)}(x|\theta) \subseteq R_{\mathcal{D}(F)}(y|\theta)$. In the proof of Proposition 1.48, we showed that there exists a B-maximal element, say x^*. Suppose that, for some $x \in X$, we have $xC_{R_F(\theta)}x^*$, i.e., $R_F(x|\theta) \subseteq P_F(x^*|\theta)$. A consequence of $P_{\mathcal{D}(F)}(x|\theta) \subseteq R_F(x|\theta)$ and $P_F(x^*|\theta) \subseteq R_{\mathcal{D}(F)}(x^*|\theta)$ is, then, that xBx^*. Since x^* is B-maximal, we therefore have x^*Bx. This implies $R_{\mathcal{D}(F)}(x|\theta) = R_{\mathcal{D}(F)}(x^*|\theta)$, and then, by richness, we have $x = x^*$, contradicting $xP_F(\theta)x^*$. Therefore, there is no such x, and we conclude that $x^* \in WUC_F(\theta)$. Now take $x \in WUC_F(\theta)$ and $y \in X$, and suppose that $yC_{R_{\mathcal{D}(F)}(\theta)}x$, i.e., $R_{\mathcal{D}(F)}(y|\theta) \subseteq P_{\mathcal{D}(F)}(\theta)$. This implies that $R_F(y|\theta) \subseteq P_F(x|\theta)$, or equivalently, $yC_{R_F(\theta)}x$. But then x is not in the weak uncovered set $WUC_F(\theta)$, a contradiction. Therefore, there is no such y, and we conclude that x is in the weak uncovered set $WUC_{\mathcal{D}(F)}(\theta)$, as required. ∎

If you're interested in continuity of the strong uncovered set, there is no general result. You can show, however, that, assuming that F is strong and LSWP is satisfied, the strong uncovered set is upper hemicontinuous at any state with a nonempty core. This follows because, at such a state, the core is an externally $P_F(\theta)$-stable singleton and will coincide with the weak and strong uncovered sets. Then upper hemicontinuity of the strong uncovered set follows from Proposition 2.59.

The above example and the last property of the weak uncovered set listed above suggest that it may be a more useful bound on social choices than the weak top cycle. The next proposition immediately tells us somewhat more generally that the uncovered set won't suffer from the above "chaos" results for the weak top cycle: it says that, under "reasonable" conditions, the elements of the weak uncovered set are always Pareto optimal. Note that the result does not rely on any structure of the set of alternatives or individual preferences.

Proposition 2.58 *Assume that $X \subseteq \mathbb{R}^d$, that $(P_F(\theta), R_F(\theta))$ satisfies thin indifference, and that each $(P_i(\theta), R_i(\theta))$ is upper semicontinuous. Let F be weakly decisive and satisfy Pareto. Then $WUC_F(\theta) \subseteq C_{SP}(\theta)$.*

Proof: Take $x, y \in X$ and $\theta \in \Theta$ such that $xP_{SP}(\theta)y$, i.e., $P(x, y|\theta) = N$. Letting $R = R_F(\theta)$, I claim that xC_Ry, i.e., $R_F(x|\theta) \subseteq P_F(y|\theta)$. Take $z \in R_F(x|\theta)$. If $z = x$, then $zP_F(\theta)y$ by Pareto. If $z \neq x$, then, by thin indifference, there is a sequence $\{z_m\}$ in $P_F(x|\theta)$ converging to z. By weak decisiveness, $G_m = R(z_m, x|\theta) \in \mathcal{D}(F)$ for all m. Since N is finite, we must have $G = G_m$ for some decisive G and infinitely many m. By upper semicontinuity, we have $G \subseteq R(z, x|\theta)$. By transitivity of individual preferences, $R(z, x|\theta) \subseteq P(z, y|\theta)$, so $zP_F(\theta)y$, as required. Therefore, $y \notin WUC_F(\theta)$. ∎

The requirement of thin social indifference in the above proposition is not usually imposed, because most analyses consider a notion of covering somewhere between C_R and C_P. This gives us a smaller uncovered set than $WUC_F(\theta)$, one that is always contained in the Pareto optimals. You can check that the strong uncovered set is contained in the Pareto optimals generally. When considering the weak uncovered set, we need the additional assumption. Let $n = 4$, $X = \{x, y, z\}$, and consider a state with individual preferences as follows.

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$	$R_4(\theta)$
x	x	z	z
y	y	x	x
z	z	y	y

Here, $xP_{SP}(\theta)y$ but $y \in WUC_{SM}(\theta)$.

Like the nonemptiness of the weak uncovered set, the upper bound of the Pareto optimals, in Proposition 2.58, relies on continuity properties of social preferences. It is easily checked, given an arbitrary PAR F, that $WUC_F(\theta) \subseteq WUC_{\mathcal{D}(F)}(\theta)$. Thus, as long as $F_{\mathcal{D}(F)}$ generates social preferences satisfying thin indifference at θ, as with relative majority rule when n is odd, at least Proposition 2.58 holds more generally.

Though Proposition 2.58 gives us an upper bound on the weak uncovered set, the set of Pareto optimal alternatives can be quite large and does not give us very tight bounds on collective choices — even when the core is "close" to being nonempty (in the sense of the Hausdorff metric defined above.) The next result, on continuity of the weak uncovered set correspondence solves that problem. The result assumes continuity of the mapping $R_F \colon \Theta \to \mathcal{R}_c$. Note that, from Proposition 2.7, we have conditions under which this is satisfied by all simple PARs.

Proposition 2.59 *Assume $X \subseteq \mathbb{R}^d$ is compact, $PR(\Theta) \subseteq \mathcal{R}_c^N$, and $R_F(\Theta) \subseteq \mathcal{R}_c$. Let $R_F \colon \Theta \to \mathcal{R}_c$ be continuous at θ. Then $WUC_F(\theta)$ is closed and WUC_F is upper hemicontinuous at θ.*

Proof: Take any open set U with $WUC_F(\theta) \subseteq U$, and note that $WUC_F(\theta) = UC(R_F(\theta))$. We know from Proposition 1.50 that UC is upper hemicontinuous on $\mathcal{R}_c$, so there exists an open set W containing $R_F(\theta)$ such that, for all $R \in W$, $UC(R) \subseteq U$. Assuming R_F is continuous at θ, there is an open set V containing θ such that, for all $\theta' \in V$, $R_F(\theta') \in W$. Thus, $WUC_F(\theta') \subseteq U$ for all $\theta' \in V$, as required. I'll let you prove that $WUC_F(\theta)$ is closed. ∎

Proposition 2.59 tells us that, when we move slightly from a state with a nonempty core, the weak uncovered set does not "blow up" discontinuously. We next give more global bounds for the weak uncovered set under the preference restriction of Euclidean domain.

The Yolk

We can tighten bounds on the weak uncovered set, if we focus on a strong PAR F and on Euclidean preferences. Let $\mathbf{B}_F(\theta)$ consist of the closed balls $B \subseteq \mathbb{R}^d$ that intersect F-median hyperplanes with all normals: that is, $B \in \mathbf{B}_F(\theta)$ if and only if

$$\forall y \in \mathbb{R}^d, \forall c \in \mathbb{R} : H_{y,c} \in \mathbf{M}_F(\theta) \text{ implies } B \cap H_{y,c} \neq \emptyset.$$

Let $B_F(\theta)$ denote the unique element of $\mathbf{B}_F(\theta)$ with smallest radius. We call $B_F(\theta)$ the *yolk* at θ. Let it have center y^* and radius r. Thus, $B_F(\theta) = B_r(y^*)$. See Figure 2.24.

(Uniqueness of this ball is not obvious. It seems to be widely accepted, though I do not know of a published or circulated proof. I had a proof sketched out last year, but I got too busy to write it up. So I think the claim is true.)

Figure 2.24: The yolk

To see that the yolk is generally well-defined only for strong rules, consider Figure 2.25. Here there are four individuals with Euclidean preferences, ideal points arranged in a rectangle. Drawn are two elements of $\mathbf{B}_F(\theta)$, assuming simple majority rule, of minimal radius.

Figure 2.25: No yolk

It turns out that uniqueness of the yolk is not such a critical issue. Proposition 2.61, below, will hold for any ball intersecting F-median hyperplanes in all directions. (Check this.)

Clearly the yolk can be a relatively small set. In fact, it becomes arbitrarily small as we move ideal points closer to radial symmetry. In the following propositions, I assume Euclidean domain, so we can identify each state with a profile $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ of ideal points. Convergence in the Hausdorff metric is just convergence of ideal points in the usual Euclidean metric.

Proposition 2.60 *Assume that $X \subseteq \mathbb{R}^d$ convex, that $PR(\Theta) \subseteq \mathbf{E}$, and that F is strong. If $C_F(\theta) \neq \emptyset$, then $B_F(\theta) = C_F(\theta)$ and B_F is upper hemicontinuous at θ.*

Proof: Take a state $\theta = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ in which the core is nonempty, so, under the conditions of the proposition, it consists of a single alternative, say x^*, that is strictly socially preferred to every other alternative. Thus, $B_F(\theta) = \{x^*\}$. Take a sequence of states $\{\theta_m\}$, where $\theta_m = (\tilde{x}_1^m, \tilde{x}_2^m, \ldots, \tilde{x}_n^m)$, converging to θ. To simplify notation, let

$B = B_F(\theta)$, and let $B_m = B_F(\theta_m)$. I will show that, for all $\epsilon > 0$, there exists m' such that, for all $m \geq m'$, $B_m \subseteq B_\epsilon(x^*)$. If not, then we can find $\epsilon > 0$ and a subsequence $\{w_m\}$, also indexed by m for convenience, such that $w_m \in B_m \setminus B_\epsilon(x^*)$ for all m. I claim that, for each m, there exists a F-median hyperplane H_m, relative to the ideal points $(\tilde{x}_1^m, \tilde{x}_2^m, \ldots, \tilde{x}_n^m)$, such that $H_m \cap B_{\epsilon/3}(x^*) = \emptyset$. To prove this, suppose not. Then $B_{\epsilon/3}(x^*)$ intersects all F-median hyperplanes, so B_m must have radius less than $\epsilon/3$. Then the F-median hyperplane with normal $w_m - x^*$ does not intersect $B_{\epsilon/3}(x^*)$ (why?), a contradiction. This establishes the claim. Denote the mth hyperplane by H_m, and let y_m and c_m satisfy $H_m = H_{y_m, c_m}$, $||y_m|| = 1$, and $c_m - y_m \cdot x^* \geq \epsilon/3$. Note that $\{(y_m, c_m)\}$ has a convergent subsequence (why?), also indexed by m. Denote the limit by (y, c). Furthermore, note that $c - y \cdot x^* \geq \epsilon/3$, i.e., $y \cdot x^* \leq c - \epsilon/3$. Since N is finite and each H_m is a F-median hyperplane, there is a blocking group G whose ideal points are on the y_m-side of infinitely many hyperplanes H_m. That is,

$$y_m \cdot \tilde{x}_i^m \geq c_m$$

for each $i \in G$. Therefore, $y \cdot \tilde{x}_i \geq c$ for all $i \in G$. Define $z = x^* + (c - y \cdot x^*)y$, i.e., z is the vector in $H_{y,c}$ closest to x^* in the y-direction. In particular, $z \neq x^*$. For each $i \in G$,

$$\begin{aligned} \frac{1}{2}(x^* + z) \cdot y &= \frac{1}{2}(x^* \cdot y + z \cdot y) \\ &= \frac{1}{2}(x^* \cdot y + c) \\ &\leq c \\ &\leq \tilde{x}_i \cdot y, \end{aligned}$$

where the first inequality uses $x^* \cdot y \leq c - \epsilon/3$ and the second uses $y \cdot \tilde{x}_i \geq c$. Therefore, $zR_F(\theta)x^*$, a contradiction. ∎

More generally, the yolk will be some "centrally" located subset of alternatives. To establish bounds on the weak uncovered set using the yolk, we first establish a result on the alternatives weakly socially preferred to y^*.

Lemma 2.9 *Assume that $X \subseteq \mathbb{R}^d$ convex, that each $(P_i(\theta), R_i(\theta))$ is Euclidean, and that F is strong. Then $R_F(y^*|\theta) \subseteq B_{2r}(y^*)$.*

Proof: Take any $x \notin B_{2r}(y^*)$ and note that the hyperplane through $(1/2)(x + y^*)$ with normal $x - y^*$ does not intersect $B_F(\theta)$. Therefore, it is not a F-median hyperplane (nor is there one closer to x), so $y^* P_F(\theta)x$, establishing the claim. ∎

And now another result on the strict upper sections of alternatives far from y^*.

Lemma 2.10 *Assume that $X \subseteq \mathbb{R}^d$ convex, that each $(P_i(\theta), R_i(\theta))$ is Euclidean, and that F is strong. For every $x \notin B_{4r}(y^*)$, $B_{2r}(y^*) \subseteq P_F(x|\theta)$.*

Proof: Take any $\hat{x} \notin B_{4r}(y^*)$, and take any $\hat{y} \in B_{2r}(y^*)$. See Figure 2.26, below. To show $\hat{y}$ is strictly socially preferred to $\hat{x}$, I will show that the hyperplane through $(1/2)(\hat{x}+\hat{y})$ with normal $\hat{x}-\hat{y}$ does not intersect $B_F(\theta)$ and is, therefore, not a F-median hyperplane. Without loss of generality, let $y^* = 0$. The closest point in $B_F(\theta)$ to that hyperplane is $\alpha(\hat{x}-\hat{y})$, where $||\alpha(\hat{x}-\hat{y})|| = r$, i.e., $\alpha = \frac{r}{||\hat{x}-\hat{y}||}$. Thus, I need to show

$$\frac{1}{2}(\hat{x}+\hat{y}) \cdot (\hat{x}-\hat{y}) \;>\; \alpha(\hat{x}-\hat{y}) \cdot (\hat{x}-\hat{y}) \;=\; r||\hat{x}-\hat{y}||.$$

Let y' and β' solve

$$\begin{array}{rl} & \min\; y \cdot (\hat{x}-\hat{y}) \\ & \\ \text{s.t.} & y = \hat{y} + \beta(\hat{x}-\hat{y}) \\ & \beta \leq 0 \\ & y \in B_{2r}(y^*) \end{array}$$

and let x' and γ' solve

$$\begin{array}{rl} & \min\; x \cdot (\hat{x}-\hat{y}) \\ & \\ \text{s.t.} & x = \hat{y} + \gamma(\hat{x}-\hat{y}) \\ & \gamma \geq 0 \\ & x \notin B^{\circ}_{4r}(y^*), \end{array}$$

where $B^{\circ}_{4r}(y^*)$ is the open ball of radius $4r$ around y^*. See Figure 2.26.

Figure 2.26: What's going on?

Observe three things: first, since $\hat{y} \in B^{\circ}_{4r}(y^*)$, we have $\gamma' > 0$; second, since $\hat{x} \notin B_{4r}(y^*)$, we have $x' \cdot (\hat{x}-\hat{y}) < \hat{x} \cdot (\hat{x}-\hat{y})$; third, $||y'|| = 2r$ and $\|x'\| = 4r$. From

$$\begin{aligned} x' &= \hat{y} + \gamma'(\hat{x}-\hat{y}) \\ y' &= \hat{y} + \beta'(\hat{x}-\hat{y}), \end{aligned}$$

we deduce

$$\hat{x}-\hat{y} \;=\; c(x'-y'),$$

where, from the first observation above, $c = \frac{1}{\gamma'-\beta'} > 0$. Thus, I need to show

$$\frac{1}{2}(\hat{x}+\hat{y}) \cdot c(x'-y') \;>\; r||c(x'-y')||,$$

or equivalently,

$$\frac{1}{2}(\hat{x}+\hat{y})\cdot(x'-y') \quad > \quad r||x'-y'||.$$

From the construction of y' and x', and from the second observation above, this would be implied by

$$\frac{1}{2}(x'+y')\cdot(x'-y') \geq r||x'-y'||,$$

or equivalently,

$$\frac{1}{2}\left(||x'||^2-||y'||^2\right) \quad \geq \quad r||x'-y'||.$$

From the third observation above, namely, $||x'|| = 4r$ and $||y'|| = 2r$, this is

$$6r \quad \geq \quad ||x'-y'||,$$

which follows since, by the triangle inequality, $||x'-y'|| \leq ||x'|| + ||y'|| = 6r$. ∎

The preceding lemmas immediately yield the following result.

Proposition 2.61 *Assume $X \subseteq \mathbb{R}^d$ convex and each $(P_i(\theta), R_i(\theta))$ is Euclidean. Then $WUC_F(\theta) \subseteq B_{4r}(y^*)$.*

Note the implication, already proved for the general case, that the weak uncovered set is upper hemicontinuous at states with a nonempty core. That is, suppose x^* is the unique core point when ideal points are $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$. If we perturb one ideal point slightly, so that the core becomes empty, the weak uncovered set will not "blow up." It will get bigger, but continuously. Contrast this with the weak top cycle: if we perturb one ideal point and the core becomes empty, the weak top cycle set blows up to the entire set of alternatives.

2.10 Variable Feasible Sets

[More to do here.]

2.11 Exercises

2.1. Prove that a profile $((P_1, R_1), \ldots, (P_n, R_n))$ is single-peaked only if the following holds: for every triple $\{x, y, z\}$ of distinct alternatives,

($\vee$) There is some alternative that no individual ranks last among the three, i.e., there exists $w \in \{x, y, z\}$ such that, for every $i \in N$, there exists $v \in \{x, y, z\}$ such that wP_iv.

2.2. We say a profile $((P_1, R_1), \ldots, (P_n, R_n))$ is *relatively single-peaked* if every triple $\{x, y, z\}$ of distinct alternatives can be indexed, as in $\{x_1, x_2, x_3\}$, so that the following holds:

($\wedge$) For each $i \in N$, $x_1R_ix_2$ implies $x_2P_ix_3$, and $x_3R_ix_2$ implies $x_2P_ix_1$.

Prove that the conditions ($\vee$) and ($\wedge$) are equivalent.

2.3. Explain how relative single-peakedness is weaker than single-peakedness. (I can think of three ways.) What about weak single-peakedness?

2.4. Show that (R_1, R_2, R_3') preceding Proposition 2.1 is not single-peaked.

2.5. Prove the remaining direction of Proposition 2.1.

2.6. Prove Corollary 2.1.

2.7. In the one-dimensional public good provision problem from Example 2, monotonicity is violated. Does this mean that monotonicity is an unreasonable condition in public good economies? Does this mean that the example is unreasonable?

2.8. In multiple public good provision problems, why does strictly monotonic domain rule out the possibility of private good components in the definition of an alternative x?

2.9. Consider a multiple public good provision problem in which a society has a fixed budget m to spend and each good k is priced at some price $p_k > 0$. As in Example 3, let $X = \{x \in \mathbb{R}^\ell_+ \mid p \cdot x = m\}$. Individual preferences on X are continuous and strictly convex, and each individual has an ideal point. Are individual preferences monotonic? Strictly so?

2.10. Prove that, if $X \subseteq \mathbb{R}^d$ is convex and each $(P_i(\theta), R_i(\theta))$ is strictly convex, then LSWP holds.

2.11. Prove that LSWP holds in the divide-the-dollar environment. (What about private good exchange economies?)

2.12. Calculate weak social preferences for the PARs defined in Section 2.3.

2.13. Check that the PARs defined in Section 2.3 are well-defined, i.e., strict social preference is asymmetric and weak preference is complete.

2.14. I claimed in Section 2.3 that, whenever G is a proper subset of N, F_G is not a q-rule; and I claimed that simple majority is not a G-rule. Why?

2.15. Let $n = 5$, let $X = \{a, b, c, d\}$, let $q = .55$, and let $w_1 = .25$, $w_2 = .24$, $w_3 = w_4 = w_5 = .17$. Suppose individual preferences in θ are as follows:

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$	$R_4(\theta)$	$R_5(\theta)$
a	$d\ c$	b	b	$c\ d$
d	a	d	c	a
c	b	$c\ a$	a	b
b			d	

Graph the strict and weak social preference relations $P_{w,q}(\theta)$ and $R_{w,q}(\theta)$ of the corresponding weighted quota rule.

2.16. Let $n = 5$, and consider the PAR F defined as follows: $xP_F(\theta)y$ if and only if either

$$\{1, 2, 3\} \subseteq P(x, y|\theta) \text{ or } \{1, 2, 4\} \subseteq P(x, y|\theta) \text{ or } \{3, 4, 5\} \subseteq P(x, y|\theta).$$

Prove that F is not a weighted quota rule, i.e., there do not exist weights w and a quota q such that $F = F_{w,q}$.

2.17. Assume X is finite. Prove that, for all $\theta \in \Theta$, $\sum_{x \in X} B^x(\theta) = 0$.

2.18. Assume that $B^y(\theta) \neq 0$ for some $y \in X$. Prove that, if $x \in C_{RM}(\theta)$, then $B^x(\theta) > \min_{y \in X} B^y(\theta)$. Give an example showing that we may have $C_{SM}(\theta) \cap C_B(\theta) = \emptyset$, even if $C_{SM}(\theta) \neq \emptyset$. Could we weaken the assumption of this problem to merely that $P_{RM}(\theta) \neq \emptyset$?

2.19. Suppose that x is a "relative majority Condorcet loser," i.e., for all $y \neq x$, $yP_{RM}(\theta)x$. Prove that $x \notin C_B(\theta)$.

2.20. Why does part 2 of Proposition 2.3 imply that $\mathcal{D}(F)$ is proper? Is the converse of part 4 true?

2.21. Prove that weighted majority rule is strong if and only if there is no partition I, J of N such that $\sum_{i \in I} w_i = \sum_{i \in J} w_i = n/2$.

2.22. What are $\mathcal{D}(F_B)$, $\mathcal{B}(F_B)$, $\mathcal{D}(F_T)$, $\mathcal{B}(F_T)$, $\mathcal{D}(F_C)$, and $\mathcal{B}(F_C)$?

2.23. Prove that a PAR F satisfies IIA if and only if, for all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$[P(x, y|\theta) = P(x, y|\theta') \text{ and } R(x, y|\theta) = R(x, y|\theta') \text{ and } xR_F(\theta)y] \Rightarrow xR_F(\theta')y.$$

2.24. Prove that a PAR F satisfies IIA if and only if, for all $x, y \in X$ and all $\theta, \theta' \in \Theta$,

$$[PR(\theta)|_{\{x,y\}} = PR(\theta')|_{\{x,y\}}] \quad \Rightarrow \quad [R_F(\theta)|_{\{x,y\}} = R_F(\theta')|_{\{x,y\}}],$$

where $R_i|_Y = R_i \cap (Y \times Y)$ is the "restriction" of R_i to Y, and $(R_1, \dots, R_n)|_Y = (R_1|_Y, \dots, R_n|_Y)$.

2.25. Prove Lemma 2.1.

2.26. Give an example showing that F_B, F_T, and F_C do not satisfy IIA generally. What if $|X| = 2$?

2.27. Which PARs in Section 2.3 aren't anonymous? Why not?

2.28. Assume there is a "very free pair," $\{x, y\}$, such that, for every pair of disjoint groups $G, G' \subseteq N$, there exists a state $\theta \in \Theta$ such that $G = P(x, y|\theta)$ and $G' = P(y, x|\theta)$. Prove that, if a PAR F is neutral and monotonic, then it is weakly decisive. Why is a free pair not enough?

2.29. Prove that, under linear domain, F strong implies F is decisive.

2.30. Assume unrestricted domain. Prove that if F is anonymous and strong, then n is odd and F satisfies the weak Condorcet principle: for all $\theta \in \Theta$ and all $x, y \in X$, $xP_{SM}(\theta)y \Rightarrow xP_F(\theta)y$.

2.31. Give an example of an anonymous and strong PAR F that is not F_{SM} or F_{RM}.

2.32. Show by example that F strong does not imply F is decisive, and that F decisive does not imply F strong.

2.33. Prove that a PAR is simple if and only if it is decisive.

2.34. Is F_B simple when $|X| = 2$? Why or why not? How does your answer depend on the domain of preferences?

2.35. Prove that a PAR is anonymous and simple if and only if it is q-rule for some q.

2.36. Prove that, given a simple PAR F, $\mathcal{G}\uparrow$ and $\mathcal{G}\downarrow$ are unique maximal and minimal (respectively) representations.

2.37. For Lemma 2.2, prove part 1, prove the remaining direction of part 3, and prove the rest of part 5.

2.38. Prove Corollary 2.2.

2.39. Prove Corollary 2.3.

2.40. Prove parts 2 and 3 of Proposition 2.6.

2.41. Assume that $X \subseteq \mathbb{R}^d$, and that each $(P_i(\theta), R_i(\theta))$ is continuous. Let F be a simple PAR. Prove that the weak upper contour correspondence $R_F(\cdot|\theta)\colon X \rightrightarrows X$ is upper hemicontinuous.

2.42. Now add the assumption that $PR(\theta)$ satisfies LSWP, and prove that $R_F(\cdot|\theta)$ is lower hemicontinuous.

2.43. In the proof of Proposition 2.7, why does LSWP imply that, for each k, there exists $x_k \in P_G(y|\theta)$ such that $||x_k - x|| < 1/k$? In the following line, why does $yR_i(\theta^m)x_k$ for infinitely many m imply $yR_i(\theta)x_k$?

2.44. Prove the other direction in Proposition 2.8.

2.45. Prove the other direction in Proposition 2.9.

2.46. We say a PAR F satisfies *positive responsiveness* if, for all $\theta, \theta' \in \Theta$ and all $x, y \in X$, the following holds: Suppose $xR_F(\theta)y$, $R(x,y|\theta) \subseteq R(x,y|\theta')$, and $P(x,y|\theta) \subseteq P(x,y|\theta')$ with at least one inclusion strict; then $xP_F(\theta')y$. Assume unrestricted domain.

(a) Prove that F_{RM} is the only PAR satisfying anonymity, neutrality, and positive responsiveness.

(b) Give an example showing that F_{RP} violates positive responsiveness.

(c) Prove that, assuming $n \geq 2$, no simple PAR satisfies positive responsiveness.

2.47. Prove McGarvey's Theorem, Proposition 2.10.

2.48. Verify that the claim in Proposition 2.11 holds for the preference profile constructed in the proof.

2.49. Prove the $n = 4$ case of Proposition 2.11.

2.50. Prove that, if we add IIA to the assumptions of Lemma 2.3, then we can deduce that $G \cap G'$ is semi-blocking.

2.51. In the proof of Lemma 2.4, fill in the omitted step: If G is semi-decisive for x over y at θ, then G is semi-decisive for every $w \neq y$ over y.

2.52. In the proof of Proposition 2.12, why does $R_F(\theta) = X \times X$ in the first case?

2.53. Does Arrow's theorem still hold if $|X| = 2$? What if we drop Pareto? IIA?

2.54. Do Propositions 2.12 and 2.13 hold if we merely assume $\mathbf{L} \subseteq PR(\Theta) \subseteq \mathbf{U}$?

2.55. Show by example that the existence of a dictator does not generally imply that F either is R-transitive or satisfies IIA.

2.56. Assume $PR(\Theta) = \mathbf{L}$.

(i) Characterize the anonymous and neutral PARs in terms of a set $Q \subseteq \{0, 1, \ldots, n\}$ of "quotas." What conditions must Q satisfy?

(ii) Assume $|X| \geq 3$. Using your answer to (i), what are the PARs satisfying anonymity, neutrality, and P-transitivity?

(iii) Now strengthen P-transitivity to R-transitivity. Are there any PARs satisfying the three conditions? What if we also add Pareto?

(iv) Repeat parts (ii) and (iii) after replacing neutrality by Pareto.

(v) How would your answers change if $PR(\Theta) = \mathbf{U}$?

2.57. Assume $|X| \geq 3$, $PR(\Theta) \supseteq \mathbf{L}$, and F is neutral and monotonic. Also assume that F inherits the resoluteness of individuals: for all $x, y \in X$ and all $\theta \in \Theta$ such that $N = P(x, y|\theta) \cup P(y, x|\theta)$, either $xP_F(\theta)y$ or $yP_F(\theta)x$. (i) Prove that F is strong. (ii) Demonstrate that F need not be simple, i.e., give an example of X, N, and a PAR F satisfying these conditions that is not simple. (iii) What can you conclude about F if it is P-acyclic?

2.58. We say F satisfies *strong Pareto* if, for all $x, y \in X$ and all $\theta \in \Theta$, $xP_{RP}(\theta)y$ implies $xP_F(\theta)y$. It satisfies *Pareto indifference* if $xI_i(\theta)y$ for all $i \in N$ implies $xI_F(\theta)y$. We say F is a *lexicographic dictatorship* if there is some ordering, say $(i_1, i_2, \ldots, i_n)$, of individuals such that: for all $x, y \in X$ and all $\theta \in \Theta$, $xP_F(\theta)y$ if and only if there exists some j such that $xP_{i_j}(\theta)y$ and, for all $k < j$, $xI_{i_k}(\theta)y$. That is, the PAR first checks i_1's preferences over x and y; if i_1 prefers x to y, then so does society; if i_1 is indifferent, it checks i_2's preferences, and so on. Assume $|X| \geq 3$ and unrestricted domain. Use Arrow's theorem to prove the following: if F satisfies strong Pareto, Pareto indifference, IIA, and R-transitivity, then it is a lexicographic dictatorship. Give examples showing that this conclusion does not hold if either strong Pareto or Pareto indifference are dropped.

2.59. Prove that, if we add IIA to the assumptions of Lemma 2.5, then we can deduce that $G \cap G'$ is semi-decisive.

2.60. Fill in the omitted step in the proof of Lemma 2.6.

2.61. In the proof of Proposition 2.14, why is G nonempty?

2.62. Assuming unrestricted domain, identify the oligarchies for F_{SP} and F_{RP}.

2.63. Does Gibbard's theorem, Proposition 2.14, still hold if we drop Pareto or IIA?

2.64. Does Gibbard's theorem hold if we assume merely $\mathbf{L} \subseteq PR(\Theta) \subseteq \mathbf{U}$?

2.65. Show that an oligarchical PAR need not satisfy IIA or P-transitivity.

2.66. Assume linear domain, and let F satisfy Pareto, IIA, and P-transitivity. Prove that F is neutral.

2.67. Prove that the Security Council PAR, F_{SC}, is P-acyclic. What if the total number of individual strict preferences needed were six instead of seven?

2.68. Construct an example in which F_{SC} violates P-transitivity.

2.69. Prove that every oligarchy is a collegium.

2.70. Consider weighted q-rule with $n = 5$ and weights $w_1 = w_2 = .35$ and $w_3 = w_4 = w_5 = .1$. Letting q take values in $(.5, 1]$, identify the ranges of q for which the PAR is oligarchic, collegial, or strong.

2.71. Assume unrestricted domain and F is simple. Prove that, if F is not P-acyclic, then there exists $\theta \in \Theta$ at which social preferences contain a "universal" cycle: for all $x, y \in X$, there exist k and $x_1, x_2, \ldots, x_k \in X$ such that

$$xP_F(\theta)x_1P_F(\theta)x_2 \cdots P_F(\theta)x_{k-1}P_F(\theta)x_k = y.$$

2.72. Consider the example in item 4 preceding Proposition 2.16. Show that F is P-acyclic.

2.73. Prove that a PAR is not collegial if and only if it satisfies virtual unanimity.

2.74. Prove the last step of part 3 of Proposition 2.20, namely, that $\mathcal{N}(F_{SM}) = 4$ when $n = 4$.

2.75. Prove Corollary 2.18.

2.76. In the proof of Proposition 2.21, why may we assume without loss of generality that the alternatives $x_1, x_2, \ldots, x_{k-1}$ are distinct?

2.77. The fifteen member European Union recently adopted the following simple PAR: a group of states is decisive if and only if it consists of at least eight members and the total population of its members exceeds half the total population of all fifteen states. Formally, let w_i denote the population of state i as a fraction of the total. Then a group G is decisive if and only if $|G| \geq 8$ and $\sum_{i \in G} w_i > 1/2$. Assume $w_i > 0$ for all i. Without loss of generality, let the states be indexed in order of population size, so state 1 is the largest, 2 next largest, and so on. Clearly, $G = \{1,2,3,4,5,6,7,8\}$ is decisive. Assume it is possible to partition G into two groups of four, say H and I, such that

$$\sum_{i \in H} w_i < \frac{1}{2} \quad \text{and} \quad \sum_{i \in I} w_i < \frac{1}{2}.$$

That is, we can partition the top eight states into two groups of four states, each group with less than half of the total population. What is the Nakamura number of this voting rule?

2.78. Referring to the previous question, what is the Nakamura number of the European Union voting rule for the general case, assuming only positive weights?

2.79. Assume unrestricted domain, and let F satisfy Pareto. We say a collection $G_1, \ldots, G_k$ of groups is a *decisive chain* for a collection $x_1, \ldots, x_k, x_{k+1}$ of distinct alternatives if, for all $h = 1, \ldots, k$,

$$G_h \text{ is } \textit{decisive for } x_h \textit{ over } x_{h+1},$$

in the sense that, for all $\theta \in \Theta$, $G_h \subseteq P(x_h, x_{h+1})$ implies $x_h P_F(\theta) x_{h+1}$. In this definition, k is the *length* of the decisive chain. If there is no decisive chain with empty intersection, i.e., $\bigcap_{h=1}^{k} G_h = \emptyset$, then define $\mathcal{N}^* = 2^{|X|}$. Otherwise, define $\mathcal{N}^*$ to be the length of the smallest decisive chain. Prove that, if $|X| \geq \mathcal{N}^* + 1$, then F violates P-acyclicity. Compare this result to Nakamura's theorem.

2.80. Assume $|X| \geq 4$ and unrestricted domain. Let F satisfy Pareto and the following condition. There are distinct individuals, i and j, with "positive rights" over distinct pairs of alternatives: there exist distinct $x, y, z, w \in X$ such that, for all $\theta \in \Theta$, $x P_i(\theta) y$ implies $x P_F(\theta) y$ and $z P_j(\theta) w$ implies $z P_F(\theta) w$. Prove that F violates P-acyclicity. (Hint: You might use the result from the previous exercise.)

2.81. Prove that, for every PAR F and every state θ, we must have $C_F(\theta) \subseteq C_{\mathcal{D}(F)}(\theta)$.

2.82. Prove part 2 of Proposition 2.22.

2.83. In the proof of Proposition 2.23, why does weak single-peakedness imply $P(x_{h+1}, x_h|\theta) \subseteq R(x_j, x_{j+1}|\theta)$? How do we conclude that $x_k \prec x_1$?

2.84. In the discussion before Proposition 2.27, why must we have $G_1 = N_{\prec}^{+}(\tilde{x}^i|\theta) \in \mathcal{D}(F)$?

2.85. In the proof of Proposition 2.28, how do we know $\tilde{x}^{i_1} = \tilde{x}^{i_2}$? How is Proposition 2.22 used in the proof?

2.86. Assume F is strong and $PR(F)$ is single-peaked. Prove that $R_F(\theta)$ is single-peaked. Is it true that $R_F(\theta) = R_{i_1}(\theta)$ (in Proposition 2.26)? Prove or provide a counterexample.

2.87. Assuming $X \subseteq \mathbb{R}$, prove that, if (P, R) is upper semicontinuous, then $M(R)$ is closed. Use this to show that, if $\mathbf{R}(\theta)$ is single-peaked with respect to $\leq$, then $M_F(\theta) = [\underline{x}(\theta), \overline{x}(\theta)]$.

2.88. Check that, if $X \subseteq \mathbb{R}$ and $PR(\theta)$ is single-peaked with respect to $\leq$, then $[\underline{x}(\theta), \overline{x}(\theta)] = M_F(\theta)$ (as originally defined).

2.89. Prove Proposition 2.29.

2.90. Prove that, assuming $PR(\theta)$ is weakly single-peaked, $x \in M_F(\theta)$ if there is a selection $(\tilde{x}^1, \ldots, \tilde{x}^n)$ such that $\tilde{x}^i \in M(R_i(\theta))$ for all $i \in N$ and x is a median with respect to $(\tilde{x}^1, \ldots, \tilde{x}^n)$.

2.91. Let $X = \mathbb{R}$, let $U \colon \mathbb{R} \to \mathbb{R}$ be a strictly quasi-concave function with maximum at zero, and suppose

($\star$) each individual i has a preference relation represented by $u_i(x) = U(x - \tilde{x}^i)$.

Thus, in the above, $\tilde{x}^i \in \mathbb{R}$ is i's ideal point. Suppose F is strong, and let individual k satisfy $C_F(\theta) = \{\tilde{x}^k\}$.

(a) Prove that, if F is simple, then the social preference relation is represented by u_k (i.e., the core voter is "representative").

(b) Redo part (a), replacing the assumption that F is simple with the assumptions that F is weakly decisive and that, for all $i \neq j$, $\tilde{x}^i \neq \tilde{x}^j$ (distinct ideal points). Why is the assumption of distinct ideal points needed?

(c) Again assume F is simple. Show that the core voter k is not generally representative if ($\star$) is weakened to the assumption merely that individual preferences are single-peaked (with respect to $\leq$).

(c) Now assume that $X = \mathbb{R}^d$, that $U \colon \mathbb{R}^d \to \mathbb{R}$, with $\tilde{x}^i \in \mathbb{R}^d$, and that ($\star$) continues to hold. Is the social preference relation necessarily represented by u_k?

2.92. We say a preference profile is *order restricted* if the individuals can be numbered so that, for all $x, y \in X$,

$$\{i \mid xP_iy\} < \{i \mid xI_iy\} < \{i \mid yP_ix\}$$

or

$$\{i \mid yP_ix\} \;<\; \{i \mid xI_iy\} \;<\; \{i \mid xP_iy\},$$

where "<" here indicates that all elements of the lefthand set are less than all elements of the righthand set. Austen-Smith and Banks give an example showing that a profile may be order restricted but not single-peaked, and they give an example of the other direction: there are single-peaked profiles that are not order restricted. But consider a weakening of order restriction. We say a profile is *relatively order restricted* if, for every triple $a, b, c \in X$, the individuals can be numbered so that, for all $x, y \in \{a, b, c\}$, either

$$\{i \mid xP_iy\} \;<\; \{i \mid xI_iy\} \;<\; \{i \mid yP_ix\}$$

or

$$\{i \mid yP_ix\} \;<\; \{i \mid xI_iy\} \;<\; \{i \mid xP_iy\}.$$

Prove that, if a profile is single-peaked, then it is relatively order restricted. What if, instead of single-peakedness, $(\vee)$ holds for every triple of distinct alternatives?

2.93. Austen-Smith and Banks prove the following result: if a preference aggregation rule, F, is neutral and monotonic, and if a profile $PR(\theta)$ is order restricted, then $P_F(\theta)$ is transitive.

(a) How does this result bear on part 1 of Proposition 2.22?

(b) Let F be a strong simple PAR, and let $PR(\theta)$ be order restricted. Prove that some individual is a representative, i.e., there is some $i \in N$ such that $P_F(\theta) = P_i(\theta)$.

(c) How is the result from (b) related to the above exercise on representative voters?

(d) Now drop the assumption that F is strong from (b) and assume Pareto. Prove that there is some group $G \subseteq N$ such that $P_F(\theta) = P_G(\theta)$.

(e) How are the conclusions of parts (b) and (d) altered if we drop the assumption that F is simple and add monotonicity and neutrality?

2.94. Assume $X \subseteq \mathbb{R}$ and that every profile $((P_1, R_1), \ldots, (P_n, R_n)) \in PR(\Theta)$ is single-peaked with respect to $\leq$. We have shown that the correspondence $C_{SM}\colon \Theta \rightrightarrows X$ is nonempty-valued, and we have characterized $C_{SM}(\theta)$ for each state. We know from Proposition 1.49 that C_{SM} is upper hemicontinuous. Prove that it is actually continuous. (Hint: We had an exercise in Chapter 1 on convergence of Euclidean preferences that may be helpful.)

2.95. Consider a divide-the-dollar environment, where X is the simplex in $\mathbb{R}^n$ and where each $R_i(\theta)$ has the representation $u_i(x) = x_i$. Prove that, if F is non-collegial, then $C_F(\theta) = \emptyset$.

2.96. Let $X \subseteq \mathbb{R}^d$ be convex, let F be a simple PAR, and let $R = F(\theta)$ and $P = P_F(\theta)$. Prove that, if F is strong and each $(P_i(\theta), R_i(\theta))$ is strictly convex, then (P, R) is "strictly star-shaped," meaning: for all $x \in X$, all $y \in R(x)$, and all $\alpha \in (0, 1)$, $\alpha y + (1 - \alpha)xPx$. Use this to prove that (P, R) satisfies thin indifference.

2.97. Assuming $n = 5$, consider the weighted q-rule with $w_1 = w_2 = .35$ and $w_3 = w_4 = w_5 = .1$. Assume $X \subseteq \mathbb{R}^d$ is convex, and suppose that each $R_i(\theta)$ is Euclidean with ideal points arranged in a pentagon. Assume that individuals 1 and 2 are not adjacent. Identify the core of the weighted q-rule for all values of $q \in (.5, 1]$.

2.98. Prove Proposition 2.33.

2.99. Assume that $X = \mathbb{R}^d$, that each $(P_i(\theta), R_i(\theta))$ is continuous and strictly convex. Let F be a strong simple PAR. Prove that $\{x\} \cup P_F(x|\theta) = \{x\} \cup \mathrm{int} R_F(x|\theta)$.

2.100. Consider the proof of Plott's theorem, Proposition 2.38.

(i) Why does there exist $p \neq 0$ such that, for all $i \in N$, $p \cdot \nabla u_i(x^*) \neq 0$?

(ii) Prove that, if $G_1 \cup G_2 \cup \{i\} \subsetneq N$, then there exists a non-zero $q \in \mathbb{R}^d$ such that

$$\{j \in N \mid \nabla u_j(x^*) \cdot q = 0\} \quad = \quad G_1 \cup G_2 \cup \{i\}.$$

(iii) How do we know $|G_1| + |H_1| \geq (n + 1)/2$?

2.101. Extend part (i) of Proposition 2.38 to every strong PAR. How is this result related to Proposition 2.33?

2.102. In stating Proposition 2.38, we assume that $\nabla u_i(x) = 0$ for at most one individual. I gave an example showing the theorem isn't generally true if we drop that qualification. Suppose we just assume $\nabla u_i(x) = 0$ for at most *two* individuals. How would we weaken the definition of "radial symmetry at x" so as to maintain the truthfulness of the theorem? What if more individuals could have a zero gradient at the same alternative?

2.103. Assume $X \subseteq \mathbb{R}^d$ and each $(P_i(\theta), R_i(\theta))$ is represented by differentiable u_i such that there are no shared critical points. Let $x^* \in \mathrm{int}X \cap C_{SM}(\theta)$. Prove that, if n is even, then there exists $y \in X \setminus \{x^*\}$ such that $yR_{SM}(\theta)x^*$.

2.104. Assume $X \subseteq \mathbb{R}^d$ and each $(P_i(\theta), R_i(\theta))$ is represented by differentiable u_i. Let $x \in \mathrm{int}X$, let $G \subseteq N$, and suppose that zero is not in the semi-positive span of the gradients of members of G, i.e., there do not exist $\{\alpha_i \in \mathbb{R}_+ \setminus \{0\} \mid i \in G\}$ such that $\sum_{i \in G} \alpha_i \nabla u_i(x) = 0$. Prove that there exists $y \in X$ such that $yP_G(\theta)x$. (In answering this, you may use the following result, which is known as a "theorem of the alternative."

Let A denote an $m \times m$ matrix. Either the equation $xA = 0$ has a semi-positive solution $x \in \mathbb{R}^m \setminus \{0\}$ or the inequality $Ay > 0$ has a solution.)

2.105. Prove that, if u_i is a pseudo-concave representation of (P, R), then (P, R) is convex.

2.106. Let $X \subseteq \mathbb{R}^d$. When preferences do not have differentiable utility representations, gradients are not well-defined. But given $x \in X$, say $p \in \mathbb{R}^d$ *supports* $R_i(x)$ *at* x if, for all $y \in R_i(x)$, we have $p \cdot (y - x) \geq 0$. In the absence of differentiability, $R_i(x)$ could be supported by many different vectors. Now let $x^* \in \mathrm{int}X$ and assume each $(P_i(\theta), R_i(\theta))$ is strictly convex. Prove that $x^* \in C_{SM}(\theta)$ if:

> There exist $p^1, p^2, \ldots, p^n \in \mathbb{R}^d$ such that each p^i supports $R_i(x^*)$ at x^* and, for all $y \in \mathbb{R}^d$, $|\{i \in N \mid p^i \cdot y > 0\}| < n/2$.

Give an example showing that the converse is not generally true.

2.107. Prove that, if u_i is a strictly pseudo-concave, then it is strictly quasi-concave. Can you find a counter-example to the converse direction?

2.108. Consider the proof of Proposition 2.43.

(i) Where do I use the assumption that the domain of u_i is $\mathbb{R}^d$?

(ii) Give the argument for the claim that, if $d \geq 2$, then each C_{jh} is closed and nowhere dense. What if $d = 1$?

(iii) Verify the claim that $R_i^m \to R_i$ in the Hausdorff metric.

2.109. Assume that $X = \mathbb{R}^d$ and that $PR(\Theta)$ consists of all profiles of Euclidean preferences satisfying $C_{SM}(\theta) \neq \emptyset$. Give an example demonstrating that, when n is even, the correspondence C_{SM} is not necessarily lower hemicontinuous. (Hint: Think about $n = 4$ and $d = 2$.) What if n is odd?

2.110. Explain how strict convexity is used in the proof of Proposition 2.46.

2.111. Assume $X = \mathbb{R}^d$ and individual preferences are Euclidean. Prove the following claims: for all $y \in \mathbb{R}^d$, there exists $c \in \mathbb{R}$ such that $H_{y,c} \in \mathbf{M}$; when F is strong, there exists exactly one c for which this is true; the set of such c's is a compact interval.

2.112. Prove part 2 of Proposition 2.48.

2.113. In the proof of Lemma 2.8, how do we know that $y_j \cdot \hat{x} < c_j$? Why does x_ϵ solve the constraints of the maximization problem for small enough ϵ?

2.114. In the proof of Proposition 2.49, verify that $t_k = s + (y_i \cdot t_k)y_i$ and $t_{k+1} = s - (y_i \cdot t_k + c_i)y_i$. How do we ensure, at the end of the proof, that $vR_F(\theta)t_k$?

2.115. Prove Corollary 2.8.

2.116. How would you adapt Proposition 2.49 (and proof) for the strong top cycle?

2.117. In the discussion following the proof of Proposition 2.49, why is it that, assuming individual preferences are continuous and F is simple, $T_P(x)$ is open? Also, why is it that $y, z \in \mathrm{bd}T_P(x)$ implies $yI_F(\theta)z$?

2.118. Prove that, for every $Y \subseteq X \subseteq \mathbb{R}^d$,

(i) $\mathrm{fr}Y$ is a closed subset of the boundary of Y

(ii) $\mathrm{fr}Y = \mathrm{fr}(\mathrm{int}Y) = \mathrm{fr}(\mathrm{clos}Y)$

(iii) $\mathrm{fr}Y \neq \emptyset$ if and only if $\mathrm{int}Y$ and $\mathrm{int}\overline{Y}$ are nonempty.

2.119. Check that, in the example following Proposition 2.56, the weak uncovered set for simple majority rule is $[-2\epsilon, 2\epsilon]$.

2.120. Complete the proof of Proposition 2.59 by showing that $WUC_F(\theta)$ is closed.

2.121. Assume that $n = 3$, that $X = \mathbb{R}^2$, and that each $(P_i(\theta), R_i(\theta))$ is Euclidean, with ideal points arranged in an equilateral triangle. Let $\hat{x}$ be the point at the center of the triangle. Prove that $\hat{x} \in WUC_{SM}(\theta)$.

2.122. Consider a divide-the-dollar environment, where X is the unit simplex in $\mathbb{R}^n$ and where each $(P_i(\theta), R_i(\theta))$ has the representation $u_i(x) = x_i$. What is $WUC_{SM}(\theta)$? (You can assume $n = 3$ if it helps.)

2.123. Let F be a weakly decisive PAR satisfying Pareto. Prove that, regardless of whether social preferences satisfy thin indifference, the strong uncovered set is contained in the Pareto optimal alternatives: $UC(P_F(\theta)) \subseteq C_{SP}(\theta)$.

2.124. In the proof of Proposition 2.60, how do we know that the median hyperplane with normal $w_m - x^*$ does not intersect $B_{\epsilon/3}(x^*)$?

2.125. Give the argument for the claim, in the proof of Proposition 2.60, that $\{(y_m, c_m)\}$ has a convergent subsequence.

2.126. Check that Proposition 2.61 holds for any ball intersecting the F-median hyperplanes in all directions, even if the yolk is not uniquely defined.

Chapter 3

Implementation Theory

3.1 The Implementation Problem

3.2 Dominant Strategy Equilibrium

3.3 Nash Equilibrium

3.4 Nash Refinements

3.4.1 Strong Nash Equilibrium

3.4.2 Undominated Nash Equilibrium

3.4.3 Subgame Perfect Equilibrium

3.5 Bayesian Equilibrium