Vector fields

\[ \mathbf{V}(p) = (v_1(p), v_2(p), v_3(p))_p \], where \( v_i \) are real-valued functions.

\[
\begin{align*}
\mathbf{V}(p) &= v_1(p)(1, 0, 0)_p + v_2(p)(0, 1, 0)_p + v_3(p)(0, 0, 1)_p \\
&= v_1(p)\mathbf{U}_1(p) + v_2(p)\mathbf{U}_2(p) + v_3(p)\mathbf{U}_3(p)
\end{align*}
\]

Claim: If \( \mathbf{V} \) is a vector field on \( \mathbb{R}^3 \), there are three uniquely determined real-valued functions \( v_1, v_2, v_3 \) on \( \mathbb{R}^3 \).

Thus \( \mathbf{V} = \sum v_i \mathbf{U}_i \) where \( \mathbf{U}_i \) are vector fields on \( \mathbb{R}^3 \).

For each point \( p \in \mathbb{R}^3 \) we call them the natural frame fields on \( \mathbb{R}^3 \).

Let \( x = x_1, \ y = x_2, \ z = x_3 \).

If \( \mathbf{V} = x\mathbf{U}_1 - y\mathbf{U}_2 \), then \( \mathbf{V}^I = x\mathbf{U}_1^I - y\mathbf{U}_2^I \) (for \( \mathbf{V} = (x, 0, -z) \) and \( \mathbf{V}(p) = x\mathbf{U}_1 + y\mathbf{U}_2 + z\mathbf{U}_3 \)).

\[ \mathbf{V}^I = x\mathbf{U}_1^I - y\mathbf{U}_2^I \]

But \( \mathbf{U}_i(p) = \left( \frac{\partial}{\partial x_i} \right) \mathbf{r}(p) \), and \( \mathbf{V}(p) \mathbf{I}^j = \frac{\partial x_i}{\partial x_j} \mathbf{r}(p) \) (where \( x^I = (x, y, z) \).

\[
\begin{align*}
\mathbf{V}^I &= \sum u_i \frac{\partial}{\partial x_i} \\
&= \frac{\partial f}{\partial x_i} \\
\end{align*}
\]

Therefore, \( \mathbf{V}^I = 2x_1y - 3y^2z^2 \) and \( \mathbf{V}(p) \mathbf{I}^j = 2p_1^2p_2 - 3(p_2)^2(p_3)^2 \).
A 1-form \( \phi \) on \( \mathbb{R}^2 \) is a real-valued function on the set of all tangent vectors to \( \mathbb{R}^2 \), \( \phi \) is linear at each point.

\[ \phi_p : T_p(\mathbb{R}^2) \rightarrow \mathbb{R} \]

\[ \text{Def: } \text{If } f \text{ is a differentiable real-valued function on } \mathbb{R}^3, \text{ the differential of the 1-form } \omega \]

\[ d\phi (v_p) = \omega (v_p) \]

\[ \text{Indeed real valued from } T_p(\mathbb{R}^3) \rightarrow \mathbb{R} \text{ and linear since } \]

\[ v_p \omega (v_p) = \dddot{v}_i \frac{\partial f}{\partial x^i} \]

Recall that \( v_p \omega (v_p) = \frac{d}{dt} \left( f(p + tv) \right) \)

So \( d\phi (v_p) \) knows all initial rates of change of \( f \) in all directions on \( \mathbb{R}^3 \), so it's not surprising that differentials are fundamental to the calculus on \( \mathbb{R}^3 \).

Lemma: If \( \phi \) is a 1-form on \( \mathbb{R}^2 \), then \( \phi = \sum f_i dx^i \), where \( f_i = \phi(U_i) \).

These functions \( f_1, f_2, f_3 \) are called the Euclidean coordinate functions of \( \phi \).

Proof: \( \phi \) and \( \sum f_i dx^i \) are equal iff \( \phi(v_p) = (\sum f_i dx^i)(v_p) \leftrightarrow v_0 = v_p^3(\mathbb{R}^3) \)

But \( (\sum f_i dx^i)(v_p) = \sum f_i (v_p) dx_i(v_p) \)

But \( dx_i(v_p) = v_0(x_i) = \frac{v_i}{\frac{25}{9}} \frac{d}{ds} e_0 |_{s} = v_i \) (independent of \( p \)).

Giving \( (\sum f_i dx^i)(v_p) = \sum v_i \phi(U_i(p)) \)

On the other hand, \( \phi(v_p) = \phi(\sum v_i U_i(p)) = \sum v_i \phi(U_i(p)) = \sum v_i \phi(U_i(p)) \)

\( \square \)
Corollary: If \( f \) is a differentiable function on \( \mathbb{R}^2 \), then
\[
\sum_{i} \frac{\partial f}{\partial x_i} \frac{df}{dx_i}
\]
Proof: \( df(\nu) = \nu_i \frac{df}{dx_i} \)
\[
\left( \sum_{i} \frac{\partial f}{\partial x_i} \frac{df}{dx_i} \right)(\nu) = \sum_{i} \frac{\partial f}{\partial x_i} \nu_i \frac{df}{dx_i} \]
\[
\frac{df(\nu)}{\nu} = \sum_{i} \frac{\partial f}{\partial x_i} \frac{df}{dx_i} \]
\[
\text{Ex: If } \phi \text{ is a 1-form, find } \phi(V) \text{ for any vector field } V.
\Rightarrow \phi = \sum f_i dx_i; \text{ where } f_i = \phi(U_i)
\]
\[
V = \sum v_i U_i
\]
Since \( \phi(V) \) is linear, i.e., \( \phi(gV + gW) = g\phi(V) + g\phi(W) \)
\[
\phi(V) = \phi(\sum v_i U_i) = \sum \phi(v_i U_i) = \sum v_i \phi(U_i) = \sum v_i f_i
\]
Ex: Evaluate the 1-form \( \phi = x^2 dx - y^3 dy \) on the vector field
\[
V = xy(U_2 - U_3) + yz(U_1 - U_2)
\]
\[
\Rightarrow V = \left( \frac{xy + yz}{v_z} 
\right) U_1 + \left( -y^2 \right) U_2 + \left( -xy \right) U_3
\]
\[
\phi = \frac{x^2 dx - y^3 dy}{v_z}
\]
\[
\Rightarrow \phi(V) = \left( \frac{xy + yz}{v_z} \right) x^2 + (-y^2) 0 + (-xy) (1 - y^2)
\]
\[
= x^2 y + x^2 y^2 + xy^2
\]
Mappings

Def. Given a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$, let $f_1, f_2, \ldots, f_m$ denote the real-valued functions on $\mathbb{R}^n$ of $F$.

$$F(\vec{p}) = (f_1(\vec{p}), f_2(\vec{p}), \ldots, f_m(\vec{p})) \quad \forall \vec{p} \in \mathbb{R}^n$$

We write $F = (f_1, f_2, \ldots, f_m)$.

The function $F$ is differentiable provided its coordinate functions are differentiable. A differentiable function $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Ex. $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\vec{p} \mapsto F(\vec{p}) = (x - y, x + y, 2z)$

Then $F(\vec{p}) = (x(\vec{p}) - y(\vec{p}), x(\vec{p}) + y(\vec{p}), 2z(\vec{p}))$, where $\vec{p} = (p_1, p_2, p_3)$

$$= (p_1 - p_2, p_1 + p_2, 2p_3)$$

Def. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a mapping. $\vec{v}$ is a tangent vector to $\mathbb{R}^n$ at $\vec{p}$ (basically $\vec{v} \in T_p(\mathbb{R}^n)$), let $F_\vec{v}(\vec{p})$ be the initial velocity of the curve $\vec{p}(t) = F(\vec{p} + t\vec{v})$.

Then $F_\vec{v}: T_p(\mathbb{R}^n) \rightarrow \mathbb{T}_{F(\vec{p})}(\mathbb{R}^m)$ is called the tangent map of $F$.

Proposition. Let $F = (f_1, f_2, \ldots, f_m)$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $\vec{v} \in T_p(\mathbb{R}^n)$, then

$$F_\vec{v}(\vec{p}) = (f_1(\vec{p} + t\vec{v}), f_2(\vec{p} + t\vec{v}), \ldots, f_m(\vec{p} + t\vec{v}))$$

(Chain rule: $\vec{p}(t) = F(\vec{p} + t\vec{v}) = (f_1(\vec{p} + t\vec{v}), \ldots, f_m(\vec{p} + t\vec{v}))$)

Then $\vec{p}'(t) = (\frac{df_1}{dt}(\vec{p} + t\vec{v}), \ldots, \frac{df_m}{dt}(\vec{p} + t\vec{v}))|_{t=0}$.

$F_\vec{v}(\vec{p})$ is the initial velocity of the curve $\vec{p}(t)$

so $F_\vec{v}(\vec{p}) = \vec{p}'(0) = (\frac{df_1}{dt}(\vec{p} + t\vec{v})|_{t=0}, \ldots, \frac{df_m}{dt}(\vec{p} + t\vec{v})|_{t=0}) \circ \vec{0}$.
Main idea behind the push forward is to study how a map acts on tangent vectors.

Mappings preserve velocities of curves.

Corollary: Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\beta = F(\alpha)$ is the image of a curve $\alpha$ in $\mathbb{R}^n$, then

$$\beta' = F_\ast (\alpha')$$

Proof: $\alpha'(t) = \left( \frac{d\alpha_1(t)}{dt}, \ldots, \frac{d\alpha_n(t)}{dt} \right)$

Then $F_\ast (\alpha'(t)) = \left( \alpha_1'(t) [f_1], \ldots, \alpha_n'(t) [f_m] \right) = F(\alpha(t))$

So $F_\ast (\alpha'(t)) = \left( \frac{d}{dt} F(\alpha_1(t)), \ldots, \frac{d}{dt} F(\alpha_n(t)) \right)$

Since $\beta(t) = F(\alpha(t)) = \left( F_1(\alpha(t)), \ldots, F_m(\alpha(t)) \right)$

$$\beta' = \left( \sum_{i=1}^n \frac{d\alpha_i(t)}{dt} \frac{d}{dt} F_i(\alpha(t)), \ldots, \frac{d}{dt} F_m(\alpha(t)) \right) = \frac{d}{dt} F(\alpha(t)) \cdot \alpha'(t) \quad \text{QED}$$

Let $\{ U_i \}$ for $1 \leq i \leq n$ and $3 \overline{U_i}$ be the natural fields of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Corollary: If $F = (f_1, \ldots, f_m)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$F_\ast (U_i(p)) = \sum_{i=1}^m \frac{d}{d\alpha_i} F_i(p) \cdot U_i(p)$$
Proof: We know \( F_p(\vec{v}_p) = (v_p^1, \ldots, v_p^m) \in F_p \)

\[
F_p(U_j(p)) = (U_j(p)^1, \ldots, U_j(p)^m) \in F_p
\]

\[
= \left( \frac{\partial f_1}{\partial x_j}(p), \ldots, \frac{\partial f_m}{\partial x_j}(p) \right) F_p
\]

Since \( U_j(p)^i = \frac{\partial f_i}{\partial x_j}(p) \)

But \( U_i(F(p)) = (0, \ldots, 0, 1, 0, \ldots, 0) \)

in plane

so \( F_p(U_j(p)) = \sum_{i=1}^m U_i(F(p)) \frac{\partial f_i}{\partial x_j}(p) \)

Since we can think of \( U_j \) as \( \frac{\partial}{\partial x_j} \) in the \( T_p(M^n) \) space

and \( U_i \) as \( \frac{\partial}{\partial y_i} \) in the \( T_q(M^n) \) space

we have exactly what the push forward says.

Since then we have

\[
F_p \left( \frac{\partial}{\partial x_j}(p) \right) = \sum_{i=1}^m \left( \frac{\partial}{\partial y_i} F(p) \right) \frac{\partial f_i}{\partial x_j}(p)
\]

Jacobian get out from this

To equivalent
Example: \( F(x_1, x_2) = \left( e^{x_1 + x_2}, \sin(x_2), \cos(x_2) \right) \) is a mapping from \( \mathbb{R}^2 \to \mathbb{R}^3 \), with Cartesian coordinate \((y_1, y_2, y_3)\) for the image. Find what \( \frac{\partial}{\partial x_1} \bigg|_{\hat{p}} \) and what \( \frac{\partial}{\partial x_2} \bigg|_{\hat{p}} \) → where \( \hat{p} = (p_1, p_2) \)

Solution: \( \frac{\partial}{\partial x_1} \bigg|_{\hat{p}} = \frac{\partial}{\partial x_1} \left( e^{x_1 + x_2}, \sin(x_2), \cos(x_2) \right) \bigg|_{\hat{p}} = e^{x_1 + x_2} \left( \frac{\partial}{\partial x_1} e^{x_1 + x_2} \right) + \cos(x_2) \left( \frac{\partial}{\partial x_1} \sin(x_2) \right) + \cos(x_2) \left( \frac{\partial}{\partial x_1} \cos(x_2) \right) \bigg|_{\hat{p}} = e^{x_1 + x_2} \left( \frac{\partial}{\partial x_1} e^{x_1 + x_2} \right) + \cos(x_2) \left( \frac{\partial}{\partial x_1} \sin(x_2) \right) + \cos(x_2) \left( \frac{\partial}{\partial x_1} \cos(x_2) \right) \bigg|_{\hat{p}} \)

So \( \frac{\partial}{\partial x_1} \bigg|_{\hat{p}} \) → \( e^{x_1 + x_2} \left( \frac{\partial}{\partial x_1} e^{x_1 + x_2} \right) + \cos(x_2) \left( \frac{\partial}{\partial x_1} \sin(x_2) \right) + \cos(x_2) \left( \frac{\partial}{\partial x_1} \cos(x_2) \right) \bigg|_{\hat{p}} \)

where \( F(\hat{p}) = (e^{x_1 + x_2}, \sin(x_2), \cos(x_2)) \)

Example: \( F(r, \theta) = \left( r \cos(\theta), r \sin(\theta) \right) \) with Cartesian coordinate \((y_1, y_2)\) for the image.

Find \( \frac{\partial}{\partial r} \bigg|_{\hat{p}} \) → \( \frac{\partial}{\partial \theta} \bigg|_{\hat{p}} \) → where \( \hat{p} = (r, \theta) \)

Solution: \( \frac{\partial}{\partial r} \bigg|_{\hat{p}} = \frac{\partial}{\partial r} \left( r \cos(\theta), r \sin(\theta) \right) \bigg|_{\hat{p}} = \cos(\theta) \left( \frac{\partial}{\partial r} r \cos(\theta) \right) + \sin(\theta) \left( \frac{\partial}{\partial r} r \sin(\theta) \right) \bigg|_{\hat{p}} = \cos(\theta) \left( \frac{\partial}{\partial r} r \cos(\theta) \right) + \sin(\theta) \left( \frac{\partial}{\partial r} r \sin(\theta) \right) \bigg|_{\hat{p}} \)

So \( \frac{\partial}{\partial r} \bigg|_{\hat{p}} \) → \( \cos(\theta) \left( \frac{\partial}{\partial r} r \cos(\theta) \right) + \sin(\theta) \left( \frac{\partial}{\partial r} r \sin(\theta) \right) \bigg|_{\hat{p}} \)

where \( F(\hat{p}) = (r \cos(\theta), r \sin(\theta)) \)
Just as the derivative of a function is used to gain information about the function, the tangent map $F_p$ can be used in the study of a mapping $F$.

**Def.:** A mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ is regular provided that at every point $p$ of $\mathbb{R}^n$ the tangent map $F_p$ is one-to-one.

Since tangent maps are linear transformations, standard results of linear algebra show that the following conditions are equivalent:

1. $F_p$ is one-to-one
2. $F_p(0) = 0 \Rightarrow F_p(0) = 0$.
3. The Jacobian matrix of $F$ at $p$ has rank $n$, the dimension of $\mathbb{R}^n$ of $F$.

The following noteworthy property of linear transformations $T: V \to W$ will be useful in dealing with tangent maps. If the vector spaces $V$ and $W$ have the same dimension, then $T$ is one-to-one and onto if it is onto, so either property is equivalent to $T$ being a linear isomorphism.

A mapping that has a differentiable inverse mapping is called a diffeomorphism. The results in this section all remain valid when Euclidean spaces $\mathbb{R}^n$ are replaced by open sets of Euclidean spaces, so we can speak of a diffeomorphism from one set to another. A fundamental result of advanced calculus:

**Theorem:** Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a mapping between Euclidean spaces of the same dimension. If $F_p$ is one-to-one at a point $p$, there is an open set $U$ containing $p$ such that $F$ restricted to $U$ is a diffeomorphism of $U$ onto an open set $V$.

This is called the inverse function theorem, since it says that the restricted mapping $U \to V$ has a differentiable mapping $V \to U$. 
Ex: (a) Give an example to demonstrate that a one-to-one and onto mapping need not be a diffeomorphism.

\[ f : \mathbb{R} \to \mathbb{R} \to f(x) = x^3, \quad \lim_{\to 0} f^{-1}(x) = 3^{x/3} = g(x) \]

\[ g'(x) = \frac{1}{3} (x^{-\frac{2}{3}}) \text{ so not everywhere differentiable on } \mathbb{R} \ (x < 0) \]

(b) Prove that if a one-to-one and onto mapping \( F : \mathbb{R}^n \to \mathbb{R}^n \) is regular, then it is a diffeomorphism.

For \( F \) to be a diffeomorphism, we need \( F \) to be bijective and its inverse differentiable.

\( F \) is a bijection by assumption and is regular so \( F'(x) \to V \times EIR^n \).

By the inverse function theorem, for \( b = F(a) \), \( b \) and \( a \in \mathbb{E}IR^n \), then

\[ (F^{-1})'(b) = \frac{1}{F'(a)} \quad \text{since } (F^{-1})'(b) = \frac{1}{F'(F^{-1}(b))} \]

This is clearly well-defined, since \( F \) is regular. Thus \( F^{-1} \) is a diffeomorphism.
Gravitation is a manifestation of spacetime curvature, and that curvature shows up in the deviation of one geodesic from a nearby geodesic ("relative acceleration of test particles"). The fundamental aspect of GR for us is differential geometry is extremely useful.

You can compute the "separation" and the rate of change of "separation," of two geodesics in curved "geometry".

"Separation" between geodesics will mean "vector". But the concept of vector as employed in flat Lorentz spacetime (a bi-local object: point for head and point for tail) must be stippled up into the local concept of tangent vector, when one passes to curved spacetime. It also reveals how the passage to curved spacetime affects 1-forms and tensors.

Where a geodesic is the curve of shortest length in the given spatial geometry that passes through two given points.

Then talk about lecture plan

I. 1 person $\rightarrow$ 2h (1 chapter)

II. 2 person $\rightarrow$ 1h each (1 chapter)

III. 3 person $\rightarrow$ 2h (1\frac{1}{2} chapter)