Faster-than-light Communication Cannot be Achieved Via Quantum Entanglement

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December 15, 2021

Abstract

This paper explains why quantum entanglement does not provide a way of faster-than-light communication. To discuss the reason, the concepts of convex set and extremal point are introduced. Then it follows that density matrices form a convex set. This means that different ways of preparation could lead to the same mixed state. Hence, although via entanglement, one can immediately prepare a far away particle in different ways, that the prepared state is the same means that it is impossible to read the information.

1 Introduction

Quantum entanglement is one of the most fascinating and counterintuitive phenomena in quantum mechanics. If there is a pair of entangled particles, measuring the state of one of them, you would know the state of the other particle immediately, even if these two particles are on two opposite sides of the universe. This leads to some misconceptions in the public that we can send information faster than light using entangled particles. For example, in the science fiction *The Three Body Problem*, the aliens are able to establish real time communication with people on the Earth from 4 light years away via entangled particles. However, this is impossible. This paper will explain why quantum entanglement does not allow faster-than-light communication. The reason is the ambiguity in the preparation of mixed states. To illustrate this ambiguity, I will first define two useful mathematical concepts: convex set and extremal points.

2 Convex Set and Extremal Point

First, we define convex set. A subset S of a vector space is said to be convex if the set contains the straight line segment connecting any two points in the set. That is, for any $s_1 \in S$ and $s_2 \in S$, if $s(\lambda) = \lambda s_1 + (1-\lambda)s_2$ are in S for any $0 \leq \lambda \leq 1$, then S is convex. For example, if we consider the x-y plane as shown in fig. 1, the set S is a convex set, while the set T is not a convex set, because the line L is not contained in the set.

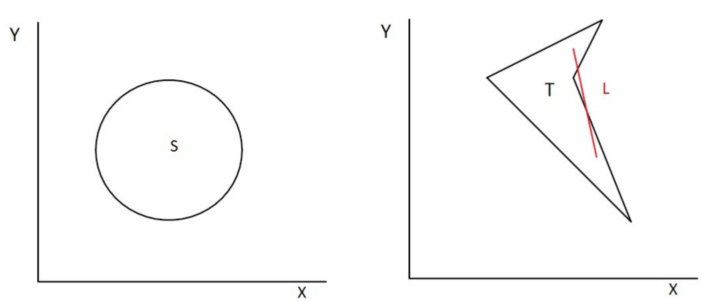

Figure 1: The set S is a convex set, but the set T is not.

Then we define extremal point. A point in a convex set S is an extremal point if it does not lie in any open line segment joining two distinct points in S. That is, the point s is called an extremal point if there do not exist two points s_1 and s_2 in S such that $s = \lambda s_1 + (1-\lambda)s_2$ for any $0 < \lambda < 1$. Consider two sets in the x-y plane again, as shown in fig. 2, the extremal points of S are all the points on its edge. The extremal points of R are a, b, and c.

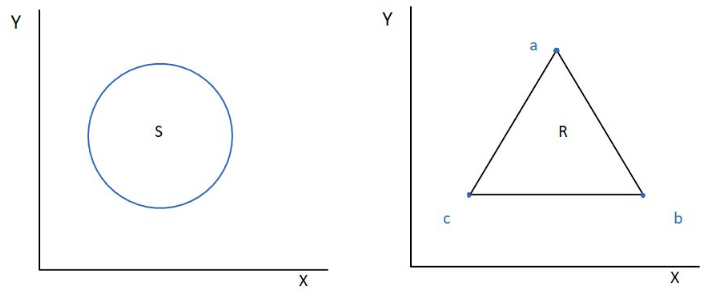

Figure 2: Examples of extremal points

Now let's go back to physics. In quantum mechanics, if a matrix ρ satisfies 1)ρ is self-adjoint, 2)ρ is positive, i.e., $\langle\psi|\rho|\psi\rangle \geq 0$ for any $|\psi\rangle$, and 3)$tr(\rho) = 1$, then ρ is called a density matrix and represents the state of a system. To show that density matrices form a convex set, we can pick two density matrices ρ_1 and ρ_2, then check if their convex sum $\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2$ satisfies the three properties. It is easy to see that $\rho(\lambda)$ satisfies 1) and 3). To check 2), we evaluate:

$$\langle\psi|\rho(\lambda)|\psi\rangle = \lambda\langle\psi|\rho_1|\psi\rangle + (1-\lambda)\langle\psi|\rho_2|\psi\rangle \geq 0 \tag{1}$$

This inequation holds for any $|\psi\rangle$. Hence, density matrices form a convex set.

3 Ambiguity in the Preparation of Mixed State

Forming a convex set means that if a density matrix ρ is not an extremal point, it can be expressed as a convex sum of two other density matrices $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$. Interpreting this physically, it means that if we prepare the state ρ_1 with classical probability λ (the type of probability produced by toasting a coin or by a random number generator), and ρ_2 with classical probability $1-\lambda$, then we get the state ρ. This is called the ambiguity in preparation. But what are the extremal points? In fact, we can see from the physical interpretation that the extremal points are the states that can't be prepared as the classical probabilistic combination of two other states. These are pure states, since the pure state $\rho = |\psi\rangle\langle\psi|$ is the only state that will guarantee the outcome 1 if we measure the projection $E = |\psi\rangle\langle\psi|$.

Let's consider an example. In the two dimensional Hilbert space, a density matrix can be thought of as a vector in the Bloch ball. In fact, there is a one to one correspondence between a vector $\vec{P}$ with norm ≤ 1 in $\mathbb{R}^3$ and a density matrix $\rho(\vec{P})$ given by

$$\rho(\vec{P}) = \frac{1}{2}(\mathbf{I} + \vec{P}\cdot\vec{\sigma}) \tag{2}$$

We can see that if we pick any two states in the Bloch ball, the line connecting them is in the ball, and the pure states, which are on the surface of the ball, are not contained in any open line segment.

Now consider the density matrix $\rho = \frac{1}{2}\mathbf{I}$. It is right at the center of the Bloch ball, since $\frac{1}{2}\mathbf{I} = \frac{1}{2}(\mathbf{I} + 0\cdot\vec{\sigma})$. Lines that go through this point also go through two antipodal points on the surface of the ball.

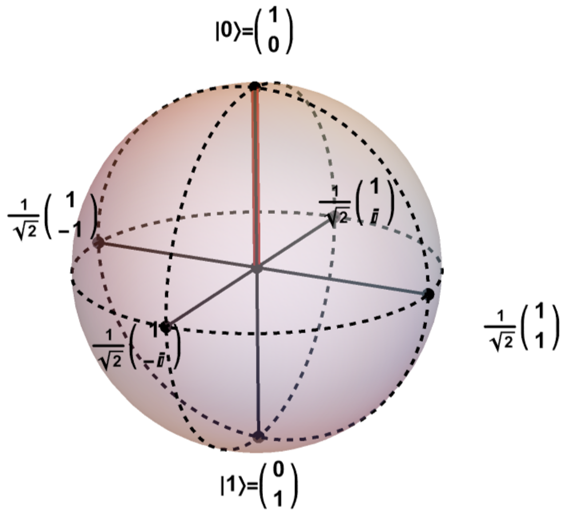

Figure 3: ρ is in the center of the Bloch ball. Lines that go through it also go through two points that are exactly opposite of each other on the surface of the ball

Hence, ρ is the convex sum of any two orthogonal pure states. For example,

$$\rho = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x| \tag{3}$$

Also,

$$\rho = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z| \tag{4}$$

We will see this ambiguity of ρ comes in handy in the explanation of why faster-than-light communication is not allowed.

4 No Faster-Than-Light Communication

The misconception of faster-than-light communication starts with a pair of entangled particles A and B. Suppose their states is the bipartite pure state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow_x\rangle_A|\uparrow_x\rangle_B + |\uparrow_x\rangle_A|\uparrow_x\rangle_B) \tag{5}$$

By measuring B in the $\{|\uparrow_x\rangle_B, |\downarrow_x\rangle_B\}$ basis, we have $\frac{1}{2}$ probability of getting the state $|\uparrow_x\rangle_A$ and $\frac{1}{2}$ probability of getting the state $|\downarrow_x\rangle_A$, so the state of A is prepared as

$$\rho_{A0} = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x| \tag{6}$$

Moreover, since $\rho_A 0$ has degenerate nonzero eigenvalues, we can apply unitary transformations on $\mathcal{H}_A$ and $\mathcal{H}_B$. These transformations preserve $|\psi\rangle_{AB}$. Hence, $|\psi\rangle_{AB}$ could also be expressed as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_A|\uparrow_z\rangle_B + |\uparrow_z\rangle_A|\uparrow_z\rangle_B) \tag{7}$$

Similarly, by measuring B in the $\{|\uparrow_z\rangle_B, |\downarrow_z\rangle_B\}$ basis, we also have $\frac{1}{2}$ probability of getting the state $|\uparrow_z\rangle_A$ and $\frac{1}{2}$ probability of getting the state $|\downarrow_z\rangle_A$, which means the density matrix of A is

$$\rho_{A1} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z| \tag{8}$$

One might come up with the following scenario of faster-than-light communication. If we prepare a many pairs of entangled particles with state $|\psi\rangle_{AB}$, then Bob takes the B particles to another galaxy, and Alice stays on the Earth with the A particles. If Bob wants to send some information to Alice, he could measure his particles along either the x or the z axis one by one. This will correspondingly prepare the spin of Alice's particles along either x or z axis. If they agree that spin along the x axis stands for 0, and spin along the z axis stands for 1, then Alice could check the spin of her particles to read the information.

However, as shown before, ρ_{A0} and ρ_{A1} are the same state. Although their preparations are different, Alice will not be able to distinguish between them no matter how she observes.

5 Summary

In this paper we introduce the concepts of convex set and extremal point. Then we show that density matrices form a convex set, which means that different ways of preparation could still lead to the same state. This is exactly what happens in the conceived situation, when Bob wants to encode his information in the way he prepares Alice's particles, but he ends up with preparing the same state, so it is impossible to read the information.

References

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[2] Wolfram Demonstrations Project. Qubits on the Poincaré (Bloch) Sphere.