The GHJW Theorem

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Abstract

This paper illustrates the Schrödinger–HJW theorem (GHJW Theorem) in the field of Quantum Information Theory, which is a result about the mixed-state quantum system as a collection of pure states, as well as the relationship between the density operator purification. It expresses the phenomenon of quantum eraser in a generalized form. The presentation of the specific calculation process is based on the Section 2.5 of "Lecture Notes for Physics 229, Quantum Information and Computation", by John Preskill [1].

1. Preparing pure states in system A with measurement on system B

Consider a density matrix ρ_A can be realized as the following, which is one of the unlimited numbers of different ways realizing a mixed state of any quantum system as an ensemble of pure states,

$$\rho_A = \sum_i p_i \, |\phi_i\rangle_{A\,A}\langle\phi_i|, \quad \sum p_i = 1. \tag{1}$$

where the states $\{|\phi_i\rangle_A\}$ are normalized vectors, but not mutually orthogonal, and each pure state $|\phi_i\rangle_{A\,A}\langle\phi_i|$ occurs with the probability p_i. Then, we can construct a purification of ρ_A, which is a bipartite pure state $|\phi_1\rangle_{AB}$ yielding ρ_A during the partial trace over the space H_B, with the form:

$$|\phi_1\rangle_{AB} = \sum_i \sqrt{p_i} |\Phi_i\rangle_A |\alpha_i\rangle_B \tag{2}$$

where the vectors $|\alpha_i\rangle_B$ that belongs to the space H_B are mutually orthogonal and normalized. Therefore, tracing on the $|\phi_1\rangle_{AB}$, we have:

$$tr_B(\,|\phi_1\rangle_{AB\,AB}\langle\phi_1|) = \rho_A \tag{3}$$

Then, performing an orthogonal measurement in System B with the projection onto the $|\alpha_i\rangle_B$ basis, the outcome would occur with the probability p_i and would prepare the pure state $|\phi_i\rangle_{A\,A}\langle\phi_i|$ of system A. Therefore, with the purification of $|\phi_1\rangle_{AB}$ of ρ_A, we have a measurement performing in system B that realizes the states ensemble interpretation of ρ_A in system A. In addition, if the measurement in system B is known, one of the pure states $|\phi_i\rangle_A$ from the mixture ensemble ρ_A will be extracted.

2. Second Type of Purification

The process above is a situation of preparing $|\uparrow_Z\rangle_A$ by the measurement of spin B along the z-axis, under the conditions of two entangled qubits. However, to generalize the concept of quantum eraser, a realization of different ensemble interpretation of ρ_A is needed by performing another measurement of B, in the state $|\phi_1\rangle_{AB}$. Therefore, we let ρ_A be:

$$\rho_A = \sum_\mu q_\mu |\psi_\mu\rangle_{AA}\langle\psi_\mu| \qquad (4)$$

where the density matrix ρ_A is the same one with the previous case. For this ensemble, there is a different type of corresponding purification:

$$|\phi_2\rangle_{AB} = \sum_\mu \sqrt{q_\mu} |\psi_\mu\rangle_A \otimes |\beta_\mu\rangle_B \qquad (5)$$

where the $|\beta_\mu\rangle_B$ are orthonormal vectors in H_B. Similarly, in the state $|\phi_2\rangle_{AB}$, performing the measurement in H_B that projecting onto the $|\beta_\mu\rangle_B$ basis would give a realization of the ensemble.

3. Relation between Two Purification

The two states $|\phi_1\rangle_{AB}$ and $|\phi_2\rangle_{AB}$ can be calculated interchangeably by a unitary change of basis acting in H_B alone:

$$|\phi_1\rangle_{AB} = (1_A \otimes U_B)|\phi_2\rangle_{AB} \qquad (6)$$

or combining with Eq. (5):

$$|\phi_1\rangle_{AB} = \sum_\mu \sqrt{q_\mu} |\psi_\mu\rangle_A |\gamma_\mu\rangle_B$$

where

$$|\gamma_\mu\rangle_B = U_B |\beta_\mu\rangle_B \qquad (7)$$

Obviously, $|\gamma_\mu\rangle_B$ is another orthonormal basis of H_B.

Therefore, by Eq.(7) and Eq.(2), we could choose the appropriate measurement of observable in System B to realize either the $|\Phi_i\rangle_A$ ensemble or the $|\psi_\mu\rangle_A$

ensemble with the purification $|\phi_1\rangle_{AB}$ of ρ_A.

To give a further generalization, consider a number of ensembles that all realize the density matrix ρ_A, where the maximum number of pure states inside each any of the ensembles is n. Then we could simply choose a Hilbert Space H_B with dimension n, and a pure state $|\phi_n\rangle_{AB} \in H_A \otimes H_B$, where any one of the ensembles $|\Phi_n\rangle_A$ can be realized by measuring a proper observable B. This is the GHJW theorem, which expressed the quantum eraser phenomenon in the most general form.

4. Relation between the GHJW Theorem and the Schmidt decomposition

The GHJW Theorem, actually, is a trivial corollary to the Schmidt decomposition. Both the state $|\phi_1\rangle_{AB}$ and $|\phi_2\rangle_{AB}$ have a Schmidt decomposition in the following form:

$$|\phi_1\rangle_{AB} = \sum_k \sqrt{\lambda_k}|k\rangle_A\,|k_1'\rangle_B$$

$$|\phi_2\rangle_{AB} = \sum_k \sqrt{\lambda_k}|k\rangle_A\,|k_2'\rangle_B \qquad (8)$$

because of the fact that both two states yield the same density matrix when taking the partial trace over B. The $\sqrt{\lambda_k}$ are simply eigenvalues of ρ_A and the $|k\rangle_A$ are the corresponding eigenvectors.

Since $|k_1'\rangle_B$ and $|k_2'\rangle_B$ are both the orthonormal basis for the space H_B, from the Eq. (6), we have $|k_1'\rangle_B = U_B|k_2'\rangle_B$.

5. Effects of GHJW Theorem

In the ensemble of pure states described by Eq. (1), an observer in the system A cannot detect the relative phases of these states, which means that the pure states $|\phi_i\rangle_A$ are superposed incoherently. These states cannot interfere, because by performing a measurement in System B, a projection onto the orthonormal basis $|\alpha_i\rangle_B$, it is theoretically feasible to determine which ensemble representation is actually realized. Meanwhile, using the method of Eq. (7) that projects onto the $|\gamma_\mu\rangle_B$ basis, we can extract one of the pure states $|\psi_\mu\rangle_A$ from the ensemble by transmitting information about the measurement result to System A, even if the state is a coherent superposition of the $|\phi_i\rangle_A$'s. Therefore, measuring B in $|\gamma_\mu\rangle_B$ basis creates an effect of erasing the uncertain information that whether the state of A is $|\phi_i\rangle_A$ or $|\phi_j\rangle_A$. Thus, the GHJW theorem characterizes the general quantum eraser. The message sent by this phenomenon is that the information obtained by measuring System B modifies the physical description of a state of A when it is associated to A – information is physical.

6. Conclusion

In this paper, we demonstrate the validity of the GHJW Theorem and its relation with the quantum eraser phenomenon. We have seen that by performing different measurements on individual qubits of a system in a bipartite pure state $|\varphi\rangle_{AB}$, the decompositions of an ensemble in a single system can be realized that differ but are described by the same statistical operator. It points out the fundamental nature of mixed states and describes the effect of the choice of the measurement basis. An important result from this theorem is that through local operations and classical communication, any finite ensemble of bipartite quantum states can be remotely prepared by two agents in distant laboratories (space).

7. References

[1]. *Lecture Notes for Physics 229, Quantum Information and Computation*, by John Preskill (1998).