Notes on the Class

This class takes a theoretical view of collective choice.

Examples of collective choice problems:
- allocating goods in an economy
- deciding on tax rates
- legislative choice of bills
- voters choosing between candidates
- committee choosing job candidates
- friends picking a movie

Obviously, examples of collective choice problems span economics and political science, and they range from large scale problems to small.

The course consists of three parts

1. Foundations: we survey binary relations, preference, and theories of how preference determines choice.

   preference $\rightarrow$ choice

2. Social choice theory: this classical approach assumes individual preferences determine “group preferences,” which determine collective choice.

   individual preference $\rightarrow$ group preference $\rightarrow$ collective choice

   We will be as general as possible regarding our assumptions about the first arrow. The second arrow is the same as the one above — we essentially apply the standard theory of choice, but now using social preferences rather than individual.

3. Implementation theory: Here we adopt an alternative approach, where collective choices are determined by equilibrium behavior of individuals. We take a general perspective, analyzing the properties of equilibrium outcome correspondences, as well as considering equilibria of specific game forms.
Notes on the Notes

These notes contain more information than I will try to teach, and that information is presented at a more technical level here than it will be in the lectures. In particular, there are now a few starred sections, which I think will be interesting only to specialists in choice theory. Hopefully, the notes will help me organize the lectures better and give students a valuable reference.

The notes presume familiarity with logical connectives and quantifiers, with standard set-theoretic terminology, and with basics of deductive reasoning. I write $\overline{X}$ for the complement of $X$ and $X \setminus Y$ for the elements of $X$ not in $Y$.

I also presume some knowledge of linear algebra and real analysis in finite-dimensional Euclidean space. At different points, I use concepts of convexity, continuity, compactness, etc. I write “clos,” “int,” “bd,” and “conv” to indicate the closure, interior, boundary, and convex hull of a set, respectively.

Though many results in these notes hold at a very general level (meaning even in infinite-dimensional spaces), I have stated propositions and framed proofs in finite dimensions. (The only exception is that I do say a bit about convergence in the space of weak preference relations over a set . . . ) That’s all I will expect from students in homeworks and tests.

The notes are currently written with only a few references to the literature. I apologize for that indiscretion, which will be addressed before too long.
Chapter 1

Relations, Preference, and Choice

1.1 Binary Relations

A binary relation on a set \( X \) is a subset, say \( B \), of ordered pairs of elements of \( X \), i.e., \( B \subseteq X \times X \). Usually we write “\( xBy \)” instead of “\((x, y) \in B\)” and “\( xByBz \)” instead of “\( xBy \) and \( yBz \).” In words, we might express \( xBy \) as “\( x \) bears the relation \( B \) to \( y \).”

Common examples are the greater-than-or-equal-to relation, \( \geq \), and the greater-than relation, \( > \), on \( \mathbb{R} \). These can both be extended to relations on \( \mathbb{R}^d \) as follows: given \( x, y \in \mathbb{R}^d \), say \( x \geq y \) if and only if, for all \( i \), \( x_i \geq y_i \); and say \( x > y \) if and only if \( x \geq y \) and, for some \( i \), \( x_i > y_i \). The greater-than relation can also be extended to \( \gg \): say \( x \gg y \) if and only if, for all \( i \), \( x_i > y_i \).

A binary relation can literally be interpreted as a subset of \( X \times X \). See Figure 1.1 for some examples with \( X = \mathbb{R} \):

![Figure 1.1: Examples of binary relations](image)

Sometimes (when \( X \) is finite) it can be helpful to represent \( B \) by a directed graph. An example is shown in Figure 1.2.

![Figure 1.2: A directed graph](image)

In a directed graph, dots (or nodes) represent points in \( X \) and arrows (or arcs) represent ordered pairs in \( B \). (An arrow from \( x \) to \( y \) means \((x, y) \in B\).)
Properties of Relations

Here are some useful properties of binary relations. Here, $x$, $y$, and $z$ range over the set $X$, and $k$ ranges over the natural numbers.

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetric</td>
<td>$\forall x, y : xBy \Rightarrow yBx$</td>
</tr>
<tr>
<td>reflexive</td>
<td>$\forall x : xBx$</td>
</tr>
<tr>
<td>irreflexive</td>
<td>$\forall x : \neg xBx$</td>
</tr>
<tr>
<td>anti-symmetric</td>
<td>$\forall x, y : xBy \land yBx \Rightarrow x = y$</td>
</tr>
<tr>
<td>total</td>
<td>$\forall x, y : x \neq y \Rightarrow xBy \lor yBx$</td>
</tr>
<tr>
<td>complete</td>
<td>$\forall x, y : xBy \lor yBx$</td>
</tr>
<tr>
<td>asymmetric</td>
<td>$\forall x, y : xBy \Rightarrow \neg yBx$</td>
</tr>
<tr>
<td>transitive</td>
<td>$\forall x, y, z : xByBz \Rightarrow xBz$</td>
</tr>
<tr>
<td>negatively transitive</td>
<td>$\forall x, y, z : \neg xBy \land \neg yBz \Rightarrow \neg xBz$</td>
</tr>
<tr>
<td>acyclic</td>
<td>$\forall k, x_0, \ldots, x_k : x_0Bx_1 \cdots Bx_k \Rightarrow x_k \neq x_0$</td>
</tr>
<tr>
<td>negatively acyclic</td>
<td>$\forall k, x_0, \ldots, x_k : \neg x_0Bx_1 \land \cdots \land \neg x_{k-1}Bx_k$ $\Rightarrow x_k \neq x_0$</td>
</tr>
</tbody>
</table>

As a notational convention, when a relation is assumed to be complete, we always denote it by $R$, $R'$, etc. And when a relation is assumed to be asymmetric, we always denote it by $P$, $P'$, etc.

Note that transitivity says that, if we can get from $x$ to $z$ in two $B$-steps, then we must be able to get from $x$ to $z$ directly. An equivalent definition is: for all $k$ and all $x_0, \ldots, x_k$, if $x_0Bx_1 \cdots Bx_k$, then $x_0Bx_k$. That is, if we can get from $x_0$ to $x_k$ in any finite number of $B$-steps, then we can get from $x_0$ to $x_k$ directly. That this definition implies the first is seen by setting $k = 2$. You should prove the other direction by induction.

Negative transitivity may be more intuitive if written contrapositively: $xBz$ implies $xBy$ or $yBz$. 

2
We call a set \( \{x_0, \ldots, x_k\} \) a cycle of length \( k \) if \( x_0Bx_1 \cdots Bx_{k-1}Bx_k = x_0 \), i.e., if we can get from \( x_0 \) back to itself in \( k \) \( B \)-steps. We use the notational convention that \( x_{h-1} \) refers to \( x_k \) when \( h = 1 \). The condition of acyclicity precludes the existence of cycles of any length. Compare this to asymmetry and irreflexivity: asymmetry rules out cycles of length one or two, and irreflexivity rules out cycles of length one.

Note that the example in Figure 1.2 violates all of these conditions. The relation \( \geq \) on \( \mathbb{R} \) is reflexive, anti-symmetric, total, complete, transitive, negatively transitive, and negatively acyclic. I’ll let you think about \( > \) and the extensions of these relations to \( \mathbb{R}^d \).

**Proposition 1.1** Let \( B \) be a relation.

1. \( B \) is complete if and only if it is reflexive and total.
2. \( B \) is asymmetric if and only if it is irreflexive and anti-symmetric.
3. If \( B \) is irreflexive and transitive, then it is acyclic.
4. If \( B \) is reflexive and negatively transitive, then it is negatively acyclic.
5. If \( B \) is total and transitive, then it is negatively transitive.
6. If \( B \) is anti-symmetric and negatively transitive, then it is transitive.

**Proof:** I will just prove part 5. Take any \( x, y, z \in X \), and suppose not \( xBy \) and not \( yBz \). If either \( x = y \) or \( y = z \), then not \( xBz \) follows immediately. So suppose \( x \neq y \) and \( y \neq z \). Then, since \( B \) is total, we have \( zByBx \). Then \( xBz \) would imply, with transitivity, that \( xBy \), a contradiction. Therefore, not \( xBz \).

The above properties can be defined using the familiar set-theoretic operations and the following “derivative” relations:

\[
\begin{align*}
xB^{-1}y & \iff yBx \\
\overline{B}y & \iff \neg xBy \\
xB^k y & \iff \exists x_1, \ldots, x_k : xBx_1Bx_2 \cdots Bx_k = y \\
xT_B y & \iff x \bigcup_{k=1}^{\infty} B^k y.
\end{align*}
\]

The last relation, \( T_B \), is the transitive closure of \( B \). An equivalent definition is the following: \( xT_B y \) if and only if there exist \( k \) and \( x_1, \ldots, x_k \) such that \( xBx_1 \cdots x_{k-1}Bx_k = y \).

In words, \( xT_B y \) if and only if we can get from \( x \) to \( y \) in a finite number of “\( B \)-steps.” The transitive closure of \( B \) is the smallest transitive relation containing \( B \). You should check that

\[
T_B = \bigcap \{B' \subseteq X \times X \mid B' \text{ is transitive and } B \subseteq B'\}.
\]
Moreover, $B$ is transitive if and only if $B = T_B$.

Note that $B$ is negatively transitive if and only if $\overline{B}$ is transitive, i.e., $\overline{B} = T_{\overline{B}}$. Taking complements of both sides, this is

$$B = T_{\overline{B}}.$$  

The relation $T_{\overline{B}}$ is the negatively transitive interior of $B$. It is the largest subrelation of $B$ that is negatively transitive: you can check that

$$T_{\overline{B}} = \bigcup \{ B' \subseteq X \times X \mid B' \text{ is negatively transitive and } B' \subseteq B \}.$$  

It would be a good exercise to check that $x \overline{T_{\overline{B}} y}$ if and only if $x$ can be “separated” from $y$ in the following sense: there is a partition $\{Z, W\}$ of $X$ such that, for all $z \in Z \cup \{x\}$ and all $w \in W \cup \{y\}$, $z B w$.

Let $\Delta = \{(x, x) \mid x \in X\}$ denote the “diagonal” relation on $X$.

<table>
<thead>
<tr>
<th>Property</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetric</td>
<td>$B = B^{-1}$</td>
</tr>
<tr>
<td>reflexive</td>
<td>$\Delta \subseteq B \cap B^{-1}$</td>
</tr>
<tr>
<td>irreflexive</td>
<td>$B \cap B^{-1} \subseteq \overline{\Delta}$</td>
</tr>
<tr>
<td>anti-symmetric</td>
<td>$B \cap B^{-1} \subseteq \Delta$</td>
</tr>
<tr>
<td>total</td>
<td>$\overline{\Delta} \subseteq B \cup B^{-1}$</td>
</tr>
<tr>
<td>asymmetric</td>
<td>$B \cap B^{-1} = \emptyset$</td>
</tr>
<tr>
<td>complete</td>
<td>$B \cup B^{-1} = X \times X$</td>
</tr>
<tr>
<td>transitive</td>
<td>$B = T_B$</td>
</tr>
<tr>
<td>negatively transitive</td>
<td>$B = T_{\overline{B}}$</td>
</tr>
<tr>
<td>acyclic</td>
<td>$T_B \subseteq \overline{\Delta}$</td>
</tr>
<tr>
<td>negatively acyclic</td>
<td>$T_{\overline{B}} \subseteq \overline{\Delta}$</td>
</tr>
</tbody>
</table>
Related Relations

We can partition a relation $B$ into two parts. The *symmetric part*, denoted $I_B$, is defined as

$$I_B = B \cap B^{-1},$$

i.e., that is, $xI_By$ if and only if $xB_y$ and $yBx$. The *asymmetric part*, denoted $P_B$, is defined as

$$P_B = B \setminus I_B.$$

Equivalently, $xP_By$ if and only if $xB_y$ and not $yBx$. To see this, note that

$$P_B = B \cap (B \cap B^{-1}) = B \cap (B \cup B^{-1}) = (B \cap B) \cup (B \cap B^{-1}) = B \cap B^{-1}.$$

See Figure 1.3.

Figure 1.3: Asymmetric and symmetric parts

As the names suggest, these relations are symmetric and asymmetric, respectively. The next proposition includes some other properties of these subrelations.

**Proposition 1.2** Let $B$ be a relation.

1. $P_B$ is asymmetric, and $I_B$ is symmetric.
2. $P_B$ and $I_B$ partition $B$.
3. $B = P_B$ if and only if $B$ is asymmetric.
4. $I_B = \emptyset$ if and only if $B$ is asymmetric.

We define two relatives of the above relations. The first is the *null* of $B$, denoted $N_B$ and defined as

$$N_B = (B \cup B^{-1}),$$

where complementation here is with respect to $X \times X$. That is, $xN_By$ if neither $xB_y$ nor $yBx$. The second is the *completion* of $B$, denoted $R_B$ and defined as

$$R_B = B \cup N_B.$$
The completion of $B$ is constructed by adding reflexive arcs to $B$ wherever they are missing and adding double arcs to $B$ whenever two elements of $X$ are non-comparable. Equivalently, $xR_B y$ if and only if $xBy$ or not $yBx$. To see this, note that

$$R_B = B \cup (B \cup B^{-1})$$
$$= B \cup (B \cap B^{-1})$$
$$= (B \cup B) \cap (B \cup B^{-1})$$
$$= B \cup B^{-1}.$$

See Figure 1.4.

Figure 1.4: Null and completion

Of course, $N_B$ is symmetric and $R_B$ is complete.

**Proposition 1.3** Let $B$ be a relation.

1. $R_B$ is complete, and $N_B$ is symmetric.
2. $P_B$, $I_B$, and $N_B$ partition $R_B$.
3. $R_B = B$ if and only if $B$ is complete.
4. $N_B = \emptyset$ if and only if $B$ is complete.

Note that $B$ determines $I_B$, $P_B$, $N_B$, and $R_B$. Conversely, we can construct $B$ from these derivative relations as $B = P_B \cup I_B$ or $B = R_B \setminus N_B$.

As the next proposition states, the asymmetric part of $B$ preserves the completion of $B$, and the completion of $B$ preserves the asymmetric part of $B$.

**Proposition 1.4** For a relation $B$,

1. $R_{P_B} = R_B$,
2. $P_{R_B} = P_B$,
3. $I_{R_B} = N_{P_B} = I_B \cup N_B$. 


Proof: I will prove part 1:

\[ R_{P_B} = P_B \cup P_B^{-1} \]
\[ = [B \cap B^{-1}] \cup [(B \cap B^{-1})^{-1}] \]
\[ = [B \cap B^{-1}] \cup [B^{-1} \cap B] \]
\[ = [B \cap B^{-1}] \cup [B^{-1} \cup B] \]
\[ = B \cup B^{-1}, \]

as required. \[ \blacksquare \]

Note that the previous proof uses the fact that \((B^{-1}) = (B)^{-1}\) in the third step.

Transitivity Properties

A relation \(B\) is \(P\)-acyclic if \(P_B\) is acyclic.

**Proposition 1.5** Let \(B\) be a relation.

1. \(B\) is acyclic if and only if it is asymmetric and \(P\)-acyclic.

2. \(B\) is negatively acyclic if and only if it is complete and \(P\)-acyclic.

Proof: I will prove part 1. Suppose \(B\) is acyclic. To prove asymmetry, take any \(x, y \in X\), and suppose \(xBy\). If \(yBz\), then, setting \(k = 2\), \(x_0 = x\), \(x_1 = y\), and \(x_2 = z\) in the definition of acyclicity, we have not \(x_2 \neq x_0\), i.e., \(z \neq x\). Therefore, not \(yBx\). To prove \(P\)-acyclicity, take any \(k\) and \(x_0, \ldots, x_k\) such that \(x_0 P_B x_1 \cdots P_B x_k\). In particular, this implies \(x_0 B x_1 \cdots B x_k\). If \(x_k P_B x_0\), then \(x_k B x_0\) contradicting acyclicity. This establishes asymmetry and acyclicity. Now suppose \(B\) is asymmetric and \(P\)-acyclic. Take any \(k\) and \(x_0, \ldots, x_k\) such that \(x_0 B x_1 \cdots B x_k\). By asymmetry and \(x_h B x_{h+1}\), we have, for all \(h = 0, 1, \ldots, k - 1\), not \(x_{h+1} B x_h\). Thus, \(x_0 P_B x_1 \cdots P_B x_k\). Then \(P\)-acyclicity implies \(x_k \neq x_0\). Therefore, \(B\) is acyclic. \[ \blacksquare \]

A relation \(B\) is \(R\)-transitive if \(R_B\) is transitive; it is \(P\)-transitive if \(P_B\) is transitive.

**Proposition 1.6** For a relation \(B\),

\[ R\text{-transitive} \Rightarrow P\text{-transitive} \Rightarrow P\text{-acyclic}. \]

Proof: I prove the first implication. Suppose \(B\) is \(R\)-transitive, and take \(x, y, z \in X\) such that \(x P_B y P_B z\). Therefore, \(x R_B y R_B z\), which implies \(x R_B z\). If not \(x P_B z\), then it must be that \(z R_B x\). But then \(z P_B x R_B y\) implies \(z R_B y\), a contradiction. \[ \blacksquare \]
One might wonder whether $R$-transitivity implies that $B$ is transitive. To see that it doesn’t, suppose $X = \{x, y\}$ and consider $B = \{(x, y), (y, x)\}$.

The next proposition decomposes transitivity of $B$ into transitivity of its symmetric and asymmetric parts. We say $B$ is $I$-transitive if $I_B$ is transitive.

**Proposition 1.7** For a relation $B$,

\[
\text{transitive} \Rightarrow \begin{cases} 
I\text{-transitive} \\
P\text{-transitive}.
\end{cases}
\]

If $B$ is complete, then the converse holds as well.

*Proof:* I will prove that transitivity implies $I$- and $P$-transitivity. Suppose $B$ is transitive, and take any $x, y, z \in X$ such that $x I_B y$ and $y I_B z$ and $z B x$. By transitivity, this implies $x B z$ and $z B x$, i.e., $x I_B z$. Thus, $I_B$ is transitive. Now take any $x, y, z \in X$ such that $x P_B y$ and $y P_B z$. This implies $x B y$, and, by transitivity, $x B z$. If $z B x$, then transitivity implies $z B y$, contradicting $y P_B z$. Thus, not $z B x$, which means $x P_B z$. Thus, $P_B$ is transitive.

To see that completeness can’t be dropped in the second part of the proposition, consider Figure 1.2 above: $B$ is $P$-transitive and $I$-transitive, but not transitive ($x B y$ and $y B z$, but not $x B z$), because it is not complete (not $x B z$ and not $z B x$).

The previous two propositions have close counterparts for negative transitivity. A relation $B$ is $N$-transitive if $N_B$ is transitive; it is $R$-negatively transitive if $R_B$ is negatively transitive; it is $R$-negatively acyclic if $R_B$ is negatively acyclic.

**Proposition 1.8** For a relation $B$,

\[
P\text{-negatively transitive} \Rightarrow R\text{-negatively transitive} \Rightarrow R\text{-negatively acyclic}.
\]

To see that $P$-negative transitivity does not imply that $B$ is negatively transitive, suppose $X = \{x, y, z\}$ and consider $B = \{(x, y), (y, x)\}$. We can also decompose negative transitivity of $B$, now into negative transitivity of its null and completion.

**Proposition 1.9** For a relation $B$,

\[
negatively\ transitive \Rightarrow \begin{cases} 
N\text{-transitive} \\
R\text{-negatively transitive}.
\end{cases}
\]

If $B$ is asymmetric, then the converse holds as well.
You can check that asymmetry is needed for the second direction in the above proposition.

**Duality**

Given a complete relation $R$ and an asymmetric relation $P$, we call $(P,R)$ a dual pair if $R = R_P$ and $P = P_R$.

Note that, given a complete relation $R$, there is at most one relation $P$, namely $P_R$, such that $(P,R)$ is a dual pair. And given an asymmetric relation $P$, there is at most one relation $R$, namely $R = R_P$, such that $(P,R)$ is a dual pair.

Given complete $R$, is $(P_R,R)$ necessarily a dual pair? Or might there be no dual pairs containing $R$? Similarly, given asymmetric $P$, is $(P,R_P)$ necessarily a dual pair? We will see that $(P_R,R)$ and $(P,P_R)$ are always dual pairs.

We can think of the asymmetric part operation as a mapping, $B \overset{as}{\rightarrow} P_B$, with domain the set of all relations and range the set of all asymmetric relations. The mapping is onto but is not generally 1-1. If we restrict it to the complete relations, however, then it is 1-1. That is, given two distinct complete relations $R$ and $R'$, we must have $P_R \neq P_{R'}$. Furthermore, the asymmetric part operation, restricted to the domain of complete relations, is still onto. That is, given any asymmetric relation $P$, there is some complete relation $R$ such that $P = P_R$. Thus, the asymmetric part operation, restricted to the domain of complete relations has an inverse.

The same remarks hold for the completion operation, $B \overset{co}{\rightarrow} R_B$, restricted to the domain of all asymmetric relations. See Figure 1.5.

![Figure 1.5: Mappings](image)

The next proposition formalizes the above claims, and it reveals that these operations are inverses of each other. It follows directly from Propositions 1.2–1.4. (Can you see how?)

**Proposition 1.10** For a complete relation $R$ and an asymmetric relation $P$,

1. $R = R_{P_R}$,
2. $P = P_{R_P}$.

Thus, each complete $R$ is in the dual pair $(P_R,R)$, and each asymmetric $P$ is in the dual pair $(P,P_R)$.
Going between members of a dual pair is actually quite straightforward. The next proposition shows that, given a dual pair \((P, R)\), there is a \(P\)-arc from \(x\) to \(y\) whenever there is no \(R\)-arc from \(y\) to \(x\), and vice versa, as in Figures 1.3 and 1.4. In terms of the above notation, \(P = R^{-1}\) and \(R = P^{-1}\).

**Proposition 1.11** Let \((P, R)\) be a dual pair. Then

1. \(xRy\) if and only if not \(yPx\),
2. \(xPy\) if and only if not \(yRx\),

for all \(x, y \in X\).

**Proof:** To prove part 1, note that

\[
R = R_P \\
= P \cup N_P \\
= P \cup (P \cup (P \cup P^{-1})) \\
= P \cup (P \cap P^{-1}) \\
= (P \cup \overline{P}) \cap (P \cup \overline{P^{-1}}) \\
= P \cup \overline{P^{-1}} \\
= \overline{P^{-1}},
\]

where the last equality follows from asymmetry, which implies that \(P \subseteq \overline{P^{-1}}\). I’ll leave part 2 for you. 

The next result is a trivial consequence of Propositions 1.2–1.4: for a dual pair \((P, R)\), we have \(I_R = I_{R_R} = N_{P_R} = N_P\).

**Proposition 1.12** Let \((P, R)\) be a dual pair. Then \(I_R = N_P\).

When considering a dual pair, a restriction on \(P\) automatically entails a restriction on \(R\), and vice versa. The next proposition establishes some duality relationships among the properties defined above. These relationships shed some light on properties, such as negative acyclicity or negative transitivity, that may seem unintuitive at first. Given a complete relation \(R\) in a dual pair \((P, R)\), for example, negative acyclicity translates to acyclicity of \(P\).

**Proposition 1.13** Let \((P, R)\) be a dual pair.

1. \(R\) is anti-symmetric if and only if \(P\) is total.
2. $R$ is negatively acyclic if and only if $P$ is acyclic.

3. $R$ is negatively transitive if and only if $P$ is transitive.

4. $R$ is transitive if and only if $P$ is negatively transitive.

Proof: I’ll prove part 1. This follows simply because not both $xRy$ and $yRx$ is equivalent to $xRy$ or $yRx$, which is equivalent to $yPx$ or $xPy$.

The next proposition yields a joint restriction on $R$ and $P$, when they are transitive and negatively transitive, respectively.

**Proposition 1.14** Let $(P, R)$ be a dual pair such that $R$ is transitive and $P$ is negatively transitive. Then

1. $xRyPz$ implies $xPz$,
2. $xPyRz$ implies $xPz$,

for all $x, y, z \in X$.

Proof: I will prove part 1. Suppose $xRyPz$ but not $xPz$. Then $zRxRy$ implies $zRy$, a contradiction.

You can use Proposition 1.4 to show, given any relation $B$, that $(P_B, R_B)$ is a dual pair.

The concept of a dual pair is defined above only for a complete and an asymmetric relation, but the idea of duality is actually much more general: every condition on relations has a dual counterpart (e.g., completeness and asymmetry) and every result has a dual (e.g., parts 1 and 2 of Proposition 1.5).

**Classes of Relations**

Here are some classes of complete relations, where ‘∗’s denote defining properties and ‘+’s denote implied ones. Note that each of these classes is termed “weak,” which I use to denote completeness.

<table>
<thead>
<tr>
<th></th>
<th>weak sub-order</th>
<th>weak quasi-order</th>
<th>weak order</th>
<th>weak linear order</th>
</tr>
</thead>
<tbody>
<tr>
<td>reflexive</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>total</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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<tr>
<td>complete</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
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<td>+</td>
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<tr>
<td>transitive</td>
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<td></td>
<td>*</td>
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<tr>
<td>anti-symmetric</td>
<td></td>
<td></td>
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</table>
Note that, by Proposition 1.5, negative acyclicity implies completeness, so the use of that condition in the definition of weak sub-order is redundant. And by Proposition 1.1, reflexivity and negative transitivity together imply negative acyclicity, which implies completeness. Thus, the condition of completeness used in the definition of weak quasi-order can be weakened to reflexivity.

The weak orders are an especially important class. See Figure 1.6 for an example.

Figure 1.6: A weak order

More simply, we can write $R$ as follows.

\[
\begin{array}{cccc}
R & & & \\
x & y & z & w \\
& & & v \\
\end{array}
\]

This is actually the defining property of weak orders on finite sets: they can be represented vertically as a ranking, higher elements being “preferred,” elements at the same level being “indifferent.”

When $X$ is infinite, we can’t typically write a weak order $R$ as a ranking like this. Because $I_R$ is transitive, however, the symmetric part of $R$ partitions $X$ into equivalence classes. That is, $X = \bigcup \{ I_R(x) \mid x \in X \}$ and, letting $Y = I_R(y) \neq I_R(z) = Z$, we have $Y \cap Z \neq \emptyset$. To verify the latter claim, suppose $x \in Y \cap Z$. Then $yI_RxI_Rz$, and transitivity implies $yI_Rz$. For any $w \in Y$, we then have $wI_RyI_Rz$, and transitivity implies $wI_Rz$. Thus, $Y \subseteq Z$. Similarly, $Z \subseteq Y$, a contradiction. Though we can’t completely represent $R$ by drawing these equivalence classes (because there are typically an infinite number of them), we can convey an idea of it.

Weak linear orders are the special case where no two elements are at the same level. One example is the relation $\geq$ on $\mathbb{R}$.

Here are some classes of asymmetric relations. Note the term “strict” denotes asymmetry.
Note that, by Proposition 1.5, acyclicity implies asymmetry, so the use of that condition in the definition of strict sub-order is redundant. And by Proposition 1.1, irreflexivity and transitivity together imply acyclicity, which implies asymmetry. Thus, the condition of asymmetry used in the definition of strict quasi-order can be weakened to irreflexivity.

An example of a strict linear order on \( \mathbb{R} \) is \( > \).

The next proposition, an immediate consequence of Proposition 1.13, establishes the duality relationships among these classes of complete and asymmetric relations.

**Proposition 1.15** Let \((P, R)\) be a dual pair.

1. \(R\) is a weak suborder if and only if \(P\) is a strict suborder.
2. \(R\) is a weak quasi-order if and only if \(P\) is a strict quasi-order.
3. \(R\) is a weak order if and only if \(P\) is a strict order.
4. \(R\) is a weak linear order if and only if \(P\) is a strict linear order.

Thus, the following convention makes sense. We call a relation \(B\) a “suborder” if \(R_B\) is a weak suborder or equivalently, since \((P_B, R_B)\) is a dual pair, if \(P_B\) is a strict suborder. We use the terms “quasi-order,” “order,” and “linear order” to refer to relations such that \(R_B\) is negatively transitive (\(P_B\) is transitive), \(R_B\) is transitive (\(P_B\) is negatively transitive), and \(R_B\) is transitive and anti-symmetric (\(P_B\) is negatively transitive and total), respectively.

A word of warning: there is no universally accepted terminology for these classes of relations. For example, some authors use “pre-order” or “order” for a weak order, and some would use these terms for a strict order. Some authors use “order” or “strict order” for a weak linear order, etc. But at least the terminology I propose is coherent. When writing or when talking with others — no matter what convention we use — we have to define our terms clearly (and ask others to define theirs).
Relations are downright fun, of course, but our interest in them lies in their usefulness for talking about... 

Exercises

1.1.1. Let \( X = \{a, b, c, d, e\} \), and graph the following relation:

\[
B = \{(a, b), (a, d), (b, a), (b, b), (b, e), (c, a), (c, d), (d, e), (e, b), (e, d)\}.
\]

Which properties from Section 1.1 does this relation possess? Briefly describe why \( B \) does or does not have them.

1.1.2. Consider the relations \( \geq \), \( > \), and \( \gg \) on \( \mathbb{R}^d \). Which properties from Section 1.1 do these relations possess? Why or why not?

1.1.3. Prove the remaining parts of Proposition 1.1.

1.1.4. Prove formally, using an induction proof, that \( B \) is transitive if and only if \( B = T_B \).

1.1.5. Prove that \( T_B = \bigcap \{B' \subseteq X \times X \mid B' \text{ is transitive and } B \subseteq B'\} \).

1.1.6. Prove that \( T_B = \bigcup \{B' \subseteq X \times X \mid B' \text{ is negatively transitive and } B' \subseteq B\} \).

1.1.7. Prove that \( x T_B y \) if and only if \( x \) can be separated from \( y \) in the following sense: there is a partition \( \{Z, W\} \) of \( X \) such that, for all \( z \in Z \cup \{x\} \) and all \( w \in W \cup \{y\} \), \( zBw \).

1.1.8. Prove Proposition 1.2.

1.1.9. Prove Proposition 1.3.

1.1.10. Prove parts 2 and 3 of Proposition 1.4.

1.1.11. Prove the remaining part of Proposition 1.5.

1.1.12. Prove the second implication in Proposition 1.6.

1.1.13. Give an example showing that transitivity does not generally imply \( R \)-transitivity.

1.1.14. Prove the converse in Proposition 1.7.

1.1.15. Prove Proposition 1.8.

1.1.16. Prove the converse in Proposition 1.9.
1.1.17. Give an example showing that negative transitivity does not generally imply $P$-negative transitivity.

1.1.18. Prove Proposition 1.10.

1.1.19. Prove part 2 of Proposition 1.11.

1.1.20. Prove the remaining parts of Proposition 1.13.


1.1.22. Prove that, given any relation $B$, $(R_B, P_B)$ is a dual pair.

1.1.23. Prove that, if $B$ and $B'$ are strict quasi-orders, then $B \cap B'$ is also a strict quasi-order. Does a similar conclusion hold for the other classes of asymmetric relations we’ve seen? What about the complete relations? What about unions?

1.1.24. Classify the relations $\geq$, $>$, and $\gg$ on $\mathbb{R}^d$.

1.1.25. Prove Proposition 1.15.

1.2 Preference

Binary relations are especially convenient in representing a decision maker’s preferences over alternatives. We recognize three types of preferential judgments as fundamental. First, strict preference,

$$ P = \{(x, y) \in X \times X \mid x \text{ is strictly better than } y\}. $$

Second, indifference,

$$ I = \{(x, y) \in X \times X \mid x \text{ and } y \text{ are equally good}\}. $$

Third, weak preference, denoted $R$, meaning

$$ R = \{(x, y) \in X \times X \mid x \text{ is at least as good as } y\}. $$

We will always impose the first two of the following axioms on preference relations, whether preferences of an individual or of a group.

**Axiom 1**

- $P$ is asymmetric.
• $I$ is symmetric.
• $P \cap I = \emptyset$.
• $R = P \cup I$.

Equivalently, $P = P_R$ and $I = I_R$.

**Axiom 2**
- For all $x, y \in X$, either $xPy$ or $yPx$ or $xIy$.

Equivalently, under Axiom 1, $R = R_P$. Thus, Axioms 1 and 2 imply that weak and strict preference $(P, R)$ form a dual pair.

Note that preference relations may be individual or social preferences. They may be transitive or not, acyclic or not, etc. For individual preferences, however, we impose the following “ordering axiom.”

**Axiom 3**
- $P$ is transitive.
- $I$ is transitive.

Equivalently, under Axioms 1 and 2, $R$ is a weak order and $P$ is a strict order.

Note that, under Axioms 1-3, the indifference relation partitions $X$ into equivalence classes, $I(x)$, as discussed above. We refer to a class $I(x)$ as an *indifference class*. Or when $X \subseteq \mathbb{R}^d$, we use the term *indifference curve*. We often draw these to indicate the nature of a dual pair $(P, R)$.

An example important in political science is Euclidean preferences, i.e., there exists $\tilde{x} \in \mathbb{R}^d$ such that $xRy$ if and only if $\|x - \tilde{x}\| \leq \|y - \tilde{x}\|$. Here, for obvious reasons, we call $\tilde{x}$ the *ideal point* corresponding to $(P, R)$.

More generally, there are the *weighted Euclidean preferences*, i.e., there exists a positive definite, symmetric $d \times d$ matrix $A$ and $\tilde{x} \in \mathbb{R}^d$ such that $xRy$ if and only if $(x - \tilde{x})A(x - \tilde{x}) \leq (y - \tilde{x})A(y - \tilde{x})$.

When $A$ is diagonal, with diagonal elements $a_{hh}, h = 1, \ldots, d$, this reduces to: $xRy$ if and only if

$$\sum_{h=1}^{d} a_{hh}(x_h - \tilde{x}_h)^2 \leq \sum_{h=1}^{d} a_{hh}(y_h - \tilde{x}_h)^2$$
which generates indifference curves with an elliptical shape. Here, $a_{hh}$ can be interpreted as a “weight” on the $h$th dimension of $\mathbb{R}^d$. Setting $A$ equal to the identity matrix, we obtain Euclidean preferences as a special case.

You can check that the latter preferences are separable, in the sense that, if

$$(x_1, \ldots, x_{h-1}, x_h, x_{h+1}, \ldots, x_d) P(x_1, \ldots, x_{h-1}, y_h, x_{h+1}, \ldots, x_d),$$

then, for all $z_1, \ldots, z_{h-1}, z_{h+1}, \ldots, z_d$,

$$(z_1, \ldots, z_{h-1}, x_h, z_{h+1}, \ldots, z_d) P(z_1, \ldots, z_{h-1}, y_h, z_{h+1}, \ldots, z_d).$$

Examples from economics include Cobb-Douglas preferences, CES preferences, perfect complement preferences, and perfect substitute preferences.

**Continuity**

We will consider several properties of preference relations, more generally, dual pairs, in what follows.

Some concepts defined below involve ideas of open sets. What does it mean for a subset of $X$ to be open? The collection of open subsets of $X$ is called a “topology” on $X$. A topology can be defined on any $X$, and the only restrictions are that $\emptyset$ and $X$ are open, that finite intersections of open sets are open, and that arbitrary unions of open sets are open. A little more formally, we require $\emptyset, X \in T$, and $T$ must be closed with respect to finite intersections and arbitrary unions.

Once a topology, a collection $T$ of sets, has been defined for $X$, we can talk about continuity and compactness. For example, a function $f: X \rightarrow \mathbb{R}$ is continuous if, for every open $Y \subseteq \mathbb{R}$, we have $f^{-1}(Y) \in T$.\(^1\)

To define compactness, we say a collection $\mathcal{Y}$ of sets in $T$ is an open cover of $Y$ if $Y \subseteq \bigcup \mathcal{Y}$. We say $\mathcal{Y}'$ is a subcover of $\mathcal{Y}$ if $\mathcal{Y}' \subseteq \mathcal{Y}$. Formally, we say a set $Y \subseteq X$ is compact if every open cover of $Y$ has a finite subcover.\(^2\)

When $X$ is finite, it is universally understood that every subset of $X$ is open, i.e., $T = 2^X$. This is called the “discrete topology.” In this topology, every real-valued function is continuous, and every set is compact.

---

\(^1\)In finite-dimensional Euclidean space, continuity can also be defined in terms of convergent sequences: if $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$.

\(^2\)In finite-dimensional Euclidean space, an equivalent definition is that $Y$ is compact if every sequence in $Y$ has a subsequence that converges to an element of $Y$. 

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When $X = \mathbb{R}^d$, we define $Y \subseteq X$ to be open if it can be written as the union of “open balls” of the form

$$B_\epsilon(y) = \{ x \in \mathbb{R}^d \mid ||x - y|| < \epsilon \},$$

where $\epsilon > 0$ and $|| \cdot ||$ is the usual Euclidean norm.

More generally, when $X \subseteq \mathbb{R}^d$, we use the “relative topology” on $X$. We say $Y \subseteq X$ is open in $X$ (or relatively open in $X$) if there exists a subset $G \subseteq \mathbb{R}^d$ open in the usual sense such that $Y = X \cap G$. Thus, we generate open sets in $X$ by intersecting $X$ with open sets of $\mathbb{R}^d$.

Equivalently, $Y \subseteq X \subseteq \mathbb{R}^d$ is open in $X$ if, for every $y \in Y$, there exists $\epsilon > 0$ such that the relatively open ball,

$$B_\epsilon^X(y) = \{ x \in X \mid ||x - y|| < \epsilon \}$$

is contained in $Y$. Note the difference between this open ball and the open balls defined above: this one is restricted to $X$. When it is clear that the relative topology is intended, I will drop the superscript $X$.

Note that, if $X$ is a finite subset of $\mathbb{R}^d$, then the relative topology and the discrete topology coincide: every set is open. For this reason, when working with topological conditions, I can assume that $X$ is a subset of $\mathbb{R}^d$ and still capture finite sets as a special case.

A set $Y \subseteq X \subseteq \mathbb{R}^d$ is closed in $X$ if its complement in $X$, namely, $X \setminus Y$, is open in $X$. You can check that $Y \subseteq X$ is closed if, for every sequence $\{x_m\}$ in $Y$ converging to some $x \in X$, we have $x \in Y$.

For a relation $B$ we define the upper section and lower section at $x$ by

$$B(x) = \{ y \in X \mid yBx \}$$

$$B^{-1}(x) = \{ y \in X \mid xBy \},$$

respectively. Of course, $B(x)$ consists of the elements that bear $B$ to $x$, and $B^{-1}(x)$ consists of the elements to which $x$ bears $B$. Upper and lower sections are depicted in Figure 1.7.

Figure 1.7: Sections

Given dual pair $(P, R)$, we say $(P, R)$ is upper semicontinuous (usc) if, for all $x \in X$, the upper section $R(x)$ is closed. Equivalently, $(P, R)$ is upper semicontinuous if, for all $x \in X$,
the lower section $P^{-1}(x)$ is open. To verify this, note that
\[
R(x) = \{ y \in X \mid yRx \} \\
= \{ y \in X \mid \neg xPy \} \\
= \{ y \in X \mid y \notin P^{-1}(x) \} \\
= P^{-1}(x),
\]
which delivers the claim.

We say $(P, R)$ is lower semicontinuous (lsc) if, for all $x \in X$, the lower section $R^{-1}(x)$ is closed. Equivalently, $(P, R)$ is lower semicontinuous if, for all $x \in X$, the upper section $P(x)$ is open.

We say $(P, R)$ is continuous if it is upper semicontinuous and lower semicontinuous.

Note that, when $X$ is finite, every dual pair is continuous.

A terminological convention: The above continuity properties obviously entail restrictions on both $R$ and $P$, but, by convention, authors writing on this topic usually refer to just continuity of $R$, e.g., “suppose $R$ is upper semicontinuous . . . .” I may slip into this too.

A stronger condition is that $P$ is open in the product topology on $X \times X$, i.e., for every $(x, y) \in P$, there exist subsets $U$ and $V$ open in $X$ such that $x \in U$, $y \in V$, and $U \times V \subseteq P$. Equivalently, $R$ is closed in the product topology on $X \times X$, i.e., if a sequence $\{(x_m, y_m)\}$ converges to $(x, y)$, and if $x_mRy_m$ for all $m$, then $xRy$.

You should prove that, if $R$ is closed or $P$ is open, then $(P, R)$ is indeed continuous. The converse for weak orders is contained in the following proposition. It uses the assumption that $X$ is connected, meaning there do not exist disjoint, nonempty open sets of $X$, say $U$ and $V$, such that $X = U \cup V$.

**Proposition 1.16** Assume $X \subseteq \mathbb{R}^d$ is connected. Let $(P, R)$ be a dual pair such that $R$ is transitive and $P$ is negatively transitive. If $(P, R)$ is continuous, then $R$ is closed and $P$ is open.

**Proof:** Let $\{(x_m, y_m)\}$ be a sequence in $X \times X$ such that $x_mRy_m$ for all $m$ and $(x_m, y_m) \to (x, y)$. If not $xRy$, then $yPx$. By continuity, $P(x)$ and $P^{-1}(y)$ are open subsets of $X$. Furthermore, $X = P(x) \cup P^{-1}(y)$, for take any $z \in X$. If $z \notin P^{-1}(y)$, then $zRyPx$, then then $R$-transitivity implies $zPx$, i.e., $z \in P(x)$. Since $X$ is connected, there exists $z \in P(x) \cap P^{-1}(y)$. Because $P(z)$ is open and $y \in P(z)$, we have $y_m \in P(z)$ for high enough $m$. Because $P^{-1}(z)$ is open and $x \in P^{-1}(z)$, we have $x_m \in P^{-1}(z)$ for high enough $m$. Therefore, $y_mPzPz$ for high enough $m$. But then $R$-transitivity implies $y_mPzPz$, a contradiction. Therefore, $xRy$. \[\Box\]
To see that $R$-transitivity cannot be weakened to $P$-transitivity for the above result, let $X = [0, 2]$, and define $P$ as follows. For $x \in (1, 2)$, let
\[ P^{-1}(x) = [0, 1) \setminus [1 - (x/2), 2 - x]. \]
Let $P^{-1}(2) = [0, 1)$, and let $P^{-1}(x) = \emptyset$ for all $x \in [0, 1]$. Thus, $P$ is trivially transitive, and it is continuous too. In particular, $P^{-1}(2) = [0, 1)$ and $P(0) = (1, 2]$. So $(2, 0) \in P$. But there is no open set $U \times V \subseteq P$ containing $(2, 0)$. Why? Points arbitrarily close to 2 fail to bear $P$ to points arbitrarily close to 0.

**Convexity**

Assume $X$ is a convex subset of finite-dimensional Euclidean space, $\mathbb{R}^d$. We say a relation $B$ on $X$ is convex if, for all $x \in X$, $B(x)$ is convex. Now let $(P, R)$ be a dual pair. We say $(P, R)$ is

- **convex** if $R(x)$ and $P(x)$ are convex,
- **strictly convex** if, for all $y, z \in R(x)$ with $y \neq z$ and all $\alpha \in (0, 1)$, $\alpha y + (1 - \alpha) z \in P(x)$,
- **semi-convex** if $x \notin \text{conv} P(x)$,

for all $x \in X$. Figure 1.8 depicts a violation of semi-convexity.

**Proposition 1.17** Assume $X \subseteq \mathbb{R}^d$ is convex. Let $(P, R)$ be a dual pair.

1. **strictly convex** $\Rightarrow$ convex.
2. $P$ convex $\Rightarrow$ semi-convex.

If $R$ is transitive or $P$ is negatively transitive, then

3. semi-convex $\iff$ $P$ convex $\iff$ $R$ convex.

**Proof:** I’ll prove part 3. Assume $R$ is transitive. Note that the first $\iff$ direction follows from part 2. So suppose $(P, R)$ is semi-convex, take any $x \in X$, take any $y, z \in P(x)$, and take any $\alpha \in [0, 1]$. Let $w = \alpha y + (1 - \alpha) z$. If $w \notin P(x)$, then $x R w$. Since $y P x$ and $z P x$, transitivity implies that $y P w$ and $z P w$, i.e., $y, z \in P(w)$. But then $w \in \text{conv} P(w)$, contradicting semi-convexity. Now suppose $P$ is convex. Take any $x \in X$, any $y, z \in R(x)$, and any $\alpha \in [0, 1]$. Let $w = \alpha y + (1 - \alpha) z$. If $w \notin R(x)$, then $x P w$. Then $y R x P w$ implies $y P w$, and $z R x P w$ implies $z P w$. So $y, z \in P(w)$, and $P$ convex implies $w \in P(w)$, a contradiction. Finally, suppose $R$ is convex. Take any $x \in X$, any $y, z \in P(x)$, and any $\alpha \in [0, 1]$. Let $w = \alpha y + (1 - \alpha) z$. If $w \notin P(x)$, then $x R w$. By completeness, either $y R z$ or $z R y$. Without loss of generality, suppose $y R z$. Then $y, z \in R(z)$, and $R$ convex implies $w \in R(z)$. Then $w R z P x$ implies $w P x$, a contradiction. I’ll let you prove parts 1 and 2. ■
Thus, when $R$ is transitive or $P$ is negatively transitive, semi-convexity is equivalent to the typical convexity condition in the theory of the consumer.

Let’s check the tightness of Proposition 1.17.

- Convexity of $R$ does not imply semi-convexity unless $R$ is transitive: let $X = \mathbb{R}^2$, and define $P$ by $P^{-1}(0, 2/3) = \{ x \in \mathbb{R}^2 \mid x \cdot (1, -1) \geq -\epsilon \}$; $P^{-1}(1/2, -1/3) = \{ x \in \mathbb{R}^2 \mid x \cdot (-1, -1) \geq -\epsilon \}$; and $P^{-1}(-1/2, -1/3) = \{ x \in \mathbb{R}^2 \mid x \cdot (0, 1) \geq -\epsilon \}$. For $\epsilon > 0$ small, $(P, R)$ is a dual pair. Further, $R$ is convex (why?), but semi-convexity does not hold (consider $x = 0$). Note that this example uses a multidimensional set of alternatives.

- A consequence of the latter example, with part 2 of Proposition 1.17, is that convexity of $R$ does not imply convexity of $P$. In fact, we can show more: convexity of $P$ is not even implied by the conjunction of convexity of $R$ and semi-convexity unless $R$ is transitive, even in one dimension. Let $X = [0, 3]$, and let $P$ consist of pairs $(x, y)$ such that $x \in (1 - \epsilon, 1 + \epsilon) \cup (2 - \epsilon, 2 + \epsilon)$ and $y \in [0, \epsilon)$. For $\epsilon > 0$ small, $(P, R)$ is a dual pair. Further, it is semi-convex, and $R$ is convex, but $P$ is not convex.

- Convexity of $P$ does not imply convexity of $R$ unless $R$ is transitive, even when $X$ is unidimensional: let $X = [0, 3]$, and let $P$ consist of pairs $(x, y)$ such that $x \in (1 - \epsilon, 1 + \epsilon)$ and $y \in [0, \epsilon)$. For $\epsilon > 0$ small, $(P, R)$ is a dual pair. Further, $P$ is convex, but $R$ is not convex. (Why?)

I’ll let you construct an example showing that convexity does not imply strict convexity, even when $R$ is transitive. The next result shows that the first example above assumed a multidimensional set of alternatives out of necessity: in one dimension, convexity of $R$ does imply semi-convexity.

**Proposition 1.18** Assume $X \subseteq \mathbb{R}$ is convex. Let $(P, R)$ be a dual pair. If $R$ is convex, then $(P, R)$ is semi-convex.

**Proof:** Suppose $x \in \text{conv} P(x)$ for some $x \in X$. By Caratheodory’s theorem, there exist $y, z \in P(x)$ such that $y < x < z$. Assume without loss of generality that $z R y$. But then $y, z \in R(y)$ and convexity of $R$ implies $x \in R(y)$, a contradiction.

**Thin Indifference**

An implication of strict convexity is that an indifference class $I(x)$ cannot contain any open ball, but strict convexity implies more: indifference classes cannot even contain convex sets. For many results, this extra restriction is unnecessary, and we can work with a much weaker condition.
We say the dual pair \((P, R)\) satisfies thin indifference if, for all \(x \in X\),
\[
R(x) \subseteq \{x\} \cup \operatorname{clos}P(x).
\]
Equivalently, \(I_R(x) \setminus \{x\}\) contains an open set for no \(x\). (You should check this.)

Thin indifference is satisfied whenever \((P, R)\) is strictly convex, a condition that implicitly assumes \(X\) is infinite. It is also satisfied, regardless of any structure on \(X\), when \(R\) is anti-symmetric or, equivalently, \(P\) is total, for then \(R(x) \setminus \{x\} = P(x)\). Thus, thin indifference is consistent with \(X\) being finite, in which case it is actually equivalent to anti-symmetry of \(R\) (\(P\) being total), so the condition is quite general.

Of course, if \((P, R)\) is upper semicontinuous and satisfies thin indifference, then the inclusion in the definition of the condition holds with equality.

The next proposition establishes an implication of upper semi-continuity and thin indifference.

**Proposition 1.19** Assume \(X \subseteq \mathbb{R}^d\), and let \((P, R)\) be a dual pair. If \(R(x) = \{x\} \cup \operatorname{clos}P(x)\) and \(R(y) = \{y\} \cup \operatorname{clos}P(y)\), then \(P(x) \subseteq P(y)\) implies \(R(x) \subseteq R(y)\).

**Proof:** Suppose \(P(x) \subseteq P(y)\). Clearly, \(\{x\} \cup \operatorname{clos}P(x) \subseteq \{x\} \cup \operatorname{clos}P(y)\). Note that \(y \notin P(x)\), by asymmetry, so \(x \in R(y)\). Therefore, under the assumption of the proposition,
\[
R(x) = \{x\} \cup \operatorname{clos}P(x) \subseteq \{x\} \cup \operatorname{clos}P(y) \subseteq R(y),
\]
as required.

Given \(Y \subseteq X\), we say \((P, R)\) satisfies thin indifference in \(Y\) if, for all \(x \in Y\),
\[
R(x) \cap Y \subseteq \{x\} \cup \operatorname{clos}(P(x) \cap Y),
\]
where here “closure” refers to the relative topology on \(Y\). Obviously, setting \(Y = X\), we have the above condition. The above proposition extends to subsets as well.

**Utility Representations**

A function \(u : X \to \mathbb{R}\) is a utility representation of the dual pair \((P, R)\) if, for all \(x, y \in X\),
\[
u(x) \geq u(y)\text{ if and only if }xRy,\text{ or equivalently, }u(x) > u(y)\text{ if and only if }xPy.
\]
In this case, we say that \(u\) represents \((P, R)\).

It is often convenient to use utility representations when working with specific preference relations. Euclidean preferences with ideal point \(\bar{x}\), for example, have the natural utility representation
\[
u(x) = -||x - \bar{x}||.
\]
Weighted Euclidean preferences with matrix $A$ have the utility representation

$$u(x) = -(x - \bar{x})A(x - \bar{x}).$$

Of course, Cobb-Douglas, CES, perfect complement, and perfect substitute preferences all have utility representations too.

The next proposition tells us which dual pairs have utility representations. We say $Y \subseteq X$ is $R$-order dense if, for all $x, y \in X$, $xPy$ implies there exists $z \in Y$ such that $xRzPy$.

**Proposition 1.20** Assume $X \subseteq \mathbb{R}^d$. Let $(P, R)$ be a dual pair. There is a utility representation for $(P, R)$ if and only if $R$ is transitive, $P$ is negatively transitive, and there is a countable $R$-order dense subset $Y \subseteq X$.

**Proof:** Suppose $(P, R)$ has a utility representation $u$. Clearly, $R$ is transitive and $P$ is negatively transitive. Note that $u(X)$, as a subset of the real numbers, is separable.³ Let $V$ be a countable dense subset of $u(X)$, and construct $Y^1$ by choosing one element from each $u^{-1}(v)$, where $v \in V$.⁴ Let $G \subseteq \mathbb{R}$ consist of real numbers $v \in u(X)$ such that, for some $w \in \mathbb{R}$ with $w < v$, $(w, v) \cap u(X) \neq \emptyset$. Clearly, $G$ is a countable set. (Why?) Construct $Y^2$ by choosing one element from each $u^{-1}(v)$, where $v \in G$. I claim that $Y = Y^1 \cup Y^2$ is $R$-order dense. Suppose $xPy$, so $u(x) > u(y)$. If the set $u(X) \cap (u(y) , u(x))$ is nonempty, then there exists $z \in Y^1$ such that $u(y) < u(z) < u(x)$, implying $xPzPy$, as required. If the set is empty, then $u(x) \in G$, and there exists $z \in Y^2$ such that $u(y) < u(z) = u(x)$, implying $xRzPy$, as required. Now suppose that $R$ is transitive, equivalently $P$ negatively transitive, and let $Y = \{y_1, y_2, \ldots\}$ be an $R$-order dense set. Given $x \in X$, define

$$u(x) = \sum_{k: xRy_k} \frac{1}{2^k}.$$  

I claim that this is a utility representation. Suppose $xRy$. By transitivity of $R$, $yRy_k$ implies $xRy_k$, which implies $u(x) \geq u(y)$. Suppose $u(x) \geq u(y)$. If not $xRy$, then $yPx$, and $u(y) > u(x)$. Furthermore, there is some $z \in Y$ such that $yRzPx$, which implies $u(y) > u(x)$, a contradiction. Therefore, $xRy$, as required.

You might think about how the above result would change if I had defined an $R$-order dense set using “$xPzRy$” instead of “$xRzPy$.”

The existence of an $R$-order dense set is quite weak. It is always satisfied when the set of alternatives is finite or countably infinite. (Why?)

Nevertheless, there are weak orders with no $R$-order dense set and, therefore, with no utility representation. The canonical example is the lexicographic ordering of $\mathbb{R}_+^d$, i.e.,

³A set is separable if it has a countable dense subset.
⁴This step uses the axiom of choice. Don’t worry about it.
$(x_1, \ldots, x_d) P (y_1, \ldots, y_d)$ if and only if there is some $k$ such that (i) for all $j = 1, \ldots, k - 1$, $x_j = y_j$, and (ii) $x_k > y_k$. Can you see why this relation has no countable $R$-order dense set?

Are there weak orders with no utility representation when $X \subseteq \mathbb{R}$? Yes. Let $f: \mathbb{R} \rightarrow \mathbb{R}^2$ be a 1-1, onto mapping.\(^5\) Letting $PL$ denote the asymmetric part of the lexicographic ordering, define the relation $P$ on $\mathbb{R}$ as follows: $x P y$ if and only if $f(x) PL f(y)$. If this relation had a utility representation, say $u: \mathbb{R} \rightarrow \mathbb{R}$, then $u \circ f^{-1}$ would be a utility representation of the lexicographic ordering (Do you see why?), an impossibility.

In fact, no well-ordering of any uncountable set $X$ has a utility representation. Given a countable set $Y \subseteq X$, let $y_1$ be the least element of that set, let $y_2$ be the least element of $Y$ greater than $y_1$, etc. Note that $X \setminus Y$ is uncountable, so either $\{x \in X \setminus Y \mid x < y_1\}$ has more than one element or $\{x \in X \setminus Y \mid \forall i: y_i < x\}$ does or $\{x \in X \mid y_i < x < y_{i+1}\}$ does for some $i$. In either case, we have elements $x, z \in X$ such that $x < z$, with no element of the order dense set separating them, a contradiction.\(^6\)

We sometimes assume the existence of continuous utility representations, i.e., for every open set $V \subseteq \mathbb{R}$, $u^{-1}(V)$ is open. Clearly, a necessary condition is that $(P, R)$ is continuous. (Why?) Gerard Debreu has established very weak additional conditions that, with continuity of the dual pair, are sufficient for this. Unfortunately, the proof of the general result is very involved. Rather than provide that, I give a simple proof of a less general version of the result, assuming $X$ is a subset of Euclidean space and “indifference classes” of $(P, R)$ have Lebesgue measure zero. The latter assumption is rather strong, but it holds, for example, for consumers with monotonic preferences. The proof can be easily modified to accommodate a finite number of indifference classes with positive measure, but a countably infinite number poses problems for this approach.\(^7\)

**Proposition 1.21 (Debreu)** Assume $X \subseteq \mathbb{R}^d$ is connected. Let $(P, R)$ be a dual pair such that $R$ is transitive and $P$ is negatively transitive. If $(P, R)$ is continuous, then it has a continuous utility representation.

**Proof:** I will prove the theorem under the stronger assumptions that $X$ is the closure of its interior (in $\mathbb{R}^d$), and that, for all $x \in X$, the set $I(x) = I_R(x)$ has Lebesgue measure zero. Let $\mu$ be the probability measure corresponding to any continuous distribution on $X$ with strictly positive density (such as the normal distribution conditioned on $X$), so $\mu$ is absolutely continuous with respect to Lebesgue measure. For each $x \in X$, define $u(x) = \mu(R^{-1}(x))$. If $x R y$, then $R^{-1}(y) \subseteq R^{-1}(x)$, which implies $u(x) \geq u(y)$. If $x P y$, not

---

\(^5\)The existence of such a mapping follows from the fact that $\mathbb{R}$ and $\mathbb{R}^2$ have the same cardinality. Don’t worry about this.

\(^6\)Don’t worry about this.

\(^7\)An example of the problem is that the unit interval contains a countably infinite number of closed subintervals with Lebesgue measure one, e.g., the complement of the Cantor set modified by taking out closed “middle thirds,” rather than open. Don’t worry about this.
that $P^{-1}(x)$ and $P(y)$ are open sets and $X \subseteq P^{-1}(y) \cup P(x)$. Because $X$ is connected, it follows that $P^{-1}(y) \cap P(x) \neq \emptyset$. Because $X$ is the closure of its interior, $P^{-1}(y) \cap P(x)$ contains an open subset of $\mathbb{R}^d$, which has positive $\mu$-measure. Therefore, $u(x) > u(y)$.

To prove continuity, let $\{x_m\}$ be a sequence converging to $x$. Letting $g_m$ be the indicator function of $R^{-1}(x_m)$, we have

$$u(x_m) = \int g_m(z)f(z)\mu(dz),$$

and, letting $g$ be the indicator function of $R^{-1}(x)$, we have

$$u(x) = \int g(z)f(z)\mu(dz).$$

Take any $z \in X \setminus I(x)$. If $x \in P(z)$, then $g(z) = 1$. Furthermore, because $P(z)$ is open, we have $x_m \in P(z)$ for high enough $m$, so $g_m(z) = 1$ for high enough $m$. If $x \in P^{-1}(z)$, then $g(z) = 0$. Again, $x_m \in P^{-1}(z)$ for high enough $m$, so $g_m(z) = 0$ for high enough $m$.

Thus, since $\mu(I(x)) = 0$, $g_m$ converges pointwise $\mu$-almost everywhere to $g$, and Lebesgue’s dominated convergence theorem implies that $u(x_m)$ converges to $u(x)$.

In fact, when we use utility representations, we will be assuming differentiability. I do not know of sufficient conditions on weak and strict preference for the existence of differentiable representations.

The conditions of convexity and strict convexity have straightforward analogues in terms of utility functions. Recall that $u$ is quasi-concave if, for all $x, y \in X$ and all $\alpha \in (0, 1)$, $u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$. It is strictly quasi-concave if, for all $x, y \in X$ and all $\alpha \in (0, 1)$, $u(\alpha x + (1 - \alpha)y) > \min\{u(x), u(y)\}$.

**Proposition 1.22** Assume $X \subseteq \mathbb{R}^d$ is convex. Let $(P, R)$ be a dual pair with utility representation $u$.

1. $(P, R)$ is convex if and only if $u$ is quasi-concave.
2. $(P, R)$ is strictly convex if and only if $u$ is strictly quasi-concave.

We take preferences as fundamental, because we are all familiar with the idea from personal experience, but our interest in them lies in how they determine…

**Exercises**

1.2.1. Prove that Axiom 1 is equivalent to $P = P_R$ and $I = I_R$, and that, under Axiom 1, Axiom 2 is equivalent to $R = R_P$. 25
1.2.2. Prove that weighted Euclidean preferences are separable.

1.2.3. Prove that, when $X \subseteq \mathbb{R}^d$, $Y \subseteq X$ is open in $X$ if and only if, for every $y \in Y$, there exists $\epsilon > 0$ such that $B_{\epsilon}(y) \subseteq Y$.

1.2.4. Prove that $Y$ is open in $X$ if and only if $X \setminus Y$ is closed in $X$.

1.2.5. Prove that, if $R$ is closed or if $P$ is open, then $(P, R)$ is continuous.

1.2.6. In the example following Proposition 1.16, why is there no open set $U \times V \subseteq P$ containing $(2, 0)$?

1.2.7. Prove the remaining parts of Proposition 1.17.

1.2.8. In the first example following Proposition 1.17, why is $R$ convex?

1.2.9. In the second example following Proposition 1.17, why is $P$ not convex?

1.2.10. In the third example following Proposition 1.17, why is $R$ not convex?

1.2.11. Show that convexity does not imply strict convexity, even when $R$ is transitive.

1.2.12. Prove that, given a dual pair $(P, R)$, if $\{x\} \cup P(x) = \{x\} \cup \text{int}R(x)$ and $\{y\} \cup P(y) = \{y\} \cup \text{int}R(y)$, then $R(x) \subseteq R(y)$ implies $P(x) \subseteq P(y)$.

1.2.13. Prove that the following conditions are equivalent for a dual pair $(P, R)$:
   - for all $x \in X$, $\{x\} \cup P(x) = \{x\} \cup \text{int}R(x)$,
   - for all $x \in X$, $R^{-1}(x) = \{x\} \cup \text{clos}P^{-1}(x)$.

1.2.14. In the proof of Proposition 1.20, why is $G$ a countable set?

1.2.15. How would the statement and proof of Proposition 1.20 have to be changed if we defined the idea of an $R$-order dense set using “$xPzRy$” instead of “$xRzPy$”?

1.2.16. Why does the lexicographic ordering on $\mathbb{R}_+^d$ have no countable $R$-order dense set?

1.2.17. In the discussion prior to Proposition 1.21, why would $u \circ f^{-1}$ be a utility representation of the lexicographic ordering?

1.2.18. Suppose the dual pair $(P, R)$ has a continuous utility representation. Prove that the dual pair is continuous. Does it have to be closed?
1.2.19. Prove Proposition 1.22.

1.3 Choice

Given a dual pair \((P, R)\) representing weak and strict preferences, which alternatives are viable choices?

For strict preference \(P\), we would specify the set of “undominated alternatives.” Generally, the undominated set of relation \(B\) in set \(Y \subseteq X\) is

\[
UD(Y, B) = \{x \in Y \mid \forall y \in Y : \neg yBx\}.
\]

When \(Y = X\), we just write \(UD(B)\) for \(UD(X, B)\). Thus, given \(P\), the choice set from \(X\) is \(UD(P)\).

For weak preference \(R\), we would specify the set of “dominant alternatives.” Generally, the dominant set of relation \(B\) in set \(Y \subseteq X\) is

\[
D(Y, B) = \{x \in Y \mid \forall y \in Y : xBy\}.
\]

Write \(D(B)\) for \(D(X, B)\). Thus, given \(R\), the choice set from \(X\) is \(D(R)\).

We can actually unify these two definitions. The maximal set of relation \(B\) in set \(Y \subseteq X\) is

\[
M(Y, B) = \{x \in Y \mid \forall y \in Y : yBx \Rightarrow xBy\}.
\]

That is, \(x \in M(Y, B)\) if and only if, for all \(y \in Y\), either \(xBy\) or not \(yBx\). Write \(M(B)\) for \(M(X, B)\). Note that, if \(1 \leq |Y| \leq 2\), then \(M(Y, B) \neq \emptyset\).

**Proposition 1.23** Let \(B\), \(P\), and \(R\) be relations.

1. \(M(Y, B) = UD(Y, P)\) for all \(Y \subseteq X\) if and only if \(P = PB\).
2. \(M(Y, B) = D(Y, R)\) for all \(Y \subseteq X\) if and only if \(R = RB\).

**Proof:** I will prove part 1. For the “if” direction, note that

\[
M(Y, B) = \{x \in Y \mid \forall y \in Y : yBx \Rightarrow xBy\}
\]

\[
= \{x \in Y \mid \forall y \in Y : \neg[yBx \land \neg xBy]\}
\]

\[
= \{x \in Y \mid \forall y \in Y : \neg yPBx\}
\]

\[
= UD(Y, PB).
\]

For the “only if” direction, assume \(M(Y, B) = UD(Y, P)\) for all subsets \(Y\). If \(P \neq PB\), then there exist \(x, y \in X\) such that \((x, y) \in P \setminus PB\) or \((x, y) \in PB \setminus P\). In either case, we have \(UD(\{x, y\}, P) \neq UD(\{x, y\}, PB) = M(\{x, y\}, B)\), a contradiction. Part 2 is proved similarly. 

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Obviously, it follows that $M(Y, B) = UD(Y, P_B)$ and $M(Y, B) = D(Y, R_B)$ for all $Y \subseteq X$. Applying Proposition 1.23 again, this implies $M(Y, B) = M(Y, P_B) = M(Y, R_B)$. Another implication is the following “equivalence” result for dual pairs.

**Proposition 1.24** Let $(P, R)$ be a dual pair. For all $Y \subseteq X$,


**Proof:** From $P = P_B$ and Proposition 1.23, we have $M(Y, R) = UD(Y, P)$. Similarly, from $P = P_P$, we have $M(Y, P) = UD(Y, P)$. Finally, from $R = R_P$, we have $M(Y, P) = D(Y, R)$. □

The upshot is this: we can talk about the “choice set” of a dual pair without ambiguity.

Before considering the question of nonemptiness of these sets, we note that thin indifference is sufficient for there to be at most one maximal element. Furthermore, the condition gives us an “external stability” property of the maximal alternative (if any).

**Proposition 1.25** Let $(P, R)$ be a dual pair, and let $Y \subseteq X$. If $(P, R)$ satisfies thin indifference in $Y$, then either $M(Y, R) = M(Y, P) = \emptyset$ or, for some $x \in Y$, we have $M(Y, R) = M(Y, P) = \{x\}$. In the latter case, for all $y \in Y \setminus \{x\}$, we have $xPy$.

**Proof:** Suppose $x \in M(Y, R)$ and $yRx$ for some $y \in Y \setminus \{x\}$. This implies $y \in R(x) \cap Y \subseteq \{x\} \cup \text{clos}(P(x) \cap Y)$. Since $y \neq x$, this implies $y \in \text{clos}(P(x) \cap Y)$, implying $P(x) \cap Y \neq \emptyset$ and contradicting $x \in M(Y, R)$. Therefore, $xPy$, and the proposition follows. □

**Existence of Maximal Elements**

We have noted that maximal elements always exist in one or two element sets. When do maximal elements exist generally, i.e., when are choice sets nonempty? Not always: see Figure 1.9, for example.

**Figure 1.9:** An empty choice set

The next proposition characterizes our acyclicity conditions in terms of their implications for nonemptiness of choice sets. Recall from Proposition 1.5 that $B$ is a weak suborder if and only if it is negatively acyclic, and that $B$ is a strict suborder if and only if it is acyclic.
Proposition 1.26 A relation $B$ is...

1. $P$-acyclic if and only if $M(Y, B) \neq \emptyset$
2. a weak suborder if and only if $D(Y, B) \neq \emptyset$
3. a strict suborder if and only if $UD(Y, B) \neq \emptyset$

for all finite $Y \subseteq X$.

Proof: 1. I will prove that, if $B$ is $P$-acyclic, then $M(Y, B) \neq \emptyset$ for every finite $Y \subseteq X$. Take any such $Y$, and suppose that $M(Y, B) = \emptyset$. Take any $x_1 \in Y$. Since $x_1 \notin M(Y, B)$, there exists $x_2 \in Y$ such that $x_2 P_B x_1$. Similarly, there exists $x_3 \in Y$ such that $x_3 P_B x_2$, and so on. In this way, we can define an infinite sequence $x_1, x_2, x_3, \ldots$. Since $Y$ is finite, there must be some alternative that appears twice in this sequence, say $x_k$ and $x_{k+l}$. But then we have

$$x_{k+l} P_B x_{k+l-1} P_B x_{k+l-2} P_B \cdots x_{k+1} P_B x_k = x_{k+l},$$

contradicting $P$-acyclicity. You should prove the converse. 2. Note, from Proposition 1.5, that $B$ is negatively acyclic if and only if $B$ is complete and $P$-acyclic. This is true if and only if $B$ is complete and $M(Y, B) \neq \emptyset$ for all finite $Y$. That these conditions imply $D(Y, B) \neq \emptyset$ for all finite $Y$ follows since, by Proposition 1.24 and completeness of $B$, $M(Y, B) = D(Y, B)$. For the converse, suppose $D(Y, B) \neq \emptyset$ for all finite $Y$. Completeness of $B$ follows by letting $Y$ range over all two-element sets. And then $M(Y, B) \neq \emptyset$ also follows from Proposition 1.24. I’ll leave part 3 for you.

As a corollary it follows that, if $B$ is $P$-transitive, transitive, or $R$-transitive, then it has maximal elements in finite sets.

Corollary 1.1 Let $B$ be a relation. For all finite $Y \subseteq X$,

1. if $B$ is $P$-transitive, then $M(Y, B) \neq \emptyset$,
2. if $B$ is transitive, then $M(Y, B) \neq \emptyset$,
3. if $B$ is $R$-transitive, then $M(Y, B) \neq \emptyset$.

You can check that, as a consequence, $B$ has maximal elements in finite sets if $B$ is $P$-negatively transitive, negatively transitive, or $R$-negatively transitive.

We can also characterize our other transitivity conditions in terms of their consequences for choice sets.
Proposition 1.27 Let $B$ be a relation.

1. \[
\begin{align*}
&\text{For all } Y \text{ with } 1 \leq |Y| \leq 3: \\
&\quad D(Y, B) \neq \emptyset \text{ and } \forall x \in Y \setminus D(Y, B) \exists y \in D(Y, B) \text{ s.t. } y \mathrel{P_B} x.
\end{align*}
\Rightarrow
\begin{align*}
&B \text{ is a weak quasi-order.} \\
&\text{For all finite } Y: \\
&\quad D(Y, B) \neq \emptyset \text{ and } \forall x \in Y \setminus D(Y, B) \exists y \in D(Y, B) \text{ s.t. } y \mathrel{P_B} x.
\end{align*}
\]

2. \[
\begin{align*}
&\text{For all } Y \text{ with } 1 \leq |Y| \leq 3: \\
&\quad D(Y, B) \neq \emptyset \text{ and } \forall x \in Y \setminus D(Y, B) \forall y \in D(Y, B) \exists y \mathrel{P_B} x.
\end{align*}
\Rightarrow
\begin{align*}
&B \text{ is a weak order.} \\
&\text{For all finite } Y: \\
&\quad D(Y, B) \neq \emptyset \text{ and } \forall x \in Y \setminus D(Y, B) \forall y \in D(Y, B) \exists y \mathrel{P_B} x.
\end{align*}
\]

3. \[
\begin{align*}
&\text{For all } Y \text{ with } 1 \leq |Y| \leq 3: \\
&\quad |D(Y, B)| = 1.
\end{align*}
\Rightarrow
\begin{align*}
&B \text{ is a weak linear order.} \\
&\text{For all finite } Y: \\
&\quad |D(Y, B)| = 1.
\end{align*}
\]

Proof: I’ll prove part 1. Assume $D(Y, B) \neq \emptyset$ for every nonempty set $Y$ with three or fewer elements. Furthermore, assume that, for all $x \notin D(Y, B)$, there exists $y \in D(Y, B)$ such that $y \mathrel{P_B} x$. Take any $x, y \in X$, and note that either $x \in D(\{x, y\}, B)$ or $y \in D(\{x, y\}, B)$. Thus, either $x \mathrel{B} y$ or $y \mathrel{B} x$, so $B$ is complete. Now suppose $x \mathrel{P_B} y \mathrel{P_B} z$. Note that $D(\{x, y, z\}, B) = \{x\}$. (Why?) Then, by assumption, we have $x \mathrel{P_B} z$. Therefore, $B$ is a weak quasi-order. Now assume $B$ is a weak quasi-order. Then Proposition 1.26 implies that $D(Y, B) \neq \emptyset$ for all nonempty $Y$ with three or fewer elements. Take any such $x, y, z \in X$ and suppose $x \notin D(Y, B)$ and $z \in D(Y, B)$. Thus, either $x \mathrel{B} z$ or not $x \mathrel{B} y$. In the first case, by completeness, we have $z \mathrel{P_B} x$, as required. In the second case, $y \mathrel{P_B} x$. If $y \in D(Y, B)$, we are done. If not, then $z \mathrel{P_B} y$, and $P$-transitivity implies $z \mathrel{P_B} x$, as required. \[\blacksquare\]

You might think about how the statement of Proposition 1.27 would have to be modified if we replaced the dominant set of $B$ with the maximal set or the undominated set.

Infinite Sets of Alternatives

What about infinite $X$’s? Let $X = \{x_1, x_2, x_3, \ldots\}$. Obviously, the “infinitely ascending” relation $B$, below, does not have a maximal element, though it is a weak linear order (and is, therefore, $P$-acyclic).

\[
B
\begin{array}{l}
\vdots \\
x_3 \\
x_2 \\
x_1 \\
\end{array}
\]
The problem here is that \( X \) is non-compact (in the discrete topology).

So suppose we add compactness of \( X \). Is that enough? No. Let \( X = [0,1] \), and define the dual pair \((P,R)\) by the utility representation \( u: [0,1] \to \mathbb{R}\) as follows: \( u(x) = 1/x \) for all \( x \in (0,1] \), and \( u(0) = 0 \). The relation \( R \) is a weak linear order, and \( P \) is a strict linear order, yet \((P,R)\) has no maximal element.

The problem is that the dual pair above is not continuous, or more precisely, not upper semicontinuous. We now prove a standard result on existence of maximal elements for relations on infinite sets.

**Proposition 1.28** Assume \( X \subseteq \mathbb{R}^d \), and let \((P,R)\) be a dual pair. Assume \( Y \subseteq X \) is compact and \((P,R)\) is upper semicontinuous. If \( P \) is acyclic or \( R \) is negatively acyclic, then \( M(Y,P) = M(Y,R) \neq \emptyset \).

**Proof:** Suppose \( P \) is acyclic and \( M(Y,P) = \emptyset \), so for every \( x \in Y \) there is some \( y \in Y \) such that \( yP x \). Thus,

\[
Y \subseteq \bigcup_{y \in Y} P^{-1}(y),
\]

i.e., \( \{P^{-1}(y) \mid y \in Y\} \) is an open cover of \( Y \). By compactness of \( X \), there is a finite subcover. That is, there exist \( y_1, y_2, \ldots, y_k \in Y \), such that

\[
Y \subseteq \bigcup_{h=1}^{k} P^{-1}(y_h).
\]

From Proposition 1.26, however, there is some \( y_j \in M(\{y_1, y_2, \ldots, y_k\}, P) \). But \( y_j \in P^{-1}(y_h) \) for some \( h \), a contradiction. Of course, negative acyclicity of \( R \) is equivalent to acyclicity of \( P \).

The next corollary is immediate.

**Corollary 1.2** Assume \( X \subseteq \mathbb{R}^d \), and let \((P,R)\) be a dual pair. Assume \( Y \subseteq X \) is compact and \((P,R)\) is upper semicontinuous. If either \( R \) or \( P \) is transitive or negatively transitive, then \( M(Y,R) = M(Y,P) \neq \emptyset \).

Assuming compactness and upper semicontinuity, Proposition 1.28 gives a sufficient condition, namely \( P \)-acyclicity, for the existence of maximal elements. The condition of \( P \)-acyclicity is, however, clearly not necessary. (Do you see why?) We next consider a useful necessary and sufficient condition for existence of maximal elements.
Let \((P, R)\) be a dual pair, and let \(Y \subseteq X\). Using the terminology of Austen-Smith and Banks, we say \((P, R)\) satisfies \textit{Condition F in }\(Y\) if, for every finite subset \(Z \subseteq Y\), there exists \(x \in Y\) (not necessarily in \(Z\)) such that, for all \(y \in Z\), \(xRy\). Equivalently, there is no finite subset \(Z \subseteq Y\) such that, for all \(x \in Y\), there exists \(y \in Z\) with \(yPx\). Note that the “only if” direction in the next proposition does not use compactness.

\textbf{Proposition 1.29} Assume \(X \subseteq \mathbb{R}^d\), and let \((P, R)\) be a dual pair. Assume \(Y \subseteq X\) is compact and \((P, R)\) is upper semicontinuous. Then \(M(Y, P) = M(Y, R) \neq \emptyset\) if and only if \((P, R)\) satisfies \textit{Condition F in }\(Y\).

\textbf{Proof:} First, assume \textit{Condition F} holds in \(Y\), and suppose that \(M(Y, P) = \emptyset\). As in the proof of Proposition 1.28, we have

\[ Y \subseteq \bigcup_{x \in Y} P^{-1}(x). \]

By compactness of \(Y\), there exist \(y_1, \ldots, y_k\) such that

\[ Y \subseteq \bigcup_{h=1}^{k} P^{-1}(y_h), \]

contradicting \textit{Condition F}. Now assume \(M(Y, P) \neq \emptyset\). If \textit{Condition F} does not hold in \(Y\), then there is a finite set \(Z \subseteq Y\) such that, for every \(x \in Y\), there exists \(y \in Z\) such that \(yPx\). But then \(M(Y, P) = \emptyset\), a contradiction. \(\blacksquare\)

Thus, under some background conditions, \textit{Condition F} is necessary and sufficient for the existence of maximal elements. But how helpful is the condition as a sufficient condition for existence? That is, how far is it from simply stating the existence of maximal elements? When \(X\) is infinite, we will see that \textit{Condition F} can be quite helpful. When \(X\) is finite, however, \textit{Condition F} is little help: in that case, setting \(Z = X\) in the the condition, it just says, “...there exists \(x \in X\) such that, for all \(y \in X\), \(xRy\).” In other words, it says, “\(M(P) \neq \emptyset\)?”

We next establish the sufficiency of upper semi-continuity and semi-convexity for \textit{Condition F}, allowing us to use the previous results on nonempty choice sets to establish existence of maximal elements.

\textbf{Proposition 1.30} Assume \(X \subseteq \mathbb{R}^d\), and let \((P, R)\) be a dual pair. Assume \(Y \subseteq X\) is convex, and \((P, R)\) is upper semicontinuous and semi-convex. Then \((P, R)\) satisfies \textit{Condition F in }\(Y\).
Proof: By the KKM Theorem (see Theorem 16.40 in Aliprantis and Border’s (2000) Infinite Dimensional Analysis), if \( K = \text{conv}\{y^1, \ldots, y^m\} \subseteq \mathbb{R}^d \) and \( \{Y_1, \ldots, Y_m\} \) is a family of closed subsets of \( \mathbb{R}^d \) such that, for every \( I \subseteq \{1, \ldots, m\} \),

\[
\text{conv}\{y^j \mid j \in I\} \subseteq \bigcup_{j \in I} Y_j,
\]

then \( K \cap \bigcap_{j=1}^m Y_j \) is compact and nonempty. See Figure 1.10.

Figure 1.10: The KKM Theorem

To prove the proposition, take any convex \( Y \subseteq X \) and any finite subset \( \{y^1, \ldots, y^m\} \) of \( Y \). By upper semicontinuity, \( \{R(y^1), \ldots, R(y^m)\} \) is a family of closed subsets of \( \mathbb{R}^d \). Take any \( I \subseteq \{1, \ldots, m\} \). If it is not the case that \( \text{conv}\{y^j \mid j \in I\} \subseteq \bigcup_{j \in I} R(y^j) \) \( (1.1) \), then there is some \( z \in \text{conv}\{y^j \mid j \in I\} \) such that \( z \notin \bigcup_{j \in I} R(y^j) \), i.e., \( z \in \bigcap_{j \in I} P^{-1}(y^j) \). So \( y^j \in P(z) \) for all \( j \in I \). But then \( z \) is a convex combination of preferred elements, contradicting semi-convexity. Therefore, \( (1.1) \) holds for every \( I \subseteq \{0, 1, \ldots, m\} \), and the KKM Theorem and convexity of \( Y \) imply \( Y \cap \bigcap_{j=1}^m R(y^j) \neq \emptyset \), fulfilling Condition F in \( Y \).

Adding compactness to upper semi-continuity and semi-convexity, therefore, Proposition 1.29 yields \( M(Y, P) \neq \emptyset \).

Exercises

1.3.1. Prove part 2 of Proposition 1.23.

1.3.2. Prove part 3 and the converse of part 1 in Proposition 1.26.

1.3.3. Prove Corollary 1.1.

1.3.4. Prove parts 2 and 3 of Proposition 1.27.

1.3.5. How would you have to modify the statement of Proposition 1.27 if we replaced the dominant set of \( \mathbb{B} \) with the maximal set or the undominated set?

1.3.6. Let \( (P, R) \) be a dual pair on \( X = [0,1] \) with utility representation \( u(x) = 1/x \) for \( x > 0 \) and \( u(0) = 0 \). Show that \( (P, R) \) is not upper semicontinuous.
1.3.7. Give an example showing that $P$-acyclicity is not necessary for the existence of maximal elements.

1.3.8. Give an example of a continuous dual pair $(P, R)$ on $[0,1]$ such that $P$ is transitive and there exists $x \notin M(P)$ such that, for all $y \in M(P)$, $xRy$.

1.3.9. Given a dual pair $(P, R)$, prove that, if $M(Y, R) = M(Y, P) \neq \emptyset$ for all finite $Y \subseteq X$, then $(P, R)$ satisfies Condition F.

1.4 Top Cycle Sets

When maximal elements of a preference relation do not exist, the standard rational choice theory approach fails to generate a prediction. How might choices be determined in the absence of maximal elements? In short, are there alternatives to maximality for the construction of choice sets? Or are there at least reasonable bounds we might place on choices when maximal elements do not exist?

For the asymmetric relation in Figure 1.11, for example, there is no maximal element, but we may reasonably predict that a final choice would belong to $\{a, b, c\}$, because each of these elements is preferred to $d$. It is the smallest such set, however, so this logic cannot be used to narrow our prediction any further. This set is an example of a “top cycle,” an idea introduced by Tom Schwartz.

Figure 1.11: A top cycle

Given a relation $B$, a set $Y \subseteq X$, and $x, y \in Y$, say $xT_B^Y y$ if there exist $k$ and $x_1, \ldots, x_k \in Y$ such that

$$xBx_1Bx_2B\cdots x_{k-1}Bx_k = y.$$ That is, $xT_B^Y y$ if we can get from $x$ to $y$ in a finite number of $B$-steps within the set $Y$.

The relation $T_B^Y$ is not generally complete or asymmetric. If $B$ is complete, then $T_B^Y$ will be complete, obviously. Asymmetry of $B$ does not imply asymmetry of $T_B^Y$, however: consider $X = \{x, y, z\}$ and $B = \{(x, y), (y, z), (z, x)\}$.

The top cycle set of $B$ in $Y$ is

$$TOP(Y, B) = M(Y, T_B^Y).$$ When $Y = X$, we just write $TOP(B)$ for $TOP(X, B)$.

If $B$ is complete, so $T_B^Y$ is complete as well, then Proposition 1.23 implies that $TOP(Y, B) = D(Y, T_B^Y)$, i.e., $x \in TOP(Y, B)$ if and only if, for all $y \in Y$, we can get from $x$ to $y$ in a finite number of $B$-steps.
Note that $T^Y_B$ is transitive, so, by Corollary 1.1, $M(Y, T^Y_B) \neq \emptyset$ for all finite $Y \subseteq X$. Thus, the top cycle is always nonempty in finite sets, regardless of the underlying relation $B$. We will consider infinite sets of alternatives later.

Given a dual pair $(P, R)$, a choice set can be constructed using the idea of a top cycle in two ways, depending on whether weak preference or strict preference is used.

**The Weak Top Cycle**

Given dual pair $(P, R)$ and set $Y \subseteq X$, the *weak top cycle set* in $Y$ is $\text{TOP}(Y, R)$. When $Y = X$, we just write $\text{TOP}(R)$.

As explained above, completeness of $R$ implies completeness of $T^Y_R$, so $\text{TOP}(Y, R) = D(Y, T_R)$. Thus, $x$ is in the weak top cycle set if and only if, for all $y \in X$, we can get from $x$ to $y$ in a finite number of $R$-steps within the set $Y$.

A side note: By completeness of $T^Y_R$, it follows that $x P T^Y_R y$ if and only if not $y T^Y_R x$, i.e.,

$$P_{T^Y_R} = \left(T^Y_R\right)^{-1} = \overline{T^Y_P}.$$  

Thus, $x P T^Y_R y$ if and only if there is a partition $\{Z, W\}$ of $X$ such that $x \in Z$, $y \in W$, and, for all $z \in \overline{Z}$ and all $w \in W$, $z P y$.

There are alternative definitions of the weak top cycle. Call $Z \subseteq X$ a $P$-*dominant $R$-cycle* in $Y$ if (i) $Z \subseteq Y$, (ii) $|Z| \geq 2$, (iii) for all $x, y \in Z$, $x T^Y_R y$ and $y T^Y_R x$, and (iv) for all $x \in Z$ and all $y \in Y \setminus Z$, $x P y$. Let

$$W(Y, R) = \bigcup \{Z \subseteq Y \mid Z \text{ is a } P\text{-dominant } R\text{-cycle in } Y\}.$$  

As the next proposition shows, the weak top cycle set consists of the maximal elements and the union of all $P$-dominant $R$-cycles. Thus, this theory of choice sets always produces weaker predictions than maximality.

**Proposition 1.31** Let $(P, R)$ be a dual pair. For all $Y \subseteq X$, $\text{TOP}(Y, R) = M(Y, R) \cup W(Y, R)$.

**Proof:** Let $x \in \text{TOP}(Y, R)$, and consider $T^Y_R(x)$, which contains at least $x$. If this set contains no other elements, then $x$ is $R$-maximal. If it does contain another, I claim the set is a $P$-dominant $R$-cycle. Take any $y, z \in T^Y_R(x)$, so $y T^Y_R x$ and $z T^Y_R x$. Since $x \in M(Y, T^Y_R)$, it must be that $x T^Y_R y$ and $x T^Y_R z$. Combining these observations, we have $y T^Y_R z$ and $z T^Y_R y$. Now take $y \notin T^Y_R(x)$ and $z \in T^Y_R(x)$. If not $z P x$, then, because $R$ is complete, $y R z$. But then $y \in T^Y_R(x)$, a contradiction. This establishes the claim. We conclude that $\text{TOP}(Y, R) \subseteq M(Y, R) \cup W(Y, R)$. I’ll leave the other inclusion for you. \end{proof}
As a consequence of the next proposition, there is, in fact, at most one $P$-dominant $R$-cycle. If there is one, of course, then $M(Y, R) \subseteq W(Y, R)$.

**Proposition 1.32** Let $(P, R)$ be a dual pair. For all $Y \subseteq X$, if $Z$ and $W$ are $P$-dominant $R$-cycles in $Y$, then $Z = W$.

**Proof:** Suppose there exists $z \in Z \setminus W$, and take any $w \in W$, so that $wPz$, implying $wT^Y_R z$. Because $z$ is $T^Y_R$-maximal, we have $zT^Y_R w$, i.e., there exist $x_1, x_2, \ldots, x_k \in Y$ such that

$$zRx_1Rx_2\cdots x_{k-1}Rx_k = w.$$  

Then there exists $h$ such that $x_h \notin W$ and $x_{h+1} \in W$, but then $x_{h+1}Px_h$, which contradicts $x_hRx_{h+1}$. Thus, $Z \subseteq W$, and a symmetric argument establishes $W \subseteq Z$. 

Given a relation $B$ and set $Y \subseteq X$, say $Z$ is $B$-dominant in $Y$ if $Z \subseteq Y$ and, for all $x \in Z$ and all $y \in Y \setminus Z$, $xBy$.

The next proposition establishes that the weak top cycle is $P$-dominant and, in fact, is the smallest such set. I leave the proof to you.

**Proposition 1.33** Let $(P, R)$ be a dual pair, and let $Y \subseteq X$. The weak top cycle $TOP(Y, R)$ is $P$-dominant in $Y$, and, if $Z \subseteq X$ is $P$-dominant in $Y$, then $TOP(Y, R) \subseteq Z$.

Under thin indifference, the weak top cycle set coincides with the set of maximal elements, when the latter is nonempty. If $P$ is total or $R$ is anti-symmetric, for example, then this equivalence holds when the core is nonempty. The proof of this claim follows directly from Proposition 1.25.

**Proposition 1.34** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be a dual pair satisfying thin indifference in $Y$. If $M(Y, P) = M(Y, R) \neq \emptyset$, then $TOP(Y, R) = M(Y, P) = M(Y, R)$.

The next lemma significantly extends Proposition 1.34. It is important in analyzing the connections between the weak top cycle and the strong top cycle (below). It establishes that, under upper semi-continuity and thin indifference, the elements of the weak top cycle are typically connected to every other alternative by a chain of strict preferences.

**Lemma 1.1** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be an upper semicontinuous dual pair satisfying thin indifference in $Y$. Let $x \in Y$ be such that, for some $y \in TOP(Y, R)$, $xPy$. Then, for all $z \in Y$, we have $xT^Y_P z$.  

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Proof: Let $x \in Y$ be such that, for some $y \in \text{TOP}(Y, R)$, $xPy$, i.e., $y \in P^{-1}(x)$. Take any $z \in Y \setminus \{x\}$. Since $y$ is in the weak top cycle, there exist $x_1, \ldots, x_k$ such that
\[ xPyRx_1Rx_2 \cdots x_{k-1}Rx_k = z. \]
Without loss of generality, assume $y, x_1, \ldots, x_k$ are distinct. By thin indifference, $y \in \{x_1\} \cup \text{clos}(P(x_1) \cap Y)$. Since $y \neq x_1$, we then have $y \in \text{clos}(P(x_1) \cap Y)$, where closure refers to the relative topology on $Y$. Since $P^{-1}(x)$ is open, by upper semicontinuity, and contains $y$, there exists $y_0 \in P^{-1}(x) \cap P(x_1) \cap Y$, so that $xPy_0Px_1$. Note that $x_1 \neq x_2$. Thus, we can find $y_1 \in P^{-1}(y_0) \cap P(x_2) \cap Y$. An induction argument based on these observations yields $y_0, y_1, \ldots, y_k \in Y$ such that
\[ xPy_0Py_1 \cdots y_{k-2}Py_{k-1}Px_k = z, \]
as required. \[ \Box \]

Note that $T_R$ is complete and transitive, i.e., a weak order. Therefore, as mentioned above, nonemptiness of the weak top cycle set in any finite set $Y$ is guaranteed. For infinite $Y$, we certainly need compactness to be ensured of this. The results we have seen on existence of maximal elements also use upper semicontinuity, however, and $T_R(x)$ is not generally closed, even when $R(x)$ is.

For example, let $X = [0, 1]$, and define $R$ as follows: $xRy$ if and only if $x \in [0, 1+y/2]$. This relation is complete and upper semicontinuous, but $T_R$ is not upper semicontinuous. To see this, note that $R(0) = [0, 1/2]$, $R^2(0) = [0, 3/4]$, $R^3(0) = [0, 7/8]$, etc., and $T_R(0) = [0, 1)$, which is not closed. Nevertheless, $T_R$ does have maximal elements. (In fact, $R$ does.)

Nonemptiness of the weak top cycle will follow from more general existence results later on, but I will include a relatively simple proof with the following proposition.

**Proposition 1.35** Let $(P, R)$ be a dual pair. Assume $Y \subseteq X$ is compact and $(P, R)$ is upper semicontinuous. Then $\text{TOP}(Y, R) \neq \emptyset$.

**Proof:** If $M(Y, P) \neq \emptyset$, then, by Proposition 1.31, the weak top cycle is nonempty. Suppose $M(Y, P) = \emptyset$, so $Y \subseteq \bigcup_{x \in Y} P^{-1}(x)$. By compactness, there exist $y_1, \ldots, y_m$ such that $Y \subseteq \bigcup_{j=1}^m P^{-1}(y_j)$. Since $\{y_1, \ldots, y_m\}$ is finite, there exists $y \in \text{TOP}(\{y_1, \ldots, y_m\}, R)$. The desired result follows upon observing that $y \in \text{TOP}(Y, R)$. (Do you see why?) \[ \Box \]

**The Strong Top Cycle**

If our goal is to formulate tight bounds on choice sets in the absence of maximal elements, then, by Proposition 1.33, the logic of external stability with respect to $P$ can take us no
further than the weak top cycle set. We will see that the top cycle set based on strict preference, $P$, corresponds to a notion of external stability with respect to $R$ and will produce smaller choice sets.

Given dual pair $(P, R)$ and set $Y \subseteq X$, the strong top cycle set in $Y$ is $\text{TOP}(Y, P)$. When $Y = X$, we just write $\text{TOP}(P)$.

In words, $x$ is in the strong top cycle of $P$ if, for every $y \in X$, either we can get from $x$ to $y$ in a finite number of $P$-steps within $Y$, or we cannot get from $y$ to $x$ in a finite number of $P$-steps within $Y$. Because $P$ is not complete, it is not generally the case that $\text{TOP}(Y, P) = D(Y, T_P)$, though we will see conditions for this to hold. I do not know of a nice interpretation of $P_T Y$ in terms of separation properties.

There are alternative definitions of the strong top cycle. Call $Z \subseteq X$ an $R$-dominant $P$-cycle in $Y$ if (i) $Z \subseteq Y$, (ii) $|Z| \geq 2$, (iii) for all $x, y \in Z$, $xT_P y$ and $yT_P x$, and (iv) for all $x \in Z$ and all $y \in Y \setminus Z$, $xR y$. Let

$$S(Y, P) = \bigcup \{Z \subseteq X \mid Z \text{ is an } R\text{-dominant } P\text{-cycle in } Y\}.$$

Paralleling the result for the weak top cycle, the strong top cycle is the union of the maximal elements and the $R$-dominant $P$-cycles.

**Proposition 1.36** Let $(P, R)$ be a dual pair. For all $Y \subseteq X$, $\text{TOP}(Y, P) = M(Y, P) \cup S(Y, P)$.

In contrast to $P$-dominant $R$-cycles, there may be multiple $R$-dominant $P$-cycles. (Can you think of an example?) There may also be multiple minimal $R$-dominant sets, so we have no result that the strong top cycle is the “smallest” such set. With Proposition 1.36, however, the next proposition establishes that the strong top cycle is the union of minimal $R$-dominant sets. Thus, if $Z$ is an $R$-dominant $P$-cycle in $Y$, then there is no set $Z' \subseteq Z$ $R$-dominant in $Y$.

**Proposition 1.37** For all $Y \subseteq X$, $Z$ is a minimal $R$-dominant set if and only if $Z$ is an $R$-dominant $P$-cycle in $Y$ or $Z = \{x\}$, where $M(Y, P) = M(Y, R) = \{x\}$.

The next proposition explores the relationship between the weak and strong top cycles: the strong top cycle is generally a subset of the weak top cycle, and they coincide when $P$ is total. We’ll use the following lemma.

**Lemma 1.2** Let $(P, R)$ be a dual pair, and let $Y \subseteq X$. 38
1. If $Z$ is an $R$-dominant $P$-cycle in $Y$ and $Z'$ is a $P$-dominant $R$-cycle in $Y$, then $Z \subseteq Z'$.

2. If $P$ is total or $R$ is anti-symmetric, then $Z$ is an $R$-dominant $P$-cycle in $Y$ if and only if it is a $P$-dominant $R$-cycle in $Y$.

Proof: 1. Take $y \in Z$ and any $x \in Y$. If $x \in Z$, then $yT_{YP}x$, implying $yT_{YP}x$. If $x \in Y \setminus Z$, then $yRx$, implying $yT_{YP}x$. Thus, $y \in M(Y, T_{YP}) = \text{TOP}(Y, R)$. 2. Suppose $Z$ is a $P$-dominant $R$-cycle in $Y$, and take $y, z \in Z$. By definition, $yT_{YP}z$, so there exist $x_1, \ldots, x_m \in Y$ such that $yRx_1R \cdots x_{m-1}Rx_m = z$. If an element makes consecutive appearances in the sequence $y, x_1, \ldots, x_m$, as when $x_j = x_{j+1}$, eliminate all but the first. This gives us a sequence $y_1, \ldots, y_k$ in which no element appears twice in a row. Since $P$ is total, we have $yPy_1P \cdots y_{k-1}Py_k = z$, and, therefore, $yT_{YP}z$. If $y \in Z$ and $z \in Y \setminus Z$, then, by definition, we have $yPz$, implying $yRz$. Thus, $Z$ is an $R$-dominant $P$-cycle in $Y$. I’ll leave the other direction for you. Of course, anti-symmetry of $R$ is equivalent to $P$ being total.

Some connections between the strong top cycle and our other choice sets are then easily verified.

**Proposition 1.38** Let $(P, R)$ be a dual pair, and let $Y \subseteq X$.

1. $M(Y, P) = M(Y, R) \subseteq \text{TOP}(Y, P) \subseteq \text{TOP}(Y, R)$.

2. If $P$ is total or $R$ is anti-symmetric, then $\text{TOP}(Y, P) = \text{TOP}(Y, R)$.

A version of Proposition 1.34 for the weak top cycle also holds for the strong top cycle.

**Proposition 1.39** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be a dual pair satisfying thin indifference in $Y$. If $M(Y, P) = M(Y, R) \neq \emptyset$, then $\text{TOP}(Y, P) = M(Y, P) = M(Y, R)$.

With Proposition 1.25, the preceding proposition implies that, when thin indifference holds and there exists a (necessarily unique) maximal alternative, the strong top cycle consists of just that alternative, say $x$, and $x$ almost dominant for $T_P$. In fact, we can get from $x$ to every other alternative in one $P$-step. But we do not have $xT_Px$. The next result establishes that, when thin indifference holds and the maximal set is empty, we can characterize the strong top cycle as $T_P$-dominant elements.

**Proposition 1.40** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be an upper semicontinuous dual pair satisfying thin indifference in $Y$. If $M(Y, P) = M(Y, R) = \emptyset$, then $\text{TOP}(Y, P) = D(Y, T_P)$.
Proof: It is clear that \( D(Y, T_P) \subseteq \text{TOP}(Y, P) \). Take any \( x \in \text{TOP}(Y, P) \) and any \( y \in Y \). Since the maximal set is empty, Proposition 1.36 implies that \( x \) is an element of a \( R \)-dominant \( P \)-cycle in \( Y \). In particular, there is some \( z \in \text{TOP}(Y, P) \), implying \( z \in \text{TOP}(Y, R) \), such that \( xPz \). Then Lemma 1.1 implies that \( x \in D(T_P) \), as required.

Note that we have not verified connections between the weak and strong top cycles in the general case of thin indifference — only when \( R \) and \( P \) are anti-symmetric and total and when the core is nonempty. The next proposition establishes a strong connection in the general case of thin indifference: the two sets differ only at points of closure. In the statement of the lemma, closure refers to the relative topology on \( Y \).

**Proposition 1.41** Assume \( X \subseteq \mathbb{R}^d \), and let \( Y \subseteq X \). Let \( (P, R) \) be an upper semicontinuous dual pair satisfying thin indifference in \( Y \). Then \( \text{TOP}(Y, P) \subseteq \text{TOP}(Y, R) \subseteq \text{clos}\text{TOP}(Y, P) \).

Proof: The first inclusion is from Proposition 1.38. The second follows from Proposition 1.39 when \( M(Y, R) = M(Y, P) \neq \emptyset \), so suppose there is no maximal element. In particular, by Proposition 1.31, we have \( \text{TOP}(Y, R) = W(Y, R) \), which must contain at least two elements. Take any \( x \in \text{TOP}(Y, R) \), and let \( y \in \text{TOP}(Y, R) \) satisfy \( y \neq x \) and \( xRy \). Then, by thin indifference, \( x \in \text{clos}(P(y) \cap Y) \), so there exists \( z \in Y \) arbitrarily close to \( x \) such that \( zPy \). Then Lemma 1.1 implies \( z \in D(T_P) \subseteq \text{TOP}(Y, P) \), and we conclude that \( x \in \text{clos}\text{TOP}(Y, P) \).

Of course, \( \text{TOP}(Y, P) \) is nonempty when \( Y \) is finite. And Proposition 1.41 also yields nonemptiness of the strong top cycle, but under the assumption of thin indifference. Our last result on the strong top cycle establishes general nonemptiness under the usual compactness and continuity conditions. Note that, combined with Proposition 1.38, this gives us nonemptiness of the weak top cycle.

**Proposition 1.42** Let \( (P, R) \) be a dual pair. Assume \( Y \subseteq X \) is compact and \( (P, R) \) is upper semicontinuous. Then \( \text{TOP}(Y, P) \neq \emptyset \).

Proof: If \( (P, R) \) satisfies Condition \( F \) in \( Y \), then nonemptiness of the strong top cycle follows from Proposition 1.36. Suppose Condition \( F \) is not satisfied in \( Y \). Then there is a finite set \( Z \subseteq Y \) such that, for all \( x \in Y \), there exists \( y \in Z \) with \( yPx \). Note that \( T^Y_P \) is transitive, so there exists \( z^* \in M(Z, T^Y_P) \). I claim that \( Z^* = T^Y_P(z^*) \) is an \( R \)-dominant \( P \)-cycle in \( Y \). It is clear that, for all \( x \in Z^* \) and all \( y \in Y \setminus Z^* \), we have \( xRy \). (Why?) Now take any \( x \in Z^* \), so \( xT^Y_Pz^* \). If \( x \in Z \), then \( z^*T^Y_Px \) follows by maximality of \( z^* \). If \( x \in Z^* \setminus Z \), then there exists \( y \in Z \) such that \( yPx \), so \( yT^Y_Pz^* \). Again, by maximality of \( z^* \), we have \( z^*T^Y_Py \), and then \( z^*T^Y_PyPx \) implies \( z^*T^Y_Px \). To see that \( Z^* \) is a \( P \)-cycle, take \( x, y \in Z^* \). By the preceding argument, we have \( xT^Y_Pz^*T^Y_PyT^Y_Pz^*T^Y_Px \), as required.
In collective choice problems, with more structure than we have here, we’ll be able to say more about the weak and strong top cycles. Unfortunately, these sets can be quite large. Thus, I want to consider one more alternative to maximality.

**Exercises**

1.4.1. Prove the remaining inclusion in Proposition 1.31.

1.4.2. Prove the following.

(a) There is a \( P \)-dominant \( R \)-cycle if and only if \( |\text{TOP}(R)| \geq 2 \).

(b) Assume \((P, R)\) satisfies thin indifference. If \( M(P) = M(R) = \emptyset \), then \( |\text{TOP}(R)| \geq 3 \).

1.4.3. Prove Proposition 1.33.

1.4.4. In the proof of Proposition 1.35, how do we conclude that \( y \in \text{TOP}(Y, R) \)?

1.4.5. Prove that, if \( M(Y, R) \neq \emptyset \), then

\[
\text{TOP}(Y, R) = \{ x \in Y \mid \exists y \in M(Y, R) : xT^Y_R y \}.
\]

1.4.6. Assume \( X \) is finite, and let \( Y \) denote an \( R \)-cycle. Assume that \( |Y| \geq 2 \). Prove that there is an \( R \)-cycle through \( Y \) with no elements repeated, i.e., we can index the (say) \( m \) elements of \( Y \) in such a way that

\[y_1Ry_2R\cdots y_{m-1}Ry_m Ry_1.\]

1.4.7. Give an example in which there are multiple \( R \)-dominant \( P \)-cycles.

1.4.8. Prove that, if \( Y \) is an \( R \)-dominant \( P \)-cycle, then there is no \( R \)-dominant set \( Y' \subsetneq Y \).

1.4.9. In the proof of Proposition 1.42, why is it clear that, for all \( x \in Z^* \) and all \( y \in Y \setminus Z^* \), we have \( xRy \)?

**1.5 Uncovered Sets**

We now consider a solution known as the “uncovered set,” an idea derived from the theoretical political science literature, to the problem of empty maximal sets. Whereas the top cycle sets are defined in terms of the transitive closure of weak preference and strict preference, the uncovered sets are defined in terms of one-or-two steps of these relations.
Given a relation $B$ and a set $Y \subseteq X$, the covering relation of $B$ in $Y$ is defined as follows: $xC^Y_B y$ if $xBy$ and, for all $z \in Y$, $zBx$ implies not $yBz$. In this case, we say $x$ “covers” $y$ in $Y$. In different notation,

$$x C^Y_B y \iff xBy \text{ and } B(x) \cap Y \subseteq B^{-1}(y) \cap Y.$$ 

When $Y = X$, we just write $C_B$ for this relation.

Note that $C^Y_B$ is not generally asymmetric or complete: consider $X = \{x, y\}$ and $B = \{(x, y), (y, x)\}$. The covering relation is asymmetric if $B$ is asymmetric or reflexive: $xC^Y_B y$ and $yC^Y_B x$ implies that $xBy$ and $yBx$ and that $B(x) = B^{-1}(y)$. The first of these implications is inconsistent with asymmetry, and the two together are inconsistent with reflexivity, as $xBx$ would imply not $yBx$.

The uncovered set of $B$ in $Y$ is defined as

$$UC(Y, B) = M(Y, C^Y_B).$$

When $Y = X$, we just write $UC(B)$ for $UC(X, B)$. We say the elements of $UC(Y, B)$ are “uncovered” in $Y$.

The connection between the concept uncovered set and top cycle set is clearest when $C^Y_B$ is asymmetric. In that case,

$$UC(Y, B) = UD(Y, C^Y_B) = D(Y, R_{C^Y_B}),$$

where the completion consists of pairs $(x, y)$ in $Y \times Y$ such that not $yC^Y_B x$, i.e., either (i) not $yBx$ or (ii) there exists $z \in Y$ such that $xBzBy$. (Right?) Thus, the completion of covering looks a little like the idea of transitive closure: it says that we can get either get from $x$ to $y$ in one $B^{-1}$-step or from $x$ to $y$ in two $B$-steps (rather than any finite number).

If $B$ is actually complete, then the completion of the covering relation is further simplified, because (i) reduces to $xBy$. In that case,

$$UC(Y, B) = \{x \in Y \mid \forall y \in Y, \exists z \in Y : xBzBy\},$$

and the uncovered set is clearly a subset of the top cycle.

**The Weak Uncovered Set**

Given a dual pair $(P, R)$ and set $Y \subseteq X$, the weak uncovered set in $Y$ is $UC(Y, R)$, i.e., the maximal elements of $C^Y_R$, the covering relation of $R$ in $Y$. When $Y = X$, we just write $UC(R)$.
Note that \( xC^Y_R y \) if and only if \( xRy \) and \( R(x) \cap Y \subseteq P(y) \cap Y \). Since \( x \in R(x) \), this reduces to:

\[
xC^Y_R y \iff R(x) \cap Y \subseteq P(y) \cap Y,
\]

which implies \( xPy \). The covering relation of \( R \) is asymmetric and transitive, i.e., a strict quasi-order. Thus, the elements of the weak uncovered set are the undominated set of \( C^Y_R \), i.e., \( UC(Y, R) = UD(Y, C^Y_R) \), and we term such alternatives “uncovered.”

As explained above, the completion of the covering relation of \( R \), \( R_{C^Y_R} \), is given by

\[
xR_{C^Y_R} y \iff \exists z \in Y : xRzRy.
\]

Since \( UC(Y, R) = UD(Y, C^Y_R) = D(Y, R_{C^Y_R}) \), this means that \( x \in Y \) is weakly uncovered in \( Y \) if and only if, for every alternative \( y \in Y \), we can get from \( x \) to \( y \) in one or two weak preference steps within \( Y \). This property is sometimes called the “two step” principle.

See Figure 1.12 for an example of the covering relation of \( R \) with \( X \) finite.

Figure 1.12: Covering relation of \( R \), finite \( X \)

In the example of Figure 1.12, the weak uncovered set is \( \{a, c, e\} \), while the weak and strong top cycle sets are \( \{a, b, c, d, e\} \). See Figure 1.13 for an example of covering with infinite \( X \).

Figure 1.13: Covering relation of \( R \), infinite \( X \)

Thus, in the example of Figure 1.13, \( yC_{Rx} \), so \( x \) is not in the weak uncovered set. The next result shows that the weak uncovered set is nested between the maximal set and the weak top cycle.

**Proposition 1.43** Let \((P, R)\) be a dual pair. For all \( Y \subseteq X \), \( M(Y, R) \subseteq UC(Y, R) \subseteq TOP(Y, R) \).

The weak uncovered set coincides with the maximal set under the usual conditions.

**Proposition 1.44** Assume \( X \subseteq \mathbb{R}^d \), and let \( Y \subseteq X \). Let \((P, R)\) be a dual pair satisfying thin indifference in \( Y \). If \( M(Y, P) = M(Y, R) \neq \emptyset \), then \( UC(Y, R) = M(Y, P) = M(Y, R) \).

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Since $C_Y^R$ is a strict quasi-order, we know that the uncovered set is nonempty whenever $Y$ is finite. The next proposition establishes nonemptiness under the usual conditions. Note that, with Proposition 1.43, nonemptiness of the weak top cycle follows as a consequence.

**Proposition 1.45** Assume $X \subseteq \mathbb{R}^d$. Let $(P, R)$ be a dual pair. Assume $Y \subseteq X$ is compact and $(P, R)$ is upper semicontinuous. Then $UC(Y, R) \neq \emptyset$.

**Proof:** Define the relation $B$ on $Y$ as follows: $xBy$ if and only if either $x = y$ or both $P(x) \cap Y \subseteq P(y) \cap Y$. Clearly, $B$ is reflexive and transitive. (Right?) I will prove existence of a $B$-maximal element by Zorn’s lemma. Take any $B$-chain $\{x_\alpha\}$ in $Y$, i.e., for all $\alpha$ and $\beta$, either $x_\alpha B x_\beta$ or $x_\beta B x_\alpha$. View this as a net with direction $\geq$ defined as $\alpha \geq \beta$ if and only if $x_\alpha B x_\beta$. Then compactness of $Y$ yields a subnet, which we also index by $\alpha$ for simplicity, converging to some $x \in Y$. I claim that $x$ is a $B$-upper bound for the chain $\{x_\alpha\}$. Take any $\alpha$. If $x = x_\alpha$, then $xBx_\alpha$. If $x \neq x_\alpha$, then take any $y \in P(x) \cap Y$, so $x \in P^{-1}(y)$. By upper semi-continuity, there is some $\beta > \alpha$ such that $x_\beta \in P^{-1}(y)$, i.e., $yPx_\beta$. Then $x_\beta B x_\alpha$ implies $yPx_\alpha$, as required. Thus, Zorn’s lemma implies the existence of a $B$-maximal element, say $x^*$. To see that $x^* \in UC(Y, R)$, suppose $R(y) \cap Y \subseteq P(x^*) \cap Y$ for some $y \in Y$. In particular, $x^* \neq y$ and $P(y) \cap Y \subseteq P(x^*) \cap Y$, so maximality of $x^*$ implies $P(x^*) \cap Y \subseteq P(y) \cap Y$. But then $y \in P(x^*) \cap Y \subseteq P(y) \cap Y$ implies $yPy$, a contradiction. 

We’ll see that in collective choice problems, with more structure than available here, we can say more about the nature of the weak uncovered set.

**The Strong Uncovered Set**

Given dual pair $(P, R)$ and $Y \subseteq X$, the **strong uncovered set** in $Y$ is $UC(Y, P)$, i.e., the maximal elements of $C_Y^P$, the covering relation of $P$ in $Y$. When $Y = X$, we just write $UC(P)$.

Note that

$$xC_Y^Py \iff xPy \text{ and } P(x) \cap Y \subseteq R(y) \cap Y,$$

a much weaker relation than $C_R^Y$, in the sense that $C_R^Y \subseteq C_Y^P$: it is much easier to cover an alternative with $P$ than with $R$, and the strong uncovered set is, therefore, potentially much smaller than the weak.

The strong covering relation is clearly asymmetric. The next proposition establishes that, under thin indifference, strong covering is also transitive, i.e., a strict quasi-order.

**Proposition 1.46** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be a dual pair. Assume $(P, R)$ is upper semicontinuous and satisfies thin indifference in $Y$. Then $C_Y^P$ is transitive.
Proof: If $xC'_P yC'_P z$, then

$$P(x) \cap Y \subseteq R(y) \cap Y = \{y\} \cup \text{clos}(P(y) \cap Y) \subseteq \text{clos}(R(z) \cap Y) = R(z) \cap Y,$$

where we use $yPz$. Since $xPyPz$, we must have $x \neq z$. Thus, if $zRx$, then $z \in \text{clos}(P(x) \cap Y)$. By upper semi-continuity and $yPz$, there exists $w \in P^{-1}(y) \cap P(x) \cap Y \subseteq P(y) \cap Y$, which implies $yPwPz$, a contradiction. Therefore, $xPz$, and we conclude that $xC'_P z$. 

You can also check that the completion of the strong covering relation, $R_{C'_Y}$, is given by

$$xR_{C'_Y}y \iff xRy \text{ or } \exists z \in Y : xPzP y.$$  

Since $UC(Y, P) = UD(Y, C'_P) = D(Y, R_{C'_Y})$, this means that $x \in Y$ is weakly uncovered in $Y$ if and only if, for every alternative $y \in Y$, we can get from $x$ to $y$ in one weak preference step or in two strict preference steps within $Y$. Though not quite as tidy as the first one we saw, this property is also sometimes referred to as a “two step” principle.

The next proposition establishes some obvious set inclusions for the strong uncovered set.

**Proposition 1.47** Let $(P, R)$ be a dual pair, and let $Y \subseteq X$.

1. $M(Y, P) = M(Y, R) \subseteq UC(Y, P) \subseteq UC(Y, R) \subseteq TOP(Y, R)$.

2. If $P$ is total or $R$ is anti-symmetric, then $UC(Y, P) = UC(Y, R)$.

In addition, under thin indifference, the strong uncovered set is contained in the strong top cycle.

**Proposition 1.48** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be a dual pair. Assume $Y$ is compact and $(P, R)$ is continuous and satisfies thin indifference in $Y$. Then $UC(Y, P) \subseteq TOP(Y, P)$.

*Proof:* Note that, by Proposition 1.42, $TOP(Y, P) \neq \emptyset$. Take any $x \in UC(Y, P)$ and $y \in TOP(Y, P)$. If $x \notin TOP(Y, P)$, then it must be that $x \neq y$. Furthermore, I claim that $xRy$: if $yPx$, then, because $x$ is strongly uncovered in $Y$, there exists $z \in Y$ such that $xPzPy$, which implies $x \in TOP(Y, P)$, establishing the claim. Then $xRy$ and thin indifference yield $x \in \text{clos}(P(y) \cap Y)$ so there exists $z \in Y$ such that $zPy$. If $xPz$, then $xPzPy$ implies $x \in TOP(Y, P)$, a contradiction. Therefore, $zRx$, which implies $z \in \text{clos}(P(x) \cap Y)$. Since $P(y)$ is open, by continuity, there exists $w \in P(y) \cap P(x) \cap Y$. Since $x$ is strongly uncovered in $Y$ and $wPx$, there exists $v \in Y$ such that $xPvPw$, but then $xPvPwPy$ implies $x \in TOP(Y, P)$, a final contradiction. Therefore, $x \in TOP(Y, P)$. 

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See Figure 1.14 for an example in which thin indifference is violated and the strong uncovered set is not in the strong top cycle.

The strong uncovered set coincides with the maximal set under the usual conditions.

**Proposition 1.49** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be a dual pair satisfying thin indifference in $Y$. If $M(Y, P) = M(Y, R) \neq \emptyset$, then $UC(Y, P) = M(Y, P) = M(Y, R)$.

When $Y$ is finite, thin indifference is equivalent to $R$ being anti-symmetric or $P$ being total. Under these conditions, because $C^P Y$ is a strict quasi-order, we have $UC(Y, P) \neq \emptyset$. The next proposition establishes nonemptiness more generally.

**Proposition 1.50** Assume $X \subseteq \mathbb{R}^d$, and let $Y \subseteq X$. Let $(P, R)$ be a dual pair. Assume $Y$ is compact and $(P, R)$ is an upper semicontinuous dual pair satisfying thin indifference in $Y$. Then $UC(Y, P) \neq \emptyset$.

**Proof:** Define the relation $B$ on $Y$ as follows: $xBy$ if and only if $xRy$ and $P(x) \cap Y \subseteq R(y) \cap Y$. Clearly, $B$ is reflexive. To see that $B$ is transitive, take $x, y, z \in Y$ such that $xByBz$. Then $P(x) \cap Y \subseteq R(y) \cap Y = \{y\} \cup \text{clos}(P(y) \cap Y) \subseteq \text{clos}(R(z) \cap Y) = R(z) \cap Y,$ where we use $yRz$. Furthermore, if $zPx$, then $x \in P^{-1}(z)$, which is open by upper semi-continuity. Because $xRy$ and thin indifference implies $x \in \text{clos}(P(y) \cap Y)$, so there exists $w \in P(y) \cap Y \cap P^{-1}(z)$, i.e., $zPwPy$. But $wPy$ and $yBz$ implies $wRz$, a contradiction. Therefore, $xRz$, and it follows that $xBz$, so $B$ is transitive. I will prove existence of a $B$-maximal element by Zorn’s lemma, which is applicable because $B$ is reflexive and transitive. Take any $B$-chain $\{x_\alpha\}$ in $Y$, i.e., for all $\alpha$ and $\beta$, either $x_\alpha B x_\beta$ or $x_\beta B x_\alpha$. View this as a net with direction $\geq$ defined as $\alpha \geq \beta$ if and only if $x_\alpha B x_\beta$. Then compactness of $Y$ yields a subnet, which we also index by $\alpha$ for simplicity, converging to some $x \in Y$. I claim that $x$ is a $B$-upper bound for the chain $\{x_\alpha\}$. Take any $\alpha$. If $x = x_\alpha$, then $xB\alpha$. Suppose $x \neq x_\alpha$ and $x_\alpha Px$, so $x \in P^{-1}(x_\alpha)$, an open set by upper semi-continuity. Then there exists $\beta > \alpha$ such that $x_\beta \in P^{-1}(x_\alpha)$, contradicting $x_\beta B x_\alpha$. Thus, $xR\alpha$. Now take $y \in P(x)$, so $x \in P^{-1}(y)$. Again, there exists $\beta > \alpha$ such that $x_\beta \in P^{-1}(y)$, i.e., $yPx_\beta$, so $x_\beta B x_\alpha$ implies $yRx_\alpha$. Therefore, $P(x) \subseteq R(x_\alpha)$, and we have $xBx_\alpha$. Thus, $x$ is a $B$-upper bound, and Zorn’s lemma yields a $B$-maximal element, say $x^*$. Now suppose there exists $y \in Y$ such that $yC^P x^*$, so $yPx^*$ and $P(y) \cap Y \subseteq R(x^*) \cap Y$. This implies $yBx^*$, so, by maximality, we have $x^*Bx$. But this implies $x^* Ry$, a contradiction. Therefore, $x^* \in UC(Y, P)$. \qed
To see why thin indifference is needed for nonemptiness of the strong uncovered set, suppose \( X = \{w, x, y, z\} \), and consider \( P = \{(w, x), (x, y), (y, z), (z, w)\} \). This gives us a \( C_P \)-cycle through \( X \) and \( UC(P) = \emptyset \).

**Exercises**

1.5.1. Assuming \( C_B^Y \) is asymmetric, verify that the completion of the covering relation is defined as follows: \( xR_{C_B^Y}y \) if and only if either not \( yBx \) or there exists \( z \in Y \) such that \( xByBz \).

1.5.2. Prove that \( C_Y^R \) is a strict quasi-order.

1.5.3. Check that \( xR_{C_Y^R}y \) if and only if there exists \( z \in Y \) such that \( xRzRy \).

1.5.4. Assume \( X \) is finite and \( P \) is total. Prove that \( x \) is in the weak uncovered set if and only if there is no \( y \in X \) such that \( yPx \) and \( P(y) \subseteq P(x) \).

1.5.5. Prove Proposition 1.43.

1.5.6. Prove Proposition 1.44.

1.5.7. In the proof of Proposition 1.45, why is \( B \) reflexive and transitive?

1.5.8. Check that \( xR_{C_Y^P}y \) if and only if \( xRy \) or there exists \( z \in Y \) such that \( xPzPy \).

1.5.9. Find an example in which the weak uncovered set and strong top cycle are not nested, i.e., neither is a subset of the other. (Assume \( X \) is finite.) Can they have empty intersection?

1.5.10. Prove Proposition 1.47.

1.5.11. Prove Proposition 1.49.

1.5.12. Given a dual pair \( (P, R) \), define the “intermediate” notion of covering on \( X \), denoted \( C^* \), as follows: \( xC^*y \) if and only if

\[
xPy, \ P(x) \subseteq P(y), \ R(x) \subseteq R(y).
\]

Let \( UC^* = M(C^*) \). Prove that \( UC(P) \subseteq UC^* \subseteq UC(R) \). What do you conclude in light of Proposition 1.19?

1.5.13. Given a dual pair \( (P, R) \), prove the following:

(a) If \( x \in P^2(y) \), then \( P(y) \nsubseteq P(x) \) and \( R(y) \nsubseteq R(x) \).
(b) If $P(y) \not\subseteq P(x)$ or $R(y) \not\subseteq R(x)$, then $x \in R^2(y)$.

c) Assume $R^{-1}(x) = \{x\} \cup \text{clos}P^{-1}(x)$ for all $x \in X$. Then $x \in P^2(y)$ if and only if $P(y) \not\subseteq P(x)$.

d) Assume $R(x) = \{x\} \cup \text{clos}P(x)$ for all $x \in X$. Then $x \in P^2(y)$ if and only if $R(y) \not\subseteq R(x)$.

1.5.14. Building on the previous two exercises, prove the following. Assume $R^{-1}(x) = \{x\} \cup \text{clos}P^{-1}(x)$ and $R(x) = \{x\} \cup \text{clos}P(x)$ for all $x \in X$. If $x \in UC^*$ and $y \notin UC^*$, then $x \in P^2(y)$.

1.5.15. Let $X$ be finite, and let $(P, R)$ be a dual pair on $X$. We say $Y \subseteq X$ is a von-Neumann-Morgenstern stable set (or simply “stable set”) if (i) for all $x, y \in Y$, $xRy$ and $yRx$, and (ii) for all $z \in X \setminus Y$, there exists $x \in Y$ such that $xPz$.

(a) Suppose that $X = \{a, b, c, d\}$ and $P = \{(a, b), (b, c), (c, d), (d, a)\}$. Calculate $R$, and find all of the stable sets.

(b) Prove that, if $Y$ is a stable set, then $M(R) \subseteq Y \subseteq UC(R)$.

c) Prove that, if $P$ is acyclic, then there is at least one stable set.

d) Assuming $P$ is transitive, prove that there is a unique stable set and identify this set.

e) Show by example that there may be no stable set, even if $M(P) \neq \emptyset$.

1.6 Continuity of Choice

We now examine the continuity of the choice sets defined above as the weak preference relation, $R$, varies. In doing so, we set $Y = X$, suppressing this set notationally, and we view $M(R)$, $TOP(R)$, and $UC(R)$ as correspondences, meaning...

Given sets $X$ and $Y$, a correspondence from $X$ to $Y$, denoted $\phi$, is a mapping from $X$ to subsets of $Y$. We often write this as $\phi: X \Rightarrow Y$. There are two main notions of continuity of correspondences, upper and lower hemi-continuity. We will focus on the first.

The correspondence $\phi$ is upper hemi-continuous at $x$ if, for every open set $U \subseteq Y$ with $\phi(x) \subseteq U$, there is an open set $V \subseteq X$ such that $x \in V$ and, for all $z \in V$, $\phi(z) \subseteq U$. We say it is upper hemicontinuous if it is upper hemi-continuous at every $x \in X$. See Figure 1.15.

Figure 1.15: Upper hemi-continuity
As a digression, we define the graph of $\phi$ as
\[
\text{gr}\phi = \{(x, y) \in X \times Y \mid y \in \phi(x)\}.
\]
When $X = Y$, we can view $\text{gr}\phi$ as a relation on $X$, and then $\phi(x)$ is just the lower section of $\text{gr}\phi$ at $x$.

If $Y$ is compact, then $\phi$ is upper hemicontinuous with closed values, i.e., $\phi(x)$ is closed for all $x \in X$, if and only if $\phi$ has closed graph: for every sequence $\{(x_m, y_m)\}$ with $y_m \in \phi(x_m)$ for all $m$, $(x_m, y_m) \to (x, y)$ implies $y \in \phi(x)$.

In order to apply these ideas to the analysis of choice, we must define a notion of open sets for preference relations.

Let $X \subseteq \mathbb{R}^d$, and assume that $X$ is compact, so that $X \times X$ is also compact in $\mathbb{R}^{2d}$. Let $\mathcal{R}_c$ denote the set of complete, closed relations on $X$. Given $R, R' \in \mathcal{R}_c$, we can define the distance between them as
\[
\rho(R, R') = \max \left\{ \max_{(x,y) \in R} \min_{(w,z) \in R'} ||(x,y) - (w,z)||, \max_{(w,z) \in R'} \min_{(x,y) \in R} ||(x,y) - (w,z)|| \right\}.
\]
This is the Hausdorff metric on closed subsets of $X \times X$. It is a true metric, in the sense that (i) $\rho(R, R') \geq 0$ and $\rho(R, R') = 0$ if and only if $R = R'$, (ii) $\rho(R, R') = \rho(R', R)$, and (iii) $\rho(R, R') + \rho(R', R'') \leq \rho(R, R'')$ (the triangle inequality).

We define the open ball of radius $\epsilon > 0$ around $R$ as
\[
B_\epsilon(R) = \{ R' \in \mathcal{R}_c \mid \rho(R', R) < \epsilon \}.
\]
We then say a set $\mathcal{R}' \subseteq \mathcal{R}_c$ is open if, for all $R \in \mathcal{R}'$, there exists $\epsilon > 0$ such that $B_\epsilon(R) \subseteq \mathcal{R}'$. A set $\mathcal{R}' \subseteq \mathcal{R}_c$ is closed if its complement is open.

We say a sequence $\{R_m\}$ in $\mathcal{R}_c$ converges to the relation $R \in \mathcal{R}_c$ if $\rho(R_m, R) \to 0$. You can check that $\mathcal{R}' \subseteq \mathcal{R}_c$ is closed if and only if, for every sequence $\{R_m\}$ in $\mathcal{R}_c$ converging to some $R \in \mathcal{R}_c$, we have $R \in \mathcal{R}'$.

The next lemma gives a useful necessary condition for convergence in the Hausdorff metric. Technically, it states that every Hausdorff limit is a “closed convergence” limit.

**Lemma 1.3** Assume $X \subseteq \mathbb{R}^d$ is compact. Let $\{R_m\}$ be a sequence in $\mathcal{R}_c$ converging to $R$, and let $\{(x_m, y_m)\}$ be a sequence in $X$ converging to $(x, y)$. If $x_m R_m y_m$ for all $m$, then $x R y$. Moreover, if $x R y$, then there is a subsequence $\{R_{m_k}\}$ and there is a sequence $\{(x_k, y_k)\}$ such that $x_k R_{m_k} y_k$ for all $k$ and $(x_k, y_k) \to (x, y)$.

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Proof: If not \(xRy\), then \((x, y) \notin R\). Because \(R\) is a compact subset of \(X \times X\), we have

\[
\min_{(w, z) \in R} ||(x, y) - (w, z)|| > 0.
\]

By the Theorem of the Maximum, we have

\[
\min_{(w, z) \in R} ||(x_m, y_m) - (w, z)|| \to \min_{(w, z) \in R} ||(x, y) - (w, z)||.
\]

But, because \((x_m, y_m) \in R_m\) for all \(m\), this implies

\[
\max_{(x', y') \in R_m} \min_{(w, z) \in R} ||(x', y') - (w, z)|| \to 0,
\]

which implies \(\rho(R_m, R) \to 0\), a contradiction. Therefore, \(xRy\), as claimed. I’ll let you prove the converse.

Another lemma to establish the other direction will be helpful in our analysis of social choice. It shows that the conditions of Lemma 1.3 are actually sufficient for convergence, as well as necessary. I omit the proof.

**Lemma 1.4** Assume \(X \subseteq \mathbb{R}^d\) is compact. Let \(\{R_m\}\) be a sequence in \(\mathcal{R}_c\) and let \(R \in \mathcal{R}_c\) satisfy (i) for every sequence \(\{(x_m, y_m)\}\) converging to some \((x, y)\) with \(x_mR_my_m\) for all \(m\), we have \(xRy\), (ii) for all \(x, y \in X\) with \(xRy\), there is a subsequence \(\{R_{m_k}\}\) and a sequence \(\{(x_k, y_k)\}\) such that \(x_kR_{m_k}y_k\) for all \(k\) and \((x_k, y_k) \to (x, y)\). Then \(R_m\) converges to \(R\).

Now to upper hemicontinuity.

**Proposition 1.51** Assume \(X \subseteq \mathbb{R}^d\) is compact. Then \(M: \mathcal{R}_c \Rightarrow X\) is upper hemicontinuous with closed values.

**Proof:** It is sufficient to prove that \(M\) has closed graph. Take any sequence \(\{(R_m, x_m)\}\) with \(x_m \in M(R_m) = D(R_m)\) for all \(m\), and suppose \((R_m, x_m) \to (R, x)\). Take any \(y \in X\), so \(x_mR_my\). By Lemma 1.3, we have \(xRy\), so \(x \in D(R) = M(R)\), as required.

The weak top cycle set is not generally upper hemicontinuous, as the following example shows. Let \(X = [0, 1]\), and define \(R_m\) by \(xR_my\) if and only if \(x + 1/m \geq y\). Then, for all \(m\), we have \(\text{TOP}(R_m) = [0, 1]\). (Do you see why?) Furthermore, the sequence \(\{R_m\}\) converges to \(\geq\), and \(\text{TOP}(\geq) = \{1\}\). (Why?) Taking the (relatively!) open set \(U = (1/2, 1]\), upper hemicontinuity is violated.
In the latter example, $M(R_m) = [1 - 1/m, 1] \neq \emptyset$ for all $m$, so discontinuities of the weak top cycle may not be of much interest. A more disconcerting example is the following. Let $X = [0, 1]$, and define $R_m$ by

$$R_m(x) = [x - 2/m, x + 1/m] \cup [x + 2/m, 1].$$

Note that $yP_m x$ if and only if not $xR_m y$, if and only if $x \not\in [y - 2/m, y + 1/m] \cup [y + 2/m, 1]$, i.e., if $x$ is in

$$[y - 2/m, y + 1/m] \cup [y + 2/m, 1] = [y - 2/m, y + 1/m] \cap [y + 2/m, 1] = ((0, y - 2/m) \cup (y + 1/m, 1)) \cap (0, y + 2/m) = (0, y - 2/m) \cup (y + 1/m, y + 2/m).$$

Rewriting, $yP_m x$ if and only if $x < y - 2/m$ or if $y + 1/m < x < y + 2/m$. Equivalently, $yP_m x$ if $x + 2/m < y$ or if $x - 2/m < y < x - 1/m$. Finally, we have

$$P_m(x) = (x - 2/m, x - 1/m) \cup (x + 2/m, 1].$$

Clearly, $P_m(x) \neq \emptyset$ for all $m$ and all $x \in [0, 1]$. Thus, $M(R_m) = \emptyset$ for all $m$, and we may therefore be interested in the continuity properties of alternatives, such as the weak top cycle, to maximality. The relation $R_m$ is depicted in Figure 1.16.

Figure 1.16: The relation $R_m$

Back to the example. In contrast to the maximal set, $TOP(R_m) = [0, 1]$ for all $m$. This follows from

$$R_m^{-1}(x) = [0, x - 2/m] \cup [x - 1/m, x + 2/m].$$

You can check that $R_m$ converges to the relation $\geq$. Thus, $M(R) = TOP(R) = \{1\}$. The problem is that, given weak preference $R$, the maximal set is nonempty and, in fact, is singleton. For arbitrarily close preferences, the maximal set is empty — this is not a violation of upper hemicontinuity — but the weak top cycle set “blows up” to $[0, 1]$. It is nonempty, but, as an alternative to maximality that is supposed to give us choice sets when the maximal set is empty, the weak top cycle does nothing to narrow down the plausible choices.

The feature of the top cycle set driving the above examples is the fact that one alternative $x$ may be connected to another by an arbitrarily large number of preference-steps, and this number may go to infinity as we vary preferences. The weak uncovered set, however, is defined using just two weak preference steps, allowing us to prove upper hemicontinuity.
Proposition 1.52 Assume $X \subseteq \mathbb{R}^d$ is compact. Then $UC: \mathcal{R} \rightrightarrows X$ is upper hemicontinuous with closed values.

Proof: It is sufficient to prove that $UC$ has closed graph. Take any sequence $\{(R_m, x_m)\}$ with $x_m \in UC(R_m)$ for all $m$, and suppose $(R_m, x_m) \rightarrow (R, x)$. Take any $y \in X$. For each $m$, there exists $z_m \in X$ such that $x_m R_m z_m R_m y$. By compactness, $\{z_m\}$ has a subsequence, again indexed by $m$ for simplicity, converging to some $z \in X$. By Lemma 1.3, we have $x R z R y$, so $x \in UC(R)$.

In the above example, for each $m$, the weak uncovered set is $[1 - 4/m, 1]$. As established in Proposition 1.52, this set collapses to $\{1\}$ as $m$ goes to infinity, fulfilling upper hemicontinuity. As an alternative to maximality, the weak uncovered set gives us this desirable property: if $M(R) = UC(R)$ and we perturb the preference relation $R$, the weak uncovered set will be nonempty and will vary “continuously,” even if the maximal set becomes empty.

Exercises

1.6.1. Assume $X = \mathbb{R}^d$. Let $\{(P^m, R^m)\}$ be a sequence of Euclidean preferences with ideal points $\tilde{x}^m$, and let $(P, R)$ be a Euclidean preference with ideal point $\tilde{x}$. Prove that $R^m \rightarrow R$ in the Hausdorff metric if and only if $\tilde{x}^m \rightarrow \tilde{x}$.

1.6.2. Let $X = [0, 1]$, and define $R_m$ by the utility representation $u_m$, where

$$u_m(x) = x \sin \left( \frac{\pi}{2} + \frac{x}{m} \right).$$

What relation does $\{R_m\}$ converge to? Explain.

1.6.3. Show explicitly how the Theorem of the Maximum is applied in the proof of Lemma 1.3.

1.6.4. In the example following Proposition 1.51, why does $TOP(R_m) = [0, 1]$ for all $m$? Why does $TOP(\geq) = \{1\}$?

1.6.5. Let $\mathcal{R}_{cw}$ denote the set of closed weak orders on a compact set $X \subseteq \mathbb{R}^d$. Is this set closed in the Hausdorff metric? Why or why not?

1.6.6. Let $\mathcal{R}_{cq}$ denote the set of closed weak quasi-orders on a compact set $X \subseteq \mathbb{R}^d$. Is this set closed in the Hausdorff metric? Why or why not?

1.7 Rationalizability of Choice

Given a set $X$, a choice function on $X$, denoted $C$, is a mapping from subsets $Y \subseteq X$ to subsets $C(Y) \subseteq Y$. In terms of the preceding section, a choice function is a correspondence $C: 2^X \rightrightarrows X$ such that, for all $Y \in 2^X$, $C(Y) \subseteq Y$. 

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Choice functions give us a simple way of describing behavior from either the normative or positive perspective: \( C(Y) \) may represent the alternatives that should be chosen from \( Y \), or it may represent the alternatives that will be chosen from \( Y \).

When is the behavior described by \( C \) consistent with preference-based choice? More formally, we want to know when \( C \) is “rationalizable,” in the sense that there exists a relation \( B \) such that, for all \( Y \subseteq X \), \( C(Y) = M(Y, B) \). In this case, we say \( B \) rationalizes \( C \). When such a \( B \) exists, we say \( C \) is rationalizable. Note the implication that every rationalizable choice function has non-empty values on sets with one or two elements.

Define the weak base relation of \( C \), denoted \( R^C \), by

\[
x R^C y \iff x \in C(\{x, y\}).
\]

This relation is necessarily complete if \( C \) is rationalizable. Though it is customary to focus on the weak base relation, we can also define an asymmetric counterpart. Define the strict base relation of \( C \), denoted \( P^C \), by

\[
x P^C y \iff y \notin C(\{x, y\}).
\]

If \( C \) is rationalizable, then \( P^C \) is asymmetric, and \((P^C, R^C)\) is a dual pair.

As the next result shows, these relations play an important role in the analysis of rationalizability: if a choice function is rationalizable, then it is rationalized by the base relations \( R^C \) and \( P^C \).

**Proposition 1.53** The following are equivalent for a choice function \( C \), given a relation \( B \).

- \( C \) is rationalized by \( B \).
- \( C \) is rationalized by \( R^C \) and \( R^C = R_B \).
- \( C \) is rationalized by \( P^C \) and \( P^C = P_B \).

**Proof:** Assume \( C \) is rationalized by \( B \), and take \( x \in C(Y) = M(Y, B) \). Pick any \( y \in Y \), and note that not \( y P_B x \). Therefore, \( x \in M(\{x, y\}, B) = C(\{x, y\}) \), which implies \( x R^C y \).

Since \( y \) was an arbitrary element of \( Y \), we conclude that \( x \in M(Y, R^C) \). Since \( x \) was an arbitrary element of \( C(Y) \), we conclude that \( C(Y) \subseteq M(Y, R^C) \). Now take \( x \in M(Y, R^C) \), and take any \( y \in Y \). It follows that not \( y P_{RC} x \), so \( x \in C(\{x, y\}) \). Since \( C \) is rationalized by \( B \), we have not \( y P_B x \). Since \( y \) was arbitrary, we conclude that \( x \in M(Y, B) \). Since \( x \) was arbitrary, we conclude \( M(Y, R^C) \subseteq M(Y, B) = C(Y) \). Therefore, \( C \) is rationalized by \( R^C \).

It is clear that \( R_B = R^C \), for otherwise the relations would have different maximal elements in some pair. Now assume \( C \) is rationalized by \( R^C \) and \( R^C = R_B \). For all \( Y \subseteq X \), we have \( C(Y) = M(Y, R^C) = M(Y, P^C) \), where the second equality uses \( P^C = P_{RC} \) and Proposition
1.24. Therefore, $C$ is rationalized by $P^C$. Further, $P^C = P_{RC} = P_{RB} = P_B$. Finally, assume $C$ is rationalized by $P^C$ and $P^C = P_B$. Then, for all $Y \subseteq X$, we have $C(Y) = M(Y, P^C) = M(Y, P_B) = M(Y, B)$, where the second equality follows from Proposition 1.23, and we conclude that $C$ is rationalized by $B$. 

Using Proposition 1.53, we can see that, when $X = \{x, y, z\}$, the choice function $C$, defined by

$$C(\{x, y, z\}) = \{x\}, C(\{x, y\}) = \{y\}, C(\{x, z\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x\}) = \{x\}, C(\{y\}) = \{y\}, C(\{z\}) = \{z\},$$

is not rationalizable.

**Consistency of Choice**

We will consider the following “consistency” conditions on choice functions:

- $\alpha$ $Y \subseteq Z \Rightarrow C(Z) \cap Y \subseteq C(Y)$
- $\gamma$ $C(Y) \cap C(Z) \subseteq C(Y \cup Z)$
- $\delta^*$ $[C(W) \subseteq Y \cup Z \subseteq W] \Rightarrow [C(Y) \cap C(Z) \subseteq C(W)]$
- $\beta^*$ $[Y \subseteq Z \wedge C(Z) \cap Y \neq \emptyset] \Rightarrow [C(Y) \subseteq C(Z)]$
- ACA $[Y \subseteq Z \wedge C(Z) \cap Y \neq \emptyset] \Rightarrow [C(Y) = C(Z) \cap Y]$
- WARP $[C(Y) \cap Z \neq \emptyset \wedge C(Z) \cap Y \neq \emptyset] \Rightarrow [C(Y) \cap Z = C(Z) \cap Y]$

for all $Y, Z, W \subseteq X$.

Here, $\beta^*$ strengthens condition $\beta$, defined by Sen (1971), while $\delta^*$ substantially strengthens his condition $\delta$.

See Figure 1.17 for a visual depiction of some of these conditions.

**Figure 1.17**: Consistency conditions for choice

Given a collection $\mathcal{Y}$ of subsets of $X$, we say $C$ is *decisive on* $\mathcal{Y}$ if, for all $Y \in \mathcal{Y}$, $C(Y) \neq \emptyset$. If it is decisive on all nonempty subsets of $X$, it is simply *decisive*. 

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Given a collection \( \mathcal{Y} \), we say \( C \) is resolute on \( \mathcal{Y} \) if, for all \( Y \in \mathcal{Y} \), \(|C(Y)| = 1\). If it is resolute on all nonempty subsets of \( X \), it is simply resolute.

**Proposition 1.54** The following relationships hold between the above conditions.

1. If \( C \) satisfies \( \delta^* \), then it satisfies \( \gamma \).
2. If \( C \) is decisive and satisfies \( \beta^* \), then it satisfies \( \delta^* \).
3. \( C \) satisfies ACA if and only if it satisfies both \( \alpha \) and \( \beta^* \). In either case, it satisfies WARP.
4. If \( C \) is decisive and satisfies WARP, then it satisfies ACA, \( \alpha \), and \( \beta^* \).

**Proof:** 1. Take \( Y, Z \subseteq X \). Setting \( W = Y \cup Z \), the antecedent of \( \delta^* \) is satisfied, so \( C(Y) \cap C(Z) \subseteq C(W) = C(Y \cup Z) \). By decisiveness, there exists \( x \in C(W) \subseteq Y \). Therefore, \( Y \subseteq W \) and \( C(W) \cap Y \neq \emptyset \), so \( \beta^* \) implies \( C(Y) \subseteq C(W) \). Similarly, \( C(Z) \subseteq C(W) \), and \( C(Y) \cap C(Z) \subseteq C(W) \) then follows.

3. The equivalence of ACA and the conjunction of \( \alpha \) and \( \beta^* \) is obvious. To see that \( \alpha \) and \( \beta^* \) together imply WARP, take \( Y, Z \subseteq X \) and \( x \in C(Y) \cap Z \) and \( y \in C(Z) \cap Y \). By \( \alpha \), \( x \in C(\{x, y\}) \). Then \( \beta^* \) yields \( x \in C(Z) \) and \( y \in C(Y) \), as required. 4. To see that WARP implies \( \alpha \), take \( Y, Z \subseteq X \) such that \( Y \subseteq Z \). Take any \( x \in C(Z) \cap Y \). By decisiveness, there is some \( y \in C(Y) \). In fact, \( y \in C(Y) \cap Z \). Then WARP implies \( x \in C(Y) \cap Z \). In particular, \( x \in C(Y) \cap Z \). Therefore, \( C(Y) \subseteq C(Z) \).

See Figure 1.18 for a diagramatic representation of these relationships.

**Figure 1.18:** Consistency conditions in space of choice functions

### Rationalizability

The next theorem gives sharp characterizations, in terms of consistency conditions, of rationalizability by different classes of binary relation.

**Proposition 1.55** Let \( C \) be a choice function on \( X \).

\[
\left[ \begin{array}{l}
C \text{ satisfies } \alpha \text{ and } \gamma, \\
|X| < \infty.
\end{array} \right] \quad \Rightarrow \quad \left[ \begin{array}{l}
C \text{ is rationalizable.}
\end{array} \right] \quad \Rightarrow \quad \left[ \begin{array}{l}
C \text{ satisfies } \alpha \text{ and } \gamma.
\end{array} \right]
\]
Suppose that $C$ satisfies $\alpha$ and $\gamma$, and let $R^C$ be the weak base relation of $C$. Take any $Y \subseteq X$, take any $x \in C(Y)$, and take any $y \in Y$. Then $\alpha$ implies $x \in C(\{x, y\})$, i.e., $xR^C y$, so $x \in M(Y, R^C)$. Assuming $X$ is finite, we may index $Y = \{y_1, \ldots, y_m\}$. Take any $x \in M(Y, R^C)$, and note that $x \in \bigcap_{j=1}^m C(\{x, y_j\})$. Applying $\gamma$ a finite number of times, we have $x \in C(\bigcup_{j=1}^m \{x, y_j\}) = C(Y)$. Therefore, $C(Y) = M(Y, R^C)$ for all $Y \subseteq X$, so $C$ is rationalizable. Now suppose $C$ is rationalized by some relation $B$. Take $Y, Z \subseteq X$ such that $Y \subseteq Z$, and take any $x \in C(Z) \cap Y = M(Z, B) \cap Y$. In particular, $yP_B x$ for no $y \in Y$. Therefore, $x \in M(Y, B) = C(Y)$, establishing $\alpha$. Now take any $Y, Z \subseteq X$ and any $x \in C(Y) \cap C(Z) = M(Y, B) \cap M(Z, B)$. Thus, $yP_B x$ for no $y \in Y$, and $zP_B x$ for no $z \in Z$. Therefore, $x \in M(Y \cup Z, B) = C(Y \cup Z)$, establishing $\gamma$.

Note how finiteness is used in the proof of Proposition 1.55. I’ll let you think of a counterexample to the stronger result that drops the assumption that $X$ is finite. It should be clear how to modify $\gamma$ to handle that situation.

**Proposition 1.56** Let $C$ be a choice function on $X$.

\[
\begin{bmatrix}
C \text{ satisfies } \alpha \text{ and } \gamma \\
\text{and is decisive on finite sets, } |X| < \infty.
\end{bmatrix} \Rightarrow \begin{bmatrix}
C \text{ is rationalized by a suborder.}
\end{bmatrix} \Rightarrow \begin{bmatrix}
C \text{ satisfies } \alpha \text{ and } \gamma \\
\text{and is decisive on finite sets.}
\end{bmatrix}
\]

**Proof:** If $C$ satisfies $\alpha$ and $\gamma$, if $X$ is finite, and if $C$ is decisive on finite sets, then the weak base relation is a weak suborder. (You can check this.) The second implication follows from Proposition 1.55 and part 1 of Proposition 1.26.

Say $C$ is 123-decisive if, for every $Y \subseteq X$ with $1 \leq |Y| \leq 3$, $C(Y) \neq \emptyset$.

**Proposition 1.57** Let $C$ be a choice function on $X$.

\[
\begin{bmatrix}
C \text{ satisfies } \alpha \text{ and } \delta^* \\
\text{and is 123-decisive, } |X| < \infty.
\end{bmatrix} \Rightarrow \begin{bmatrix}
C \text{ is rationalized by a quasi-order, } |X| < \infty.
\end{bmatrix} \Rightarrow \begin{bmatrix}
C \text{ satisfies } \alpha \text{ and } \delta^* \\
\text{and is decisive on finite sets.}
\end{bmatrix}
\]

**Proof:** Suppose that $C$ satisfies $\alpha$ and $\delta^*$ and is 123-decisive, and that $X$ is finite. Since $\delta^*$ implies $\gamma$, Proposition 1.55 tells us that $C$ is rationalized by a relation $B$. I claim that $B$ is a quasi-order. Take $x, y, z \in X$ such that $xP_B yP_B z$. Then 123-decisiveness implies that there is some $w \in C(\{x, y, z\})$. Note that $w \neq z$, for otherwise, $\alpha$ would imply $z \in C(\{y, z\})$, contradicting $yP_B z$. For the same reason, $w \neq y$. Therefore, $C(\{x, y, z\}) = \{x\}$. By $\alpha$, we have $x \in C(\{x, z\})$. If not $xP_B z$, then $C(\{x, z\}) = \{x, z\}$. Setting $W = \{x, y, z\}$, $Y = \{x, z\}$, and $Z = \{z\}$, we have $C(W) \subseteq Y \cup Z \subseteq W$, and $\delta^*$ implies $C(Y) \cap C(Z) \subseteq C(W)$. Since $z \in C(Y) \cap C(Z)$, we conclude that $z \in C(\{x, y, z\})$, a contradiction. Therefore, $xP_B z$, and $B$ is a quasi-order. To prove the second implication, suppose that $C$ is rationalized by a quasi-order and that $X$ is finite. Proposition 1.56 implies that $C$ satisfies $\alpha$ and is decisive.}

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on finite sets. To establish $\delta^*$, take any $Y, Z, W \subseteq X$ such that $C(W) \subseteq Y \cup Z \subseteq W$, and take any $x \in C(Y) \cap C(Z)$. If $x \notin C(W) = M(W, B)$, then there exists $y^1 \in W$ such that $y^1 P_{Bx}$. If $y^1 \notin M(W, B)$, then there exists $y^2 \in W$ such that $y^2 P_{Bx}$, and so on. Since $X$ is finite and $P_B$ is acyclic, there exist $k$ and $y^1, \ldots, y^k \in W$ such that $y^k \in M(W, B) = C(W)$ and $y^k P_{Bx}$. Since $B$ is a quasi-order, we then have $y^k P_{Bx}$. Assume without loss of generality that $y^k \in Y$. Then $x \notin M(Y, B) = C(Y)$, a contradiction. We conclude that $x \in C(W)$. Therefore, since $x$ was arbitrary, $C(Y) \cap C(Z) \subseteq C(W)$, as required.

Proposition 1.58 Let $C$ be a choice function on $X$.

$$
\begin{align*}
C & \text{ satisfies } \alpha \text{ and } \beta^* \\
\text{and is 123-decisive.}
\end{align*}
\Rightarrow
\begin{align*}
C & \text{ is rationalized by an order.}
\end{align*}
\Rightarrow
\begin{align*}
C & \text{ satisfies } \alpha \text{ and } \beta^* \\
\text{and is decisive on finite sets.}
\end{align*}
$$

Proof: Suppose $C$ satisfies $\alpha$ and $\beta^*$ and is 123-decisive. From Proposition 1.17, $\beta^*$ implies $\gamma$. Then Proposition 1.55 tells us that $C$ is rationalized by a relation $B$. Now take any $x, y, z \in X$ such that $x R_{B} y R_{B} z$. By 123-decisiveness, there is some $w \in C(\{x, y, z\})$. If $w = y$, then, by $\alpha$ and $x R_{B} y$, we have $C(\{x, y\}) = \{x, y\}$. Then $\beta^*$ implies $x \in C(\{x, y, z\})$. If $w = z$, then the same argument establishes that $y \in C(\{x, y, z\})$, and then $x \in C(\{x, y, z\})$. Therefore, $x \in L C(\{x, y, z\})$, and $\alpha$ implies $x \in C(\{x, z\})$, i.e., $x R_{B} z$. This shows that $B$ is an order. Now suppose $C$ is rationalized by an order $B$. Proposition 1.56 implies that $C$ satisfies $\alpha$ and is decisiveness on finite sets. To establish $\beta^*$, take $Y, Z \subseteq X$ such that $Y \subseteq Z$ and $C(Z) \cap Y \neq \emptyset$. Let $y \in Y$ satisfy $y \in C(Z) = M(Z, B)$. Take any $x \in C(Y) = M(Y, B)$ and any $z \in Z$. Since $y$ is maximal in $Z$, we have $y R_{B} z$. Since $B$ is acyclic, $x R_{B} y$. By transitivity of $R_{B}$, we have $x R_{B} z$. Since $z$ was an arbitrary element of $Z$, we have $x \in M(Z, B) = C(Z)$, which establishes $\beta^*$.

I’ll leave the proof of the next proposition to you. Say $C$ is 123-resolute if, for every $Y \subseteq X$ with $1 \leq |Y| \leq 3$, $|C(Y)| = 1$.

Proposition 1.59

$$
\begin{align*}
C & \text{ satisfies } \alpha \text{ and is 123-resolute.} \\
\text{ } & \Rightarrow \text{ } \Rightarrow \\
C & \text{ is rationalized by a linear order.} \\
\text{ } & \Rightarrow \text{ } \\
C & \text{ satisfies } \alpha \text{ and is resolute on finite sets.}
\end{align*}
$$

Note that, in the second implications of Propositions 1.55–1.59, only Proposition 1.57 uses finiteness of $X$. In fact, for infinite $X$, rationalizability by a weak quasi-order does not generally imply $\delta^*$. (Can you see why?)
Exercises

1.7.1.  Prove that, if $C$ is rationalizable, then $R^C$ is complete, $P^C$ is asymmetric, and $(P^C, R^C)$ is a dual pair.

1.7.2.  Give an example of a choice function that satisfies $\beta$ (defined by Austen-Smith and Banks) but not $\beta^\ast$. Can you find an example that satisfies $\alpha$ as well?

1.7.3.  Give counter-examples for the directions of implication not stated in Proposition 1.54.

1.7.4.  Give an example showing that the result of Proposition 1.55 does not hold if $X$ is allowed to be infinite. How would you extend $\gamma$ to account for this case?

1.7.5.  In the proof of Proposition 1.56, why is the weak base relation a weak sub-order?

1.7.6.  Why is finiteness of $X$ assumed in the second implication in Proposition 1.57? Show that, when $X$ is infinite, rationalizability by a quasi-order does not generally imply condition $\delta^\ast$.

1.7.7.  Say $C$ satisfies $\delta$ if, for every $Y, Z \subseteq X$ such that $Y \subseteq Z$, $|C(Y)| > 1$ implies $|C(Z)| > 1$. Prove the following.

(a) If $C$ satisfies $\delta^\ast$, then it satisfies $\delta$.

(b) $\delta$ does not imply $\gamma$.

(c) $\delta^\ast$ can be replaced with $\delta$ in Proposition 1.57.

1.7.8.  Consider the following condition on choice functions:

\[ \pi \quad C(Y \cup Z) = C(C(Y) \cup C(Z)) \]

for all $Y, Z \subseteq X$. Prove the following.

(a) If $C$ satisfies $\pi$, then it satisfies $\alpha$.

(b) $C$ satisfies both $\alpha$ and $\delta^\ast$ if and only if it satisfies both $\alpha$ and $\pi$.

1.7.9.  Consider the following condition on choice functions:

\[ \epsilon^\ast \quad [C(Z) \subseteq Y \subseteq Z] \Rightarrow [C(Y) \subseteq C(Z)] \]

for all $Y, Z \subseteq X$. Prove the following.

(a) If $C$ is decisive and satisfies $\beta^\ast$, then it satisfies $\epsilon^\ast$.  

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(b) \( C \) satisfies \( \delta^* \) if and only if it satisfies both \( \gamma \) and \( \epsilon^* \).

(c) \( C \) satisfies \( \pi \) if and only if it satisfies both \( \alpha \) and \( \epsilon^* \).

1.7.10. Assume \( X \) is finite, and let \( C \) be a decisive choice function. Prove that \( C \) satisfies \( \pi \) if and only if: there exists a finite collection of weak linear orders, say \( R^1, \ldots, R^k \), such that

\[
C(Y) = \bigcup_{j=1}^{k} M(Y, R^j)
\]

for all \( Y \subseteq X \).

1.7.11. Assume \( X \) is finite, and let \((P, R)\) be a dual pair such that \( P \) is total. Prove that \( UC(\cdot, P) \) is the smallest choice function satisfying the following three conditions: (i) \( \gamma \), (ii) \( xPy \) implies \( x \in C(\{x, y\}) \), and (iii) \( xPyPz \) implies \( x \in C(\{x, y, z\}) \). In other words, prove that the uncovered set satisfies these conditions and, for any other choice function satisfying them, we have \( UC(Y, P) \subseteq C(Y) \) for all \( Y \subseteq X \).

1.7.12. How would you extend the result of the previous exercise to dual pairs such that \( P \) is not total? What about axioms for \( UC(\cdot, R) \)?

1.8 Pseudo-Rationalizability*

[ More to come. ]

1.9 Duality for Binary Relations*

There is a sense in which the complete relation \( R \) and the asymmetric relation \( P = P_R \) are two “versions” of the same thing: because the asymmetric part operator \( P(\cdot) \) is one-to-one when restricted to the complete relations, we can always invert to recover the original complete relation \( R \). Similarly, so do the asymmetric relation \( P \) and the complete relation \( R = R_P \) contain the same information. Indeed, the members of any dual pair \((P, R)\) are equivalent in this sense.

Here, we consider a more fundamental notion of duality between relations, a way of associating of one relation with another “equivalent” relation that translates the structure of the original relation in an enlightening way.

Define the dual of \( B \), denoted \( D_B \), as

\[
D_B = (B)^{-1},
\]
i.e., $xD_By$ if and only if not $yBx$. We call $D$ the duality operator.

As noted above, $(B)^{-1} = (B^{-1})$. Thus, we may write $D_B = B^{-1}$ without ambiguity. (It does not matter which order the inverse and complementation operations are applied.)

**Proposition 1.60** For a relation $B$,

1. $D_{D_B} = B$,
2. $D_{P_B} = R_B$,
3. $D_{R_B} = P_B$.

By part 1 of Proposition 1.60, it follows that the duality operator is one-to-one and onto the set of all binary relations on $X$. Moreover, it is idempotent, i.e., it is its own inverse.

By part 2 of Proposition 1.60, we see that the duality operator, restricted to the asymmetric relations, is equivalent to the completion operator: for all asymmetric $P$, $D_P = R_P$. Similarly, part 3 implies that the duality operator, restricted to the complete relations, is equivalent to the asymmetric part operator: for all complete $R$, $D_R = P_R$. Thus, we have indeed extended the original notion of duality.

Generally, we say $(B, B')$ is a dual pair if $B' = D_B$, or, equivalently, if $B = D_{B'}$. Parts 2 and 3 of Proposition 1.60 show that this indeed extends our original definition.

We can give an alternative characterization of the dual relation of $B$ in terms of the derivative relations defined above. It makes clear that, when $B$ is complete, so $N_B = \emptyset$, the dual of $B$ is indeed the asymmetric part; and when $B$ is asymmetric, so $I_B = \emptyset$, the dual of $B$ is indeed the completion.

**Proposition 1.61** For a relation $B$, $D_B = P_B \cup N_B = R_B \setminus I_B$.

The next result illustrates how the structure of one relation is reflected, via the duality operator, in the structure of its counterpart.

**Proposition 1.62** For a relation $B$,

1. $D_{B^{-1}} = B^{-1}$,
2. $D_{B^{-1}} = B$,
3. $D_{B \cap B'} = D_B \cup D_{B'}$, 

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4. $D_{B ∪ B'} = D_B ∩ D_{B'}$.

Furthermore, as in Proposition 1.4, the duality operator preserves asymmetric parts and completions.

**Proposition 1.63** For a relation $B$,

1. $P_{D_B} = P_B$;
2. $R_{D_B} = R_B$.

The next result extends Proposition 1.24 by establishing a general duality between undominated and dominant elements of a relation $B$.

**Proposition 1.64** Let $B$ be a relation. For all $Y ⊆ X$, $UD(Y, B) = D(Y, D_B)$ and $D(Y, B) = UD(Y, D_B)$.

A *property* is any subset of binary relations (e.g., we can identify the property of transitivity with the set of all transitive relations on $X$). We say the pair $(P_1, P_2)$ of properties is dual if

$$P_1 = \{D_B \mid B ∈ P_2\} \text{ and } P_2 = \{D_B \mid B ∈ P_1\}.$$  

The next result presents several easily verified dualities. With it, the proof of Proposition 1.13 is quick work.

**Proposition 1.65** The following pairs of properties are dual.

- (complete, asymmetric)
- (reflexive, irreflexive)
- (anti-symmetric, total)
- (symmetric, symmetric)
- (acyclic, negatively acyclic)
- (transitive, negatively transitive)

Because the duality operator preserves asymmetric parts and completions, it is easy to see that $P$-acyclicity is dual to itself; as is $P$-transitivity; as is $P$-negative transitivity; as is $R$-negative acyclicity; as is $R$-negative transitivity; as is $R$-transitivity.

It is straightforward to prove that, if $(P_1, P_2)$ are dual, and if $(P_3, P_4)$ are dual, then $(P_1 ∩ P_3, P_2 ∩ P_4)$ are dual too. With this observation, the proof of Proposition 1.15 is quick work.
1.10 Compatible Extensions*

A classic problem in the theory of binary relations is that of extending a relation with certain properties to a complete relation that inherits those properties. The next condition plays a key role in this analysis.

We say a relation \( B \) is transitive-consistent if \( P_B \subseteq P_TB \). You can check that transitive-consistency is stronger than \( P \)-acyclicity and weaker than acyclicity. It is clearly weaker than transitivity.

This concept was originally defined by Suzumura in different terms: for all \( K \) and all \( x_1, x_2, \ldots, x_K \), if \( x_1 B x_2 \) and \( x_k B x_{k+1} \) for all \( k \) with \( 2 \leq k \leq K - 1 \), then not \( x_K B x_1 \). To see the equivalence of the definitions, first suppose \( P_B \subseteq P_TB \) and take \( K \) and \( x_1, x_2, \ldots, x_K \) as above. Thus, \( x_1 P_B x_2 \), implying \( x_1 P_TB x_2 \) by supposition, and \( x_2 T_B x_K \). If \( x_K B x_1 \), then we would have \( x_2 T_B x_1 \), contradicting \( x_1 P_TB x_2 \). Therefore, not \( x_K B x_1 \), as required. Now suppose \( B \) fulfills Suzumura's definition and \( x P_B y \). In particular, \( x T_B y \). If \( y T_B x \), then there exist \( K' \) and \( x'_1, x'_2, \ldots, x'_K \) such that

\[
y B x'_1 B x'_2 \cdots B x'_K = x.
\]

Letting \( K = K' - 1 \), \( x_1 = x \), \( x_2 = y \), \( x_3 = x'_1 \), \ldots, \( x_K = x_{K' - 1} \), we violate Suzumura's condition. Therefore, not \( y T_B x \), as required.

Transitive-consistency is a property of relations, and it therefore has a dual property: a relation \( B \) is negatively transitive-consistent if \( P_B \subseteq P_{T\overline{B}} \). This condition is stronger than \( P \)-acyclicity and weaker than negative acyclicity. It is clearly weaker than negative transitivity.

To see that the two consistency conditions are indeed dual, note that \( P_B = P_{\overline{B}} = P_{T\overline{B}} \); therefore, \( P_B \subseteq P_{T\overline{B}} \) if and only if \( P_{\overline{B}}^{-1} \subseteq P_{T\overline{B}} \), if and only if \( P_{\overline{B}} \subseteq P_{T\overline{B}}^{-1} = P_{\overline{T\overline{B}}} \), i.e., \( \overline{B} \) is negatively transitive-consistent.

**Proposition 1.66** For a relation \( B \),

1. If \( B \) is acyclic, then it is transitive-consistent. If \( B \) is transitive-consistent, then it is \( P \)-acyclic. Under asymmetry, the three conditions are equivalent.
2. If \( B \) is negatively acyclic, then it is negatively transitive-consistent. If \( B \) is negatively transitive-consistent, then it is \( P \)-acyclic. Under completeness, the three conditions are equivalent.
3. If \( B \) is total and transitive-consistent, then it is negatively transitive.
4. If \( B \) is anti-symmetric and negatively transitive-consistent, then it is transitive.
Proof: To prove part 1, suppose $B$ is acyclic and $x P_B y$ but not $x T_B y$; then $x T_B x$, contradicting acyclicity. Thus, $B$ is transitive-consistent. Suppose $B$ is transitive-consistent but $x T_B y$; then there exist $x_1, x_2, \ldots, x_K$ such that

$$x P_B x_1 P_B x_2 \cdots P_B x_K = x,$$

implying $x P_B x_1 T_B x$, contradicting transitive-consistency. Thus, $B$ is $P$-acyclic. Under asymmetry, acyclicity is equivalent to $P$-acyclicity, giving equivalence of all three. Part 2 is the dual version of part 1. To prove part 3, suppose $B$ is total and transitive-consistent, and take $x, y,$ and $z$ such that $x P_B y P_B z$. If not $x T_B z$, then $x B z$; but then $z P_B y T_B z$, contradicting transitive-consistency. Therefore, $x T_B z$ and $B$ is negatively transitive. Part 4 is the dual version of part 3. 

As a consequence of parts 1 and 3 of Proposition 1.66, if $B$ is total and acyclic, then it is negatively transitive. Part 3 implies that, if $B$ is total and transitive, then it is negatively transitive.

The next concept describes one sense in which we might extend a relation: adding ordered pairs to it while preserving its asymmetric part.

Given relations $B$ and $B'$, $B'$ is a compatible extension of $B$ if (i) $B \subseteq B'$ and (ii) $P_B \subseteq P_{B'}$. Let $E(B)$ denote the class of compatible extensions of $B$.

**Proposition 1.67** For relations $B$, $B'$, and $B''$,

1. The relation $B$ is transitive-consistent if and only if $T_B$ is a compatible extension of $B$.

2. If $B'$ is a compatible extension of $B$ and $B''$ is a compatible extension of $B'$, then $B''$ is a compatible extension of $B$.

A different problem is to subtract pairs from a relation, again preserving its asymmetric part.

Given relations $B$ and $B'$, $B'$ is a compatible intension of $B$ if (i) $B' \subseteq B$ and (ii) $P_B \subseteq P_{B'}$.

In fact, the concepts of extension and intension are dual: $B'$ is a compatible extension of $B$ if and only if $\bar{B}'$ is a compatible intension of $B$. Of course, $B \subseteq B'$ if and only if $\bar{B}' \subseteq \bar{B}$. Since $P_B = P_\bar{B} = P_\bar{B}^{-1}$, it follows that $P_B \subseteq P_{B'}$ if and only if $P_\bar{B} \subseteq P_{\bar{B}'}$, finishing the claim. The next proposition is the dual of Proposition 1.67.

**Proposition 1.68** For relations $B$, $B'$, and $B''$,
1. The relation $B$ is negatively transitive-consistent if and only if $T_B$ is a compatible intension of $B$.

2. If $B'$ is a compatible intension of $B$ and $B''$ is a compatible intension of $B'$, then $B''$ is a compatible intension of $B$.

We next discuss some properties of properties themselves. Note that a chain, denoted $C$, is a class such that $B, B' \in C$ implies $B \subseteq B'$ or $B' \subseteq B$, i.e., that the set-inclusion relation is complete when restricted to $C$.

A class $B$ is closed upward if, for all chains $C$ in $B$,

$$\bigcup \{ B | B \in C \} \in B.$$ 

Note that if $B$ is closed upward, then, by Zorn’s lemma, it contains a relation, $B$, maximal with respect to set-inclusion: for all $B' \in B$, $B \subseteq B'$ implies $B = B'$.

For examples of properties that are closed upward, consider:

(*) reflexivity, irreflexivity, completeness, asymmetry, totalness, anti-symmetry, transitivity, negative transitivity, acyclicity, negative acyclicity, $P$-acyclicity, and transitive-consistency.

To see that transitive-consistency is closed upward, for example, take a chain $C$ in this class, let $B = \bigcup C$, and take $(x, y) \in P_B$. If $(x, y) \notin P_T$, then, since $(x, y) \in T_B$, it must be that $(y, x) \in T_B$, i.e., there exist $x_1, x_2, \ldots, x_K$ such that

$$yBx_1Bx_2\cdots Bx_K = x.$$ 

Since $C$ is a chain, there exists $B' \in C$ such that $(x, y) \in P_{B'}$ and

$$yB'x_1B'x_2\cdots B'x_K = x,$$

or equivalently, $(y, x) \in T_{B'}$, contradicting the transitive-consistency of $B'$.

For examples of properties that are not closed upward, consider closedness and convexity.

**Proposition 1.69** For properties $B$ and $B'$,

1. If $B$ and $B'$ are closed upward, then $B \cap B'$ is closed upward.

2. For all relations $B$, the class $E(B)$ of compatible extensions of $B$ is closed upward.
As a consequence of this proposition, all of the classes of relations we have defined are closed upward as well: weak or strict suborder, weak or strict quasi-order, weak or strict order, weak or strict linear order.

The dual of “closed upward” is “closed downward.” Specifically, we say property $B$ is closed downward if, for all chains $C$ in $B$,

$$\bigcap \{B \mid B \in C\} \in B.$$  

The dual to Proposition 1.69, omitted, is readily obtained.

A class $B$ is transitive-closed if, for all $B \in B$, $T_B \in B$. Of course, if $B$ is a subset of the transitive relations, then it is trivially transitive-closed.

**Proposition 1.70**

1. All of the properties in (*), except irreflexivity, asymmetry, and anti-symmetry, are transitive-closed.

2. The conjunction of anti-symmetry with transitive-consistency is transitive-closed.

3. If $B$ and $B'$ are transitive-closed, then $B \cap B'$ is transitive-closed.

**Proof:** For part 1, I will only prove that negative transitivity is transitive-closed. Suppose $B$ is negatively transitive, take $x$, $y$, and $z$ such that $x T_B y T_B z$. If not $x T_B z$, there exist $x_1, x_2, \ldots, x_K$ such that $xBx_1 Bx_2 \cdots Bx_K = z$. If $yBx_1$, then $yT_Bz$, a contradiction. Thus, $yBx_1$. Since $x T_B y$, it follows that $x B y$. But then $xByBx_1$ and negative transitivity imply $x B x_1$, a contradiction. Therefore, $T_B$ is negatively transitive (and trivially transitive consistent). To prove part 2, let $B$ be anti-symmetric and transitive-consistent, and suppose $T_B$ is not anti-symmetric. Then there exist distinct $x$ and $y$ such that $x T_B y T_B x$. Thus, there exist $x_1, x_2, \ldots, x_K$ such that $xBx_1 Bx_2 \cdots Bx_K = y$.

Letting $x = x_0$, take $x_k$ to be any element in this sequence (there is at least one) with $x_k \neq x_{k-1}$. By anti-symmetry, $x_{k-1} P_B x_k$. But $x_k T_B x_{k-1}$, so not $x_{k-1} P_{T_B} x_k$, contradicting transitive-consistency of $B$. The proof of part 3 is omitted.

Of course, anti-symmetry in part 2 of Proposition 1.70 can be strengthened to asymmetry, because the conjunction of asymmetry and transitive-consistency is acyclicity. To see why it cannot be weakened to irreflexivity, suppose $X$ consists of two distinct elements, $a$ and $b$, and consider $B = \{(a,b),(b,a)\}$. This relation is irreflexive and, since $P_B = \emptyset$, trivially transitive-consistent. But $T_B = \{(a,a),(a,b),(b,a),(b,b)\}$, which is clearly not irreflexive.
The dual of “transitive-closed” is “negatively transitive-closed.” Specifically, a property \( B \) is **negatively transitive closed** if, for all \( B \in B, \overline{T_B} \in B \). The dual to Proposition 1.70, omitted, is readily obtained.

Let \( B(x, y) = B \cup \{(x, y)\} \). Then we say a class \( B \) of relations is **arc-receptive** if, for all distinct \( x \) and \( y \) and for all transitive \( B \in B \), \((y, x) \notin B \) implies \( T_B(x, y) \in B \).

**Proposition 1.71**

1. All of the properties in (*), except negative transitivity, are arc-receptive.
2. If \( B \) and \( B' \) are arc-receptive, then \( B \cap B' \) is arc-receptive.

**Proof:** I just prove that asymmetry is arc-receptive. Let \( B \) be an asymmetric, transitive relation as in the definition, and suppose that \( T_B(x, y) \) is not asymmetric: there exist \( z \) and \( w \) such that \( zT_B(x, y)wT_B(x, y)z \). Thus, there exist \( x_1, x_2, \ldots, x_K \) such that

\[
zB(x, y)x_1B(x, y)x_2B(x, y)x_KB(x, y)x = w, \]

and there exist \( x_{K+1}, x_{K+2}, \ldots, x_{K+L} \) such that

\[
wB(x, y)x_{K+1}B(x, y)x_{K+2}B(x, y)x_{K+L} = z. \]

Letting \( z = x_0 \), there is at least one \( k \) such that \((x_k, x_{k+1}) = (x, y)\), for otherwise \( zBwBz \) by transitivity of \( B \), contradicting asymmetry. Let \( x_{k^*} \) be the first occurrence of \( x \) and let \( x_{l^*} \) be the last occurrence of \( y \). Then

\[
y = x_{l^*}Bx_{l^*+1}Bx_{K+L} = zBx_{l^*}Bx_{k^*}, \]

implying that \( yT_Bx \). Since \( B \) is transitive, \( yBx \), contradicting \((y, x) \notin B \). \(\square\)

To see that negative transitivity is not arc-receptive, suppose \( X \) consists of three distinct elements, \( a, b, \) and \( c \), and consider \( B = \emptyset \). This relation is trivially transitive and negatively transitive. Adding \((a, b)\) and taking the transitive closure yields \( B' = \{(a, b)\} \), which is not negatively transitive.

The dual of “arc-receptive” is “negatively arc-receptive.” Specifically, a property \( B \) is **negatively arc-receptive** if, for all distinct \( x \) and \( y \) and for all negatively transitive \( B \in B \), \((y, x) \in B \) implies \( \overline{T_B((x, y))} \in B \). The dual to Proposition 1.71, omitted, is readily obtained.

The next proposition provides sufficient conditions for the construction of (non-trivial) compatible extensions.
**Proposition 1.72** If $B$ is transitive, $x \neq y$, and $(y, x) \notin B$, then $T_{B(x,y)}$ is a compatible extension of $B$ satisfying $(x, y) \in P_{T_{B(x,y)}}$.

Proof: To prove the proposition, note that $B \subseteq B(x, y) \subseteq T_{B(x,y)}$, so condition (i) in the definition of compatible extension is fulfilled. To verify that $P_B \subseteq P_{T_{B(x,y)}}$, take any $(w, z) \in P_B$ and suppose $(w, z) \notin P_{T_{B(x,y)}}$. Since $(w, z) \in T_{B(x,y)}$, this means $(z, w) \in T_{B(x,y)}$, i.e., there exist $x_1, x_2, \ldots, x_K$ such that

$$zB(x, y)x_1B(x, y)x_2\cdots B(x, y)x_K = w.$$  

Letting $z = x_0$, there is at least one $k$ such that $(x_k, x_{k+1}) = (x, y)$, for otherwise $(z, w) \notin B$ by transitivity of $B$, a contradiction. Let $x_{k^*}$ be the first occurrence of $x$ and let $x_l^*$ be the last occurrence of $y$. Then, since $(w, z) \in P_B \subseteq B$,

$$y = x_l^*Bx_{l^*+1}\cdots Bx_K = wBzBx_1\cdots Bx_{k^*} = x.$$  

By transitivity of $B$, $(y, x) \in B$, a contradiction. Therefore, condition (ii) in the definition of compatible extension is fulfilled. If $(x, y) \notin P_{T_{B(x,y)}}$, then, since $(x, y) \in T_{B(x,y)}$, it must be that $(y, x) \in T_{B(x,y)}$, i.e., there exist $x_1, x_2, \ldots, x_K$ such that

$$yB(x, y)x_1B(x, y)x_2\cdots B(x, y)x_K = x.$$  

Let $x_{k^*}$ be the first occurrence of $x$, and note that, since $(y, x) \notin B$ and $x \neq y$, it must be that $k^* > 1$. Then

$$yBx_1Bx_2\cdots Bx_{k^*} = x$$  

and transitivity of $B$ imply $(y, x) \in B$, a contradiction. □

The next proposition uses Proposition 1.72 to give sufficient conditions for the existence of total, transitive compatible extensions.

**Proposition 1.73** Assume $B$ is closed upward and arc-receptive. If $B \in B$ is transitive, then there exists a total, transitive compatible extension of $B$ in $B$.

Proof: To prove the proposition, let $B'$ be the set of transitive compatible extensions, $B'$, of $B$ such that $B' \in B$. This is non-empty since $B \in B'$. Note that $B'$ is the intersection of $E(B)$ with the class of transitive relations and so, by Proposition 1.69, it is closed upward. By Zorn’s lemma, $B'$ has a maximal element, say $B^*$. I claim that $B^*$ is total. If not, there exist distinct $x, y \in X$ such that $(x, y) \notin B^* \cup B^{*-1}$. I claim that $T_{B^*(x,y)} \in B$ and that it is a transitive (this much is clear) compatible extension of $B$, contradicting maximality of $B^*$. That $T_{B^*(x,y)} \in B$ follows because $B^* \in B$, $B^*$ is transitive, and $B$ is arc-receptive. Since $(y, x) \notin B^*$, Proposition 1.72 implies that $T_{B^*(x,y)}$ is a compatible extension of $B^*$ and, by part 2 of Proposition 1.67, of $B$. This contradiction establishes that $B^*$ is indeed total. □
We can give a sharper result, showing when a relation not only has a compatible extension, but when it is actually the intersection of such extensions.

**Proposition 1.74** Assume \( \mathcal{B} \) is closed upward and arc-receptive. If \( \mathcal{B} \) is transitive-consistent and \( T_B \in \mathcal{B} \), then

\[
T_B = \bigcap \{ B' \in \mathcal{B} \mid B' \text{ is a total, transitive compatible extension of } B \}.
\]

*Proof:* The \( \subseteq \) inclusion follows by noting that every transitive relation that includes \( \mathcal{B} \) also includes \( T_B \). Now the \( \supseteq \) inclusion. Take any \( (y, x) \notin T_B \). It suffices to find a total, transitive compatible extension, \( B^* \), of \( \mathcal{B} \) such that \( B^* \in \mathcal{B} \) and \( (y, x) \notin B^* \). Consider two cases: \( x \neq y \) and \( x = y \). In case \( x \neq y \), let \( B' \) denote \( T_B \). Then \( T_B x, y \in \mathcal{B} \) follows because \( B' \in \mathcal{B} \), \( B' \) is transitive, and \( \mathcal{B} \) is arc-receptive. By Proposition 1.72, \( T_B (x, y) \) is a transitive compatible extension of \( B' \) satisfying \( (x, y) \in P_{T_B(x, y)} \). Proposition 1.73 yields a total, transitive compatible extension, \( B'' \), of \( T_B (x, y) \) such that \( B'' \in \mathcal{B} \) and \( (x, y) \in P_{B''} \). Part 2 of Proposition 1.67 implies that \( B'' \) is a compatible extension of \( T_B \), and, by transitive-consistency of \( \mathcal{B} \), of \( \mathcal{B} \). Setting \( B'' = B' \), the case is complete. In case \( x = y \), let \( B' \) be the class of transitive compatible extensions, \( B' \), of \( \mathcal{B} \) such that \( B' \in \mathcal{B} \) and \( (x, x) \notin B' \). That \( T_B \in \mathcal{B} \) follows by the transitive-consistency of \( \mathcal{B} \) and part 1 of Proposition 1.67, so this class is non-empty. It is clearly closed upward, and, in fact, it is arc-receptive. To verify this, consider \( B' \in \mathcal{B} \) and distinct \( z, w \in X \) such that \( (z, w) \notin B' \). Note that, because \( \mathcal{B} \) is arc-receptive, \( T_{B'(w, z)} \in \mathcal{B} \). Now suppose \( (x, x) \in T_{B'(w, z)} \). Then there exist \( x_1, x_2, \ldots, x_K \) such that

\[
xB'(w, z)x_1B'(w, z)x_2 \cdots B'(w, z)x_K = x.
\]

Letting \( x = x_0 \), there is at least one \( k \) such that \( (x_k, x_{k+1}) = (w, z) \), since \( B' \) is transitive and \( (x, x) \notin B' \). Let \( x_{l^*} \) be the first occurrence of \( w \) and let \( x_{k^*} \) be the last occurrence of \( z \). Then

\[
z = x_{l^*}B'x_{l^*+1} \cdots B'x_K = xB'x_1 \cdots B'x_{k^*} = w
\]

and transitivity of \( B' \) imply \( (z, w) \in B' \). This contradiction yields \( (x, x) \notin T_{B'(w, z)} \), with the conclusion that \( \mathcal{B} \) is arc-receptive. Proposition 1.73 yields a total, transitive compatible extension, \( B'' \), of \( T_B \) such that \( B'' \in \mathcal{B} \). Setting \( B^* = B'' \), the second case is completed. \( \blacksquare \)

A result on compatible intensions is now obtained easily.

**Proposition 1.75** Assume \( \mathcal{B} \) is closed downward and negatively arc-receptive. If \( \mathcal{B} \) is negatively transitive-consistent and \( \overline{T_B} \in \mathcal{B} \), then

\[
\overline{T_B} = \bigcup \{ B' \in \mathcal{B} \mid B' \text{ is an anti-symmetric, negatively transitive compatible intension of } B \}.
\]
Proof: If $B$ is closed downward and negatively arc-receptive, then $\overline{B}$ is closed upward and arc-receptive. Since $B$ is negatively transitive-consistent and $T_B \in B$, it follows that $\overline{B}$ is transitive-consistent and $T_B \in \overline{B}$. By the general extension theorem,

$$T_B = \bigcap \{B' \in \overline{B} \mid B' \text{ is a total, transitive compatible extension of } \overline{B} \}.$$ 

Taking the complement of both sides and applying De Morgan’s Law, the proof is complete.

A number of applications of these results are now possible.

**Corollary 1.3** For a relation $B$,

1. if $B$ is transitive-consistent, then $T_B$ is the intersection of total, transitive compatible extensions of $B$,

2. if $B$ is acyclic, then $T_B$ is the intersection of strict linear order compatible extensions of $B$,

3. if $B$ is reflexive, anti-symmetric, and transitive-consistent, then $T_B$ is the intersection of weak linear order compatible extensions of $B$,

4. if $B$ is reflexive and transitive consistent, then $T_B$ is the intersection of weak order compatible extensions of $B$.

Proof: Part 1 follows from Proposition 1.74 by letting $\mathcal{B}$ be the set of all relations on $X$. For part 2, let $\mathcal{B}$ be the class of acyclic, i.e., asymmetric and transitive-consistent, relations. By Propositions 1.69–1.71, this class is closed upward, transitive-closed, and arc-receptive. Therefore, Proposition 1.74 implies that the transitive closure of each acyclic relation $B$ is the intersection of total, transitive compatible extensions of $B$ within the class of acyclic relations. A relation is total and acyclic if and only if it is a strict linear order, as required. For part 3, let $\mathcal{B}$ be the class of reflexive, anti-symmetric relations, which is closed upward and arc-receptive. Though the class is not transitive-closed, it is established in Proposition 1.70 that, for all transitive-consistent members $B$ of this class, $T_B$ is reflexive and anti-symmetric. Thus, Proposition 1.74 implies that the transitive closure of each reflexive, anti-symmetric, and transitive-consistent relation is the intersection of total, transitive compatible extensions within this class. The result follows because a relation is reflexive, total, anti-symmetric, and transitive if and only if it is a linear order. For part 4, let $\mathcal{B}$ be the class of reflexive relations: this class is closed upward, transitive-closed, and arc-receptive; and a relation is reflexive, total, and transitive if and only if it is a weak order.
Corollary 1.3 itself has some interesting implications. Every a strict quasi-order $B$ is acyclic, so part 2 applies; furthermore, transitivity means $B = T_B$, so the corollary implies that $B$ itself is the intersection of its strict linear order compatible extensions. This is the well-known result of Spilrajn (1930) and Dushnik and Miller (1941).\footnote{In this literature, “strict partial order” is used instead of “strict quasi-order.”}

When $B$ is a reflexive, anti-symmetric, and transitive relation, i.e., a “partial order,” part 3 of Corollary 1.3 has a similar implication: every partial order is the intersection of its weak linear order extensions.

When $B$ is a reflexive and transitive relation, i.e., a “pre-order,” part 4 of Corollary 1.3 can be applied to show that every pre-order is the intersection of its weak order compatible extensions. This result is due to Donaldson and Weymark (1998).

Proposition 1.75 yields dual results.

**Corollary 1.4** For a relation $B$,

1. if $B$ is negatively transitive-consistent, then $T_B$ is the union of anti-symmetric, negatively transitive compatible intensions of $B$.

2. if $B$ is negatively acyclic, then $T_B$ is the union of weak linear order compatible intensions of $B$.

3. if $B$ is irreflexive, total, and negatively transitive-consistent, then $T_B$ is the union of strict linear order compatible intensions of $B$.

4. if $B$ is irreflexive and negatively transitive consistent, then $T_B$ is the union of strict order compatible intensions of $B$.

An implication of part 2 of Proposition 1.4 is that, if $B$ is complete and $P$-acyclic, then it admits a weak order compatible intension, i.e., we can remove arcs from $B$ in a way that preserves the asymmetric part of $B$, that maintains completeness, and that eliminates all intransitivities. If $B$ is actually $P$-transitive, making it a weak quasi-order, then it is the union of such relations.

### 1.11 Restricted Admissible Sets*

We now analyze choice when the domain of choice may not include all subsets of $X$, i.e., when a choice function $C$ is defined on a collection $\mathcal{X} \subseteq 2^X$ of subsets. Thus, we study $C: \mathcal{X} \Rightarrow X$ such that, for all $Y \in \mathcal{X}$, $C(Y) \subseteq Y$.

We say $Y \subseteq X$ is *admissible* if $Y \in \mathcal{X}$.
In particular, it may be that some pairs of alternatives are not admissible, so the base relations are not well-defined here. We extend our earlier definitions in the obvious way: we say \( xR_C y \) if \( \{ x, y \} \in X \) and \( x \in C(\{ x, y \}) \); and we say \( xP_C y \) if \( \{ x, y \} \in X \) and \( y \notin C(\{ x, y \}) \).

The next restriction on \( X \) formalizes the assumption that all pairs of alternatives are admissible.

\[ A_2 \quad \forall x, y \in X : \{ x, y \} \in X. \]

Note that \((P_C, R_C)\) is a dual pair if and only if \( A_2 \) holds.

We say a relation \( B \) \( D\)-rationalizes \( C \) if, for all \( Y \in X \), \( C(Y) = D(Y, B) \). When such a \( B \) exists, we say \( C \) is \( D\)-rationalizable. A relation \( B \) \( UD\)-rationalizes \( C \) if, for all \( Y \in X \), \( C(Y) = UD(Y, B) \). When such a \( B \) exists, we say \( C \) is \( UD\)-rationalizable.

**Proposition 1.76** A choice function \( C \) is \( D\)-rationalized by \( B \) if and only if it is \( UD\)-rationalized by \( B' = D_B \). If \( A_2 \) also holds, then \( C \) is rationalized by \( B \) and \( B' \), and \( B = R_C \) and \( B' = P_C \).

Thus, \( D\)-rationalizability and \( UD\)-rationalizability are equivalent. When \( A_2 \) is violated, a choice function may be satisfy both of these conditions and yet not be rationalizable in the sense we defined earlier: set \( X = \{ x, y, z \} \), \( X = \{ \{ x, z \}, \{ y, z \}, \{ x, y, z \} \} \), and define

\[ C(\{ x, z \}) = \{ x, z \}, C(\{ y, z \}) = \{ y, z \}, C(\{ x, y, z \}) = \{ z \}. \]

Here, \( C \) is \( D\)-rationalized by \( B = \{(x, x), (y, y), (z, z), (x, z), (z, x), (y, z), (z, y)\} \), but it is not rationalized by any relation. Given \( B \), for example, both \( x \) and \( y \) are maximal in \( X \), so rationalizability by this relation would require \( x, y \in C(\{ x, y, z \}) \).

In contrast with our earlier analysis of rationalizability, which assumed \( A_2 \), \( D\)-rationalizability (likewise \( UD\)-rationalizability) does not pin down a relation up to its completion and asymmetric parts. In the example above, since \( \{ x, y \} \) is not admissible, we could add an arc between the two alternatives without changing the fact that \( B \) \( D\)-rationalizes \( C \).

If \( C \) is \( D\)-rationalized by a complete relation, of course, then it is rationalizable in our earlier sense. (Likewise for \( UD\)-rationalizability by an asymmetric relation.)

Given \( C \), we say \( x \) is revealed weakly preferred to \( y \) if there exists \( y \in X \) such that \( y \in Y \) and \( x \in C(Y) \). In this case, we write \( xV^Cy \).

We say \( x \) is revealed strictly preferred to \( y \) if, for all \( Y \in X \), \( x \in Y \) implies \( y \notin C(Y) \). In this case, we write \( xA^Cy \). It is straightforward to verify that these relations are dual, i.e., \( D_{V^C} = A^C \).

Clearly, for all \( Y \in X \), \( C(Y) \subseteq D(Y, V^C) = UD(Y, A^C) \).
We say $C$ satisfies the $V$-axiom if, for all $Y \in \mathcal{X}$, $D(Y, V^C) \subseteq C(Y)$. Equivalently, the $V$-axiom holds if, for all $Y \in \mathcal{X}$, $UD(Y, \Lambda^C) \subseteq C(Y)$.

**Proposition 1.77** The following are equivalent for a choice function $C$.

- $C$ is $D$-rationalizable,
- $C$ is $UD$-rationalizable,
- $C$ satisfies the $V$-axiom.

In our example above, the relation $B$ is actually $V^C$, the revealed weak preference relation. Of course, the $V$-axiom is satisfied there. The example shows that, when $A2$ is violated, $V^C$ need not be a complete relation (and $\Lambda^C$ is not asymmetric). In fact, the choice function in that example cannot be $D$-rationalized by any complete relation (or $UD$-rationalized by any asymmetric relation).

The same example shows that $V^C$ need not be transitive (nor $\Lambda^C$ negatively transitive), and, in fact, the choice function in that example cannot be $D$-rationalized by any transitive relation (nor $UD$-rationalized by any negatively transitive relation).

We will see that, perhaps surprisingly, $D$-rationalizability by a complete relation will actually follow from $D$-rationalizability by a transitive one. (Likewise for $UD$-rationalizability.)

We have seen conditions that guarantee rationalizability by an ordering when all subsets of $X$ are admissible: ACA, WARP, and the conjunction of $\alpha$ and $\beta^*$. More care must be taken when the collection of admissible sets is restricted.

We say $x$ is indirectly revealed weakly preferred to $y$ if $x T_{V^C} y$. Let $W^C = T_{V^C}$.

For the dual notion, we say $x$ is indirectly revealed strictly preferred to $y$ if $x T_{\Lambda^C} y$. To see this, note that

\[
x T_{V^C}^{-1} y \iff y T_{V^C} x \\
\iff y T_{\Lambda^C}^{-1} x \\
\iff x T_{\Lambda^C} y,
\]

as claimed. Let $M^C = T_{\Lambda^C}$.

Clearly, for all $y \in \mathcal{X}$, $C(Y) \subseteq D(Y, W^C) = UD(Y, M^C)$.

We say $C$ satisfies the $W$-axiom if, for all $y \in \mathcal{X}$, $D(Y, W^C) \subseteq C(Y)$. Equivalently, the $W$-axiom holds if, for all $Y \in \mathcal{X}$, $UD(Y, M^C) \subseteq C(Y)$. 

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Proposition 1.78 Let $C$ be a choice function.

1. If $C$ is $D$-rationalized by a transitive relation, then it satisfies the $W$-axiom.

2. If $C$ is decisive and satisfies the $W$-axiom, then it is $D$-rationalized by a weak order.

Proof: Let $B$ be a transitive relation such that, for all $y \in \mathcal{X}$, $C(Y) = D(Y, B)$. Take $Y \in \mathcal{X}$ and $x \in D(Y, W^C)$, so, for all $y \in Y$, $xW^Cy$. To show that $x \in C(Y)$, it suffices to show that, for all $y \in Y$, $xBy$. Taking $y \in Y$, $xW^Cy$ implies the existence of $x_0, x_1, \ldots, x_K \in \mathcal{X}$ with $x = x_0V^Cx_1V^Cx_2 \cdots V^Cx_{K-1}V^Cx_K = y$. Thus, for each $k \leq K$, there exists $Z_k \in \mathcal{X}$ such that $x_{k-1} \in C(Z_k)$ and $x_k \in Z_k$. By assumption, it follows that $x_{k-1}Bx_k$. Therefore, $xBx_1Bx_2 \cdots Bx_K = y$, and transitivity implies $xBy$. Now suppose $C$ is decisive and satisfies the $W$-axiom, so $C$ is $D$-rationalized by the transitive relation $W^C$. By Corollary 1.3, there exists a total, transitive compatible extension of $W^C$, say $B$. Define $B' = B \cup \Delta$, i.e., $xB'y$ if and only if $xBy$ or $x = y$. This is a weak order such that $W^C \subseteq B'$ and $P_{W^C} \subseteq P_{B'}$. Clearly, for all $Y \in \mathcal{X}$, we have $C(Y) = D(Y, W^C) \subseteq D(Y, B')$. Take any $Y \in \mathcal{X}$, any $x \in D(Y, B')$, and any $y \in Y$. Since $C$ is decisive, there exists $z \in C(Y) = D(Y, W^C)$, so $zW^Cy$. I claim that $xW^Cz$, for otherwise $zP_{W^C}x$, contradicting $x \in D(Y, B')$. Then $xW^CzW^Cy$ implies $xW^Cy$, as required. 

Note that an alternative route to $D$-rationalizability by a weak order in part 2 of Proposition 1.78 is to assume that all finite sets are admissible and to weaken decisiveness to the assumption that $C$ is decisive on finite sets.

That more decisiveness is needed when not all finite sets are admissible can be seen from the following example. Let $X = \mathbb{N} \cup \{0\}$, and let $\mathcal{X}$ consist of $\mathbb{N} \cup \{0\}$ and all finite subsets not including zero; define

$$C(Y) = \begin{cases} \min Y & \text{if } Y \text{ is finite,} \\ \emptyset & \text{if } Y = X. \end{cases}$$

This choice function satisfies the $W$-axiom, but it is not rationalizable by an order.

In any case, Proposition 1.78 shows that, under decisiveness, if $C$ is $D$-rationalized by a transitive relation, then it is $D$-rationalized by a complete and transitive relation. In that case, of course, $C$ is rationalizable by an order, in our earlier sense.

The dual result follows immediately.

Proposition 1.79 Let $C$ be a choice function.

1. If $C$ is $UD$-rationalized by a negatively transitive relation, then it satisfies the $W$-axiom.
2. If \( C \) is decisive and satisfies the \( W \)-axiom, then it is \( UD \)-rationalized by a strict order.

Under a stronger decisiveness condition that that used in Propositions 1.78 and 1.79, the \( W \)-axiom has two equivalent reformulations.

We say \( C \) satisfies the strong congruence axiom (SCA) if, for all \( Y \in \mathcal{X} \) and all \( x, y \in Y \),

\[
[xW^C y \wedge y \in C(Y)] \Rightarrow x \in C(Y).
\]

For the dual version of SCA, consider the contrapositive: for all \( Y \in \mathcal{X} \) and all \( x, y \in Y \), if \( y \in C(Y) \) and \( x \not\in C(Y) \), then \( \neg xW^C y \), i.e., \( yM^C x \).

Given a choice function \( C \), the sequence \( Y_1, Y_2, \ldots, Y_K \in \mathcal{X} \) of admissible sets is a \( C \)-cycle if \( C(Y_K) \cap Y_1 \neq \emptyset \) and, for all \( k = 1, \ldots, K-1 \), \( C(Y_k) \cap Y_{k+1} \neq \emptyset \). Here, we refer to \( K \) as the “length” of the \( C \)-cycle. See Figure 1.19 for an illustration of a \( C \)-cycle.

Figure 1.19: A \( C \)-cycle

We say \( C \) satisfies Hansson’s Axiom of Revealed Preference (HARP) if, for every \( C \)-cycle \( Y_1, \ldots, Y_K \), we have \( C(Y_K) \cap Y_1 = C(Y_1) \cap Y_K \).

For the intuition behind this condition, note that the chosen alternatives from \( Y_K \) are at least as good as some chosen from \( Y_{K-1} \), which are at least as good as some chosen from \( Y_{K-2} \), and so on. Following this logic, the chosen alternatives from \( Y_2 \) are at least as good as some chosen from \( Y_1 \). By transitivity, the chosen alternatives from \( Y_K \) should be at least as good as all of those chosen from \( Y_1 \), so that \( C(Y_K) \cap Y_1 \subseteq C(Y_1) \cap Y_K \). Of course, the chosen alternatives from \( Y_1 \) are at least as good as some chosen from \( Y_K \). By transitivity, the opposite inclusion holds as well.

**Proposition 1.80** A choice function \( C \) satisfies SCA if and only if it satisfies HARP.

**Proof:** First, suppose \( C \) satisfies SCA, and take any \( C \)-cycle \( Y_1, \ldots, Y_K \). Consider any \( x_K \in C(Y_K) \cap Y_1 \). To see that \( x_K \in C(Y_1) \cap Y_K \), pick \( x_k \in C(Y_K) \cap Y_{k+1} \) for \( k = 1, \ldots, K-1 \). Note that \( x_K V^C x_{K-1} V^C x_{K-2} \cdots V^C x_1 \), so \( x_K W^C x_1 \). By SCA, \( x_K \in C(Y_1) \cap Y_K \). Therefore, \( C(Y_K) \cap Y_1 \subseteq C(Y_1) \cap Y_K \). Now consider any \( x \in C(Y_1) \cap Y_K \). By assumption there is a \( y \in C(Y_K) \cap Y_1 \). Then \( x \in Y_K, x V^C y \), and SCA imply \( x \in C(Y_K) \cap Y_1 \). Therefore, \( C(Y_1) \cap Y_K \subseteq C(Y_K) \cap Y_1 \), as required. Next, suppose \( C \) satisfies HARP, take any \( y \in \mathcal{X} \) and any \( x, y \in Y \) with \( xW^C y \) and \( y \in C(Y) \). Since \( xW^C y \), there exist \( x_3, \ldots, x_K \) with

\[
x = x_1 V^C x_K V^C x_{K-1} \cdots V^C x_2 = y.
\]
Then there exists \( Y_K \in X \) such that \( x_K \in Y_K \) and \( x = x_1 \in C(Y_K) \), and for all \( k = 2, \ldots, K - 1 \), there exists \( Y_k \in X \) such that \( x_k \in Y_k \) and \( x_{k+1} \in C(Y_k) \). Thus, for all \( k = 2, \ldots, K - 1 \), we have \( C(Y_k) \cap Y_{k+1} \neq \emptyset \). Furthermore, we have \( C(Y) \cap Y_2 \neq \emptyset \), since \( y = x_2 \in C(Y) \cap Y_2 \), and we have \( C(Y_k) \cap Y \neq \emptyset \), since \( x_K \in C(Y_k) \cap Y \). Setting \( Y_1 = Y \), we have a \( C \)-cycle, and HARP implies that \( C(Y_K) \cap Y = C(Y) \cap Y_K \). Then \( x \in C(Y_K) \cap Y \) yields \( x \in C(Y) \), as required.

The interest in these conditions stems from their connection to the \( W \)-axiom, demonstrated next.

**Proposition 1.81** Let \( C \) be a choice function.

1. If \( C \) satisfies the \( W \)-axiom, then it satisfies SCA and HARP.
2. If \( C \) is decisive and satisfies SCA or HARP, then it satisfies the \( W \)-axiom.

**Proof:** For part 1, suppose \( C \) satisfies the \( W \)-axiom. To establish SCA, take \( Y \in X \) and \( x, y \in Y \) with \( xW^C y \) and \( y \in C(Y) \). That \( x \in C(Y) \) follows from the \( W \)-axiom if \( x \in D(Y, W^C) \). Take any \( z \in Y \), and note that \( yW^C z \). Then \( xW^C y \) and \( yW^C z \) imply \( xW^C z \), as required. Then HARP follows from Proposition 1.80. For part 2, suppose \( C \) is decisive and satisfies SCA. Take \( Y \in X \) and \( x \in D(Y, W^C) \). Since \( C \) is decisive, there exists \( y \in C(Y) \). Then \( xW^C y \) and SCA imply \( x \in C(Y) \). Therefore, \( C \) satisfies the \( W \)-axiom. This, with Proposition 1.80, completes the proof.

To see that decisiveness is required for SCA and HARP to imply the \( W \)-axiom, let \( X = \mathbb{R} \), and let \( X \) consist of \( \mathbb{R}_+ \) and all finite subsets of \( \mathbb{R}_+ \); for finite \( Y \), define \( C(Y) = \min Y \), and define \( C(\mathbb{R}_+) = \emptyset \). This satisfies SCA and HARP, but not the \( W \)-axiom: here, \( 0W^C x \) (and therefore \( 0W^C x \)) for all \( x \in \mathbb{R}_+ \), but \( 0 \notin C(\mathbb{R}_+) \). Note that this choice function does not satisfy ACA, so SCA and HARP do not generally imply that condition.

SCA and HARP are equivalent to much simpler conditions when more structure is imposed on \( X \). I consider the following structural assumptions.

\[
\begin{align*}
AF & \quad \forall K, x_1, \ldots, x_K : \{x_1, \ldots, x_K\} \in X \\
A \cap & \quad \forall Y, Z \in X : \{Y \cap Z \neq \emptyset\} \Rightarrow Y \cap Z \in X \\
A \cup & \quad \forall Y, Z \in X : Y \cup Z \in X
\end{align*}
\]

We say \( C \) satisfies the weak congruence axiom (WCA) if, for all \( Y \in X \) and all \( x, y \in Y \), \( xW^C y \) and \( y \in C(Y) \) implies \( x \in C(Y) \). For the dual version of this condition, consider the contrapositive: for all \( Y \in X \) and all \( x, y \in Y \), \( y \in C(Y) \) and \( x \notin C(Y) \) implies \( \neg xW^C y \), i.e., \( yA^C x \).

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We say \( C \) satisfies the weak axiom of revealed preference (WARP) if, for all \( C \)-cycles of length 2, \( C(Y_1) \cap Y_2 = C(Y_2) \cap Y_1 \). Note that this is equivalent to our earlier definition.

The next proposition establishes the equivalence of these conditions.

**Proposition 1.82** A choice function \( C \) satisfies WCA if and only if it satisfies WARP.

**Proof:** Suppose \( C \) satisfies WCA, and take \( Y, Z \in \mathcal{X} \) such that \( C(Y) \cap Z \neq \emptyset \) and \( C(Z) \cap Y \neq \emptyset \). Let \( x \in C(Y) \cap Z \) and \( y \in C(Z) \cap Y \). Since \( x \in Z \), \( xV^C y \), and \( y \in C(Z) \), WCA implies \( x \in C(Z) \cap Y \). Since \( x \) was an arbitrary element of \( C(Y) \cap Z \), we have \( C(Y) \cap Z \subseteq C(Z) \cap Y \). The opposite inclusion follows similarly, as required. Now suppose \( C \) satisfies WARP, and take \( x, y \in \mathcal{X} \) and \( x \) such that \( xV^C y \) and \( y \in C(Y) \). Since \( xV^C y \), there exists \( Z \in \mathcal{X} \) such that \( y \in Z \) and \( x \in C(Z) \). Then \( C(Y) \cap Z \neq \emptyset \) and \( C(Z) \cap Y \neq \emptyset \), and WARP then implies \( C(Y) \cap Z = C(Z) \cap Y \), which implies \( x \in C(Y) \), as required.

Obviously, SCA implies WCA and HARP implies WARP.

**Proposition 1.83** If a choice function satisfies SCA or HARP, then it satisfies WCA and WARP.

We say \( C \) satisfies \( \cup \)-decisiveness if, for all \( Y, Z \in \mathcal{X} \) such that \( C(Y) \neq \emptyset \), \( C(Z) \neq \emptyset \), and \( Y \cup Z \in \mathcal{X} \), we have \( C(Y \cup Z) \neq \emptyset \). The next proposition offers two sets of conditions under which WCA is equivalent to SCA, or in other words, WARP is equivalent to HARP.

**Proposition 1.84** Assume (i) \( A \cup \) and \( \cup \)-decisiveness, or (ii) AF and \( C \) is decisive on finite sets. If \( C \) satisfies WCA or WARP, then it satisfies SCA and HARP.

**Proof:** First, assume \( A \cup \) and \( \cup \)-decisiveness. Suppose \( C \) satisfies WCA. Take \( Y \in \mathcal{X} \) and \( x, y \in Y \) with \( xW^C y \) and \( y \in C(Y) \), so there exist \( x_0, x_1, \ldots, x_K \) satisfying

\[
x = x_1V^C x_2 \cdots V^C x_K = y.
\]

For each \( k \leq K - 1 \), there is some \( Y_k \in \mathcal{X} \) such that \( x_{k+1} \in Y_k \) and \( x_k \in C(Y_k) \). By \( A \cup \), \( Y^* = Y_1 \cup \cdots \cup Y_K \in \mathcal{X} \), and by \( \cup \)-decisiveness there is some \( x^* \in C(Y^*) \). Say \( x^* \in Y_{k^*} \). Then \( x_{k^*}V^C x^*, x_{k^*}V^C x^*, x_{k^*} \in Y^* \), and WCA imply \( x_{k^*} \in C(Y^*) \). Then \( x_{k^*-1}V^C x_{k^*}, x_{k^*-1} \in Y^* \), and WCA imply \( x_{k^*-1} \in C(Y^*) \). An induction argument based on these observations shows \( x = x_1 \in C(Y^*) \). Since \( y \in Y^* \), we have \( xV^C y \). Then \( y \in C(Y) \) and WCA imply \( x \in C(Y) \), establishing SCA, and Propositions 1.80 and 1.82 deliver the desired result. Now assume \( C \) satisfies AF and is decisive on finite sets. Suppose \( C \) satisfies WCA. Take \( Y \in \mathcal{X} \) and \( x, y \in Y \) with \( xW^C y \) and \( y \in C(Y) \), so there exist \( x_0, x_1, \ldots, x_K \) satisfying

\[
x = x_1V^C x_2 \cdots V^C x_K = y.
\]

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For each $k \leq K - 1$, there is some $Y_k \in \mathcal{X}$ such that $x_{k+1} \in Y_k$ and $x_k \in C(Y_k)$. By AF, we have $\{x_0, \ldots, x_K\} \in \mathcal{X}$, and decisiveness on finite sets yields $C(\{x_0, \ldots, x_K\}) \neq \emptyset$. Let $x_{k^*} \in C(\{x_0, \ldots, x_K\})$. Since $x_{k^*-1} \in C(\{x_0, \ldots, x_K\})$ as well, and so on. An induction argument based on these observations yields $x = x_0 \in C(\{x_0, \ldots, x_K\})$, which implies $x \in C(Y)$, establishing SCA. Again, the desired result follows from Propositions 1.80 and 1.82. 

To see that $A \cap$ is not sufficient for the result of Proposition 1.84, let $X = \{x, y, z\}$, let $\mathcal{X}$ consist of all pairs and singletons, and define $C(\{x, y\}) = C(\{x\}) = \{x\}$, $C(\{y, z\}) = C(\{y\}) = \{y\}$, and $C(\{x, z\}) = C(\{z\}) = \{z\}$. Here, $A \cap$ is satisfied, but this choice function satisfies WCA and violates SCA.

In Proposition 1.54, we saw that, when $C$ is decisive, WARP implies ACA. In fact, the proof given there holds for an arbitrary domain $\mathcal{X}$ of admissible sets, so the result extends to the environment of this section.

**Proposition 1.85** *If a choice function $C$ satisfies WARP, then it satisfies ACA.*

To see that some kind of decisiveness must be assumed for the latter result, let $X = \{x, y\}$, $\mathcal{X} = \{\{x, y\}, \{x\}, \{y\}\}$, and define $C$ by $C(\{x, y\}) = \{x\}$ and $C(\{x\}) = C(\{y\}) = \emptyset$. This choice function satisfies WARP but not ACA.

**Proposition 1.86** *Assume (i) $A \cap$, or (ii) $A \cup$, or (iii) $A \cup$ and $C$ is $\cup$-decisive. If $C$ satisfies ACA, then it satisfies WARP.*

**Proof:** First, assume $A \cap$, and take any $Y, Z \in \mathcal{X}$ with $C(Y) \cap Z \neq \emptyset$ and $C(Z) \cap Y \neq \emptyset$. Thus, there exist $x \in C(Y) \cap Z$ and $y \in C(Z) \cap Y$. By $A \cap$, $\{x, y\} \in \mathcal{X}$. Then ACA implies $x \in C(Y) \cap \{x, y\} = C(\{x, y\})$ and $y \in C(Z) \cap \{x, y\} = C(\{x, y\})$. Thus, $C(\{x, y\}) = \{x, y\}$, so $x \in C(Y) \cap Y$. Since $x$ was an arbitrary element of $C(Y) \cap Z$, we conclude that $C(Y) \cap Z \subseteq C(Z) \cap Y$. Similarly, $C(Z) \cap Y \subseteq C(Y) \cap Z$, as required. Second, assume $A \cap$, and take any $Y, Z \in \mathcal{X}$ with $C(Y) \cap Z \neq \emptyset$ and $C(Z) \cap Y \neq \emptyset$. So $\emptyset \neq Y \cap Z \in \mathcal{X}$. By ACA, $C(Y) \cap Z = C(Y) \cap (Y \cap Z) = C(Y \cap Z) = C(Y \cap Z) \cap (Y \cap Z) = C(Y \cap Z) \cap Y$, as required. Third, assume $A \cup$ and $C$ is $\cup$-decisive, and take any $Y, Z \in \mathcal{X}$ with $C(Y) \cap Z \neq \emptyset$ and $C(Z) \cap Y \neq \emptyset$. By $A \cup$, $Y \cup Z \in \mathcal{X}$. By $\cup$-decisiveness, $C(Y \cup Z) \neq \emptyset$. Assume without loss of generality that $C(Y \cup Z) \cap Y \neq \emptyset$. Then ACA yields $C(Y) = C(Y \cup Z) \cap Y$. This means that $C(Y) \cap Y \cap Z \neq \emptyset$, and in particular that $C(Y \cup Z) \cap Z \neq \emptyset$. Then ACA yields $C(Z) = C(Y \cup Z) \cap Z$. Therefore,

\[
C(Y) \cap Z = [C(Y \cup Z) \cap Y] \cap Z = [C(Y \cup Z) \cap Z] \cap Y = C(Z) \cap Y,
\]

as required. 

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It is interesting that, unlike SCA and HARP (and therefore WCA and WARP), the W-axiom implies ACA with no background assumptions.

**Proposition 1.87** If a choice function $C$ satisfies the W-axiom, then it satisfies ACA.

**Proof:** Take any $Y, Z \in \mathcal{X}$ such that $Y \subseteq Z$ and $C(Z) \cap Y \neq \emptyset$, and take any $x \in C(Y)$. Let $y \in C(Z) \cap Y$, so that $xV^C y$, which implies $xW^C y$. Take any $z \in Z \setminus C(Y)$. Since $xV^C yV^C z$, we have $xW^C z$. Since $y$ was an arbitrary element of $C(Z)$, we have $x \in D(Y, W^C)$, which implies, by the W-axiom, that $x \in C(Y)$. Therefore, $C(Y) \subseteq C(Z) \cap Y$. Now take $x \in C(Z) \cap Y$. Since $Y \subseteq Z$, we have $xV^C y$ for all $y \in Y$, and $x \in D(Y, W^C) \subseteq C(Y)$ by the W-axiom, as required.

The next figure summarizes some of the implications derived in this section.

**Figure 1.20:** Connections

Proposition 1.54 holds in the general framework of this section, with no restrictions on the collection of admissible sets. In particular, ACA is equivalent with the conjunction of $\alpha$ and $\beta$. 