CORRESPONDENCE BETWEEN LIE GROUPS AND LIE ALGEBRAS

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Abstract. We will study the Lie groups and Lie algebras, and build up to establish the Lie group-Lie algebra correspondence. Along the way we will introduce and explore some interesting and important concepts such as coverings of Lie groups and the exponential map.

Contents

1. A Review of Smooth Manifolds

We first review some of the main concepts in differential topology. For readers unfamiliar with these definitions and theorems, [5] is a self-contained textbook and strongly recommended to read. For concepts relevant to *flows*, I recommend [4] instead.

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1.1. **Some Preparations.** A real-valued function $f : U \to \mathbb{R}$ defined on an open subset of $\mathbb{R}^n$ is **smooth** or C^∞ at a point p if its partial derivatives of all orders exists at p. Similarly, a function $f : U \to \mathbb{R}^m, f = (f_1, ..., f_m)$ defined on an open subset of $\mathbb{R}^n$ is **smooth** or C^∞ at a point p if each real-valued f_i is smooth at p. We say f is smooth on U if it is smooth at any point in U.

Fix $p \in \mathbb{R}^n$. We define S_p to be the set of all pairs (f, U) where U is a neighborhood[1] of p and $f : U \to \mathbb{R}$ a smooth function defined on U. We define an equivalence relation on S_p by $(f, U) \sim (g, V)$ iff f, g agree on some neighborhood of p, i.e. $f|_W = g|_W$ for some neighborhood $W \subset U \cap V$ of p. The family of equivalence classes, denoted $C_p^\infty := S_p/\sim$, is called **the germs of C^∞ functions at** p. Conventionally, we just say $f \in C_p^\infty$.

For any vector $v \in \mathbb{R}^n$, let D_v denote the **directional derivative at** p, so for $f \in C_p^\infty$, $D_v f = v \cdot (Df)_p$ where $(Df)_p = (\frac{\partial f}{\partial x^1}|_p, ..., \frac{\partial f}{\partial x^n}|_p)$[2]. In particular, note $D_v(fg) = g(p)(D_v f) + f(p)(D_v g)$. This is called the **Leibniz rule**. We say a linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the **Leibniz rule** is a **derivation at** p or a **point-derivation** of C_p^∞. We define the **tangent space of $\mathbb{R}^n$ at** p to be the set of all derivations at p, denoted $T_p\mathbb{R}^n$. We have seen that each directional derivative D_v at p is a derivation at p. Hence there is a natural linear map

$$\phi : \mathbb{R}^n \to T_p\mathbb{R}^n, v \mapsto D_v$$

Theorem 1.1. *ϕ is a linear isomorphism.*

It follows that the directional derivatives D_{e_i} form a basis for $T_p\mathbb{R}^n$, where e_i are the standard basis for $\mathbb{R}^n$. Note $D_{e_i} = \frac{\partial}{\partial x^i}|_p$, so $v = \sum v^i e_i \in \mathbb{R}^n$ corresponds to $\sum v^i \frac{\partial}{\partial x^i}|_p$.

1.2. **Manifolds.** A space M is a **topological manifold** if it is Hausdorff, second countable and locally Euclidean, where locally Euclidean means, for some $n \in \mathbb{N}$, every point $p \in M$ has a neighborhood U s.t. there is a homeomorphism ϕ from U onto some open subset of $\mathbb{R}^n$. The integer n is the **dimension** of M. We call the pair (U, ϕ) a **chart**, U a **coordinate neighborhood**, and ϕ a **coordinate map or system** on U.

Two charts $(U, \phi), (V, \psi)$ are **C^∞-compatible** (or just compatible) if the two maps $\phi \circ \psi^{-1}, \psi \circ \phi^{-1}$ are smooth at points where they are defined. An **atlas** on a topological manifold M is a collection of pairwise compatible charts that cover M. An atlas $\mathcal{U}$ on M is always contained in a unique maximal atlas, namely the set of all charts that are compatible with every chart in $\mathcal{U}$. A **smooth manifold** is a topological manifold M with a maximal atlas. The maximal atlas is also called a **differentiable structure** on M. From now on, by a "manifold" we always mean a smooth manifold, and by a "chart" on a manifold we always mean a chart in the differentiable structure.

Example 1.2. The real numbers $\mathbb{R}$ has a natural global coordinate, namely the identity map. Note that the map $f : x \mapsto x^3$ is not compatible with the identity map, as $f^{-1} : x \mapsto x^{1/3}$ is not smooth at $p = 0$. But f is also a global coordinate on $\mathbb{R}$. Hence f and the identity map generate two different differentiable structures.

[1]By "a neighborhood of p" we mean an open subset containing p.

[2]Note here superscripts are used instead of subscrips for x^i. This is very common in differential topology.

Example 1.3. If M, N are manifolds, then $M \times N$ is also a manifold, with atlas $\{(U_i \times V_j, \phi_i \times \psi_j)\}$ where $\{(U_i, \phi_i)\}, \{(V_j, \psi_j)\}$ are atlases of M, N.

A Lie group is a manifold G with a group structure s.t. the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are both *smooth*.

Let M, N be smooth manifolds. A function $f : N \to M$ is **smooth** if $\psi \circ f \circ \phi^{-1}$ is smooth for any chart (U, ϕ) on N and any chart (V, ψ) on M. It follows that charts on an n-dimensional manifold M are diffeomorphisms (between open subsets of M and $\mathbb{R}^n$), and diffeomorphisms (between open subsets of M and $\mathbb{R}^n$) are also charts.

Given a chart (U, ϕ) on M of dimension n, usually we write $\phi = (x^1, ..., x^n)$. The standard coordinates on $\mathbb{R}^n$ is denoted by $(r^1, ..., r^n)$, so $x^i = r^i \circ \phi$. Let $f : M \to \mathbb{R}$ be a smooth function. For $p \in U$, we define the **partial derivative** $\frac{\partial f}{\partial x^i}$ of f w.r.t. x^i to be

$$\frac{\partial f}{\partial x^i}|_p := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}|_{\phi(p)}$$

Let $F : N \to M$ be a smooth map and $(U, \phi = x^1, ..., x^n), (V, \psi = y^1, ..., y^m)$ be charts on N, M respectively. We denote by $F^i := y^i \circ F$ the i-th component of F in the chart (V, ψ). Then the matrix

$$[\frac{\partial F^i}{\partial x^j}]$$

is called the **Jacobian matrix** of F relative to the charts $(U, \phi), (V, \psi)$. If M, N have the same dimension, the determinant of the Jacobian matrix is the **Jacobian determinant**.

Theorem 1.4. *Let $F : N \to M$ be a smooth map between manifolds of the same dimension. F is locally a diffeomorphism at p iff its Jacobian determinant* $\det[\frac{\partial F^i}{\partial r^j}(p)]$ *relative to $(U, \phi), (V, \psi)$ is nonzero.*

1.3. **The Tangent Space.** Let M be a manifold. Similar to $\mathbb{R}^n$, we define an equivalence relation on real-valued smooth functions defined on a neighborhood of $p \in M$ by identifying them if they agree on some neighborhood of p. The set of equivalence classes is denoted $C_p^\infty(M)$.

Similarly, we define a **derivation at** p or a **point derivation** of C_p^∞ to be a linear map from C_p^∞ to $\mathbb{R}$ satisfying the Leibniz rule. A **tangent vector** at $p \in M$ is a derivation at p. The **tangent space**, denoted T_pM, is the set of all tangent vectors at p.

Let $F : N \to M$ be a smooth map between manifolds. At each $p \in M$, F induces a linear map of tangent spaces, called its **differential at** p,

$$F_* : T_pN \to T_{F(p)}M$$

as follows. For $X_p \in T_pN$, let $F_*(X_p)$ be the derivation at $F(p)$ defined by

$$(F_*(X_p))f = X_p(f \circ F) \text{ for } f \in C^\infty_{F(p)}(M)$$

Usually we write $F_{*,p}$ to stress its dependence on p. Other common notations for the differential include $(dF)_p, (DF)_p$.

We have chain rules, meaning $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$. As a corollary of the chain rule, diffeomorphisms induce isomorphisms on the tangent spaces. Let $(U, \phi) = (U, x^1, ..., x^n)$ be a chart about p in a manifold M. Since a chart is a diffeomorphism, the partial derivatives $\frac{\partial}{\partial x^i}|_p$ (which are derivations at p) form a basis for the tangent space T_pM. Relative to such bases, the differential of a smooth map $F : N \to M$ at $p \in N$ is represented by the Jacobian matrix $[\frac{\partial F^i}{\partial x^j}(p)]$.

A smooth map $F : N \to M$ is an **immersion** (resp. **submersion**) at p if its differential is injective (resp. surjective). A **regular value** is a point c in M s.t. F is a submersion at all points in $F^{-1}(c)$.

A subset S of a manifold N of dimension n is a **regular submanifold** of dimension k if for any $p \in S$, there is some chart $(U, x^1, ..., x^n)$ about p s.t. $U \cap S$ is precisely the vanishing of $x^{k+1}, ..., x^n$. We also say S is a regular submanifold of codimension $n - k$

Theorem 1.5. *Let $F : N^n \to M^m$ be smooth, where the superscripts denote dimensions. If $c \in M$ is a regular value and $F^{-1}(c)$ nonempty, then $F^{-1}(c)$ is a regular submanifold of N of codimension m.*

We also call the image of an injective immersion F with the topology and differentiable structure inherited from F to be an **immersed submanifold.**

The **tangent bundle** of M is the union of all the tangent spaces $TM = \cup_{p \in M} T_pM$. It is also a manifold. Its charts are of the form $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ where (U, ϕ) is a chart on M. There is a natural projection $\pi : TM \to M$. A **smooth vector field** is a smooth map $X : M \to TM$ s.t. $\pi \circ X = id$. We usually write X_p to denote the value of X at p.

A **smooth curve** in M is a smooth map $c : (a, b) \to M$. Usually we assume $0 \in (a, b)$ and say c is a curve starting at $p = c(0)$. We define the **velocity vector** at $t_0 \in (a, b)$

$$c'(t_0) := c_*(\frac{d}{dt}|_{t_0}) \in T_{c(t_0)}M$$

Let X be a smooth vector field on a smooth manifold M. An **integral curve** of X is a smooth curve $c : (a, b) \to M$ s.t. $c'(t) = X_{c(t)}$ for all $t \in (a, b)$. For any smooth vector field X and any $p \in M$, an integral curve of X starting at p always exists.

Let M be a smooth manifold. A **flow domain** $\mathcal{D} \subset \mathbb{R} \times M$ is an open subset s.t. each $\mathcal{D}^p = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing 0 for all $p \in M$. A **flow** on M is a continuous map $\theta : \mathcal{D} \to M$ such that for any $p \in M$,

$$\theta(0, p) = p$$

and for all s, t,

$$\theta(t, \theta(s, p)) = \theta(t + s, p)$$

when both sides are defined.

We say a flow is **global** if $\mathcal{D} = \mathbb{R} \times M$.

Fix $p \in M$. We have a curve $\theta^p : \mathcal{D}^p \to M$ where $\theta^p(t) = \theta(t, p)$.

A **maximal integral curve** is one that cannot be extended to an integral curve on any larger open interval, and a **maximal flow** is a flow that admits no extension to a flow on a larger flow domain.

Theorem 1.6. *Let X be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ such that, for any $p \in M$, the curve $\theta^p : \mathcal{D}^p \to M$ is the unique maximal integral curve of X starting at p.*

In particular, ${\theta^p}'(0) = X_p$.

We call the flow θ in the above theorem the flow **generated by** X.

We have the following nice result for Lie groups. The term "left invariant" will be defined later when we use it.

Theorem 1.7. *A left invariant vector field on a Lie group generates a global flow.*

In particular, for any left invariant vector field X on a Lie group G and for any $g \in G$, the unique maximal integral curve of X starting at g is defined for all $t \in \mathbb{R}$.

Given a smooth vector field X and a real-valued smooth function f, Xf is also a smooth function where $(Xf)(p) := X_p f$. For vector fields X, Y, we define their **Lie bracket** $[X, Y]$ to be the vector field s.t. $[X, Y]_p f := X_p(Yf) - Y_p(Xf)$.

With these preparations, we now proceed to our discussion on Lie groups.

2. Lie Groups: Definitions and Examples

The roots of Lie theory can be traced back to the early 19th century, when mathematicians such as Évariste Galois and Niels Henrik Abel were working on the theory of equations and the solvability of algebraic equations by radicals. Galois in particular used group theory to study polynomial equations.

Inspired by Galois, Sophus Lie, a Norwegian mathematician, began working on the theory of continuous symmetry groups of differential equations, and showed how to use them to study differential equations.

Lie's work on Lie groups and Lie algebras was continued by a number of mathematicians in the early 20th century, including Élie Cartan, Wilhelm Killing, and Hermann Weyl. They developed the theory of semisimple Lie algebras and their representation theory, which has become a central topic in Lie theory.

We will review the definition of Lie groups and discuss several examples. At the end of the section, we will prove two lemmas about Lie groups, which will be used later.

2.1. Definition of Lie Groups.

We recall the definition of Lie groups.

Definition 2.1. A **Lie group** is a group and meanwhile a smooth manifold s.t. the group operation

$$\times : G \times G \to G, (g, h) \mapsto gh$$

and inversion

$$\iota : G \to G, g \mapsto g^{-1}$$

are both smooth maps.

A **map** between two Lie groups G and H (or a Lie group map) is a map $\rho : G \to H$ that is a group homomorphism and smooth.

Recall that there are two types of submanifolds - regular and immersed. We define a **Lie subgroup** of a Lie group G to be a subgroup that is also a regular submanifolds; and we define an **immersed subgroup** of G to be the image of an injective immersion of Lie groups $\rho : H \to G$.

The general linear group $GL_n\mathbb{R}$ of invertible $n \times n$ real matrices is a basic example of Lie groups. $GL_n\mathbb{R}$ is an open subset of $\mathbb{R}^{n^2}$. The matrix multiplication is smooth, and the inverse map is also smooth as $A^{-1} = \frac{\operatorname{adj} A}{|A|}$, where $\operatorname{adj} A$ is the adjugate of A or the transpose of the cofactor matrix of A.

If without explanation, by V we will always mean a finite dimensional real vector space. We will use $GL_n\mathbb{R}$ and $GL(V)$ interchangeably; the notation $GL_n\mathbb{R}$ emphasizes that its elements are real matrices.

Definition 2.2. A **representation** of a Lie group G is a morphism from G to $GL(V)$.

We will study the representation theory of Lie groups and Lie algebras. Before diving into this subject, I want to quickly mention its relation with representation theory of finite groups.

In representation theory of finite groups, we study how finite groups can act linearly on vector spaces, or, equivalently, how they can be mapped to $GL(V)$, the group of automorphisms of a vector space V (usually V is either real or complex). To study representation of Lie groups and Lie algebras is much more complicated, as they come with a topology and additionally a smooth structure. But the finite group case is still helpful when studying Lie groups and Lie algebras - many of the ideas used in representation theory of finite groups can be applied to Lie groups and Lie algebras. In fact, [1], the main reference of this paper, begins by spending six chapters on representation theory of finite groups and then goes on to discuss Lie groups and Lie algebras.

2.2. **Examples.** Many subgroups of $GL_n\mathbb{R}$ are also Lie groups. For example, we have:

Example 2.3. The subgroup of $GL_n\mathbb{R}$ consisting of matrices with positive determinants, $GL_n^+\mathbb{R}$, is an open subset and actually connected [3], so it is the connected component of the identity matrix in $GL_n\mathbb{R}$.

Example 2.4. The special linear group $SL_n\mathbb{R}$ of $n \times n$ real matrices with determinant 1 is a connected Lie subgroup of codimension 1 by regular level set theorem.

Example 2.5. The group B_n of upper triangular matrices is a Lie subgroup (of $M_n\mathbb{R}$) of codimension $\frac{n(n-1)}{2}$. Similarly the group of invertible upper triangular matrices (i.e. nonzero on the diagonal) is a Lie subgroup of $GL_n\mathbb{R}$ of the same codimension.

Example 2.6. The group $\mathrm{Sym}_n\mathbb{R}$ of symmetric matrices is a Lie subgroup (of $M_n\mathbb{R}$) of codimension $\frac{n(n-1)}{2}$. This can be seen by noting that it is the zero set of the submersion $f : M_n\mathbb{R} \to \mathbb{R}^{\frac{n(n-1)}{2}}, (a_{ij})_{1\leq i,j\leq n} \mapsto (a_{ij} - a_{ji})_{i\neq j}$. Similarly the group of invertible symmetric matrices is a Lie subgroup of $GL_n\mathbb{R}$ of the same codimension.

Example 2.7. The orthogonal group $O(n)$ of orthogonal matrices (equivalently, matrices that preserve the inner product on $\mathbb{R}^n$) is a subgroup of $GL_n\mathbb{R}$. To see it is a Lie subgroup, we consider the smooth map $f : GL_n\mathbb{R} \to \mathrm{Sym}_n\mathbb{R}, A \mapsto A^T A$ where it is worth noting that $f = f \circ L_X$ for $X \in O(n)$, L_X being the left-multiplication-by-X map. Show Df_I is surjective, from which we can deduce that $f^{-1}(I)$ is a regular level set. Hence $O(n)$ is a Lie subgroup of $GL_n\mathbb{R}$ of dimension $\frac{n(n-1)}{2}$. As $SO(n)$ is an open subset of $O(n)$, $SO(n)$ is a Lie subgroup of the same dimension.

[3]Adding a scalar multiple of a row to another row can be achieved via a path $\gamma : [0,1] \to GL_n^+\mathbb{R}, t \mapsto A + t\alpha R_i^j$ where R_i^j has the i-th row of A on its j-th row. Similar for columns. Thus we get a path from A to a diagonal matrix. We can even assume that elements on the diagonal are positive. Finally, any diagonal matrix with positive diagonal is path-connected to the identity matrix. The same proof works for $SL_n\mathbb{R}$.

We will show later in Example 4.8 that their tangent spaces at the identity - their Lie algebras - have dimension $\frac{n(n-1)}{2}$ without knowing the dimension of the Lie groups.

Example 2.8. There are complex manifolds and hence complex Lie groups. For example $GL_n\mathbb{C}, U(n), SU(n)$, etc. Complex Lie groups are naturally also real Lie groups; their (complex) Lie algebras also naturally real. In the context of this paper, we only care about real ones.

In the representation theory of Lie groups, once we establish the correspondence between Lie groups and Lie algebras, one approach is to focus our attention on classification of Lie algebras, and complex Lie algebras are easier to classify, which we will not be able to discuss here. In fact, simple complex Lie algebras are completely classified.

2.3. **Two Lemmas.** Though with a more complicated structure, in some sense, Lie groups become a lot "cuter" than groups. Here are two interesting lemmas, which will be helpful later. We prove them under the Lie group setting, but the same proofs will work for topological groups in general.

Lemma 2.9. *Let G be a connected Lie group, and $U \subset G$ any neighborhood of the identity. Then U generates G.*

Proof. Let $U^{-1} = \{g^{-1} : g \in U\}$ and $V = U \cap U^{-1}$. Note U^{-1} is open as $g \mapsto g^{-1}$ is a diffeomorphism and hence a homeomorphism, so V is open. The purpose of this construction is that we now have $V^{-1} = V$. Consider $G' = \cup_{n\geq 1} V^n$. We show that G' is clopen (closed and open). Hence $G' = G$ and U generates G.

For any $x \in G$, xV is open as $g \mapsto xg$ is a diffeomorphism. Hence $V^n = \cup_{x\in V^{n-1}} xV$ is open for any $n \geq 1$. Thus G' is open. Now for any $x \notin G'$, $xV \cap G' = \emptyset$ (if $xv \in xV \cap G'$, then $xv \in V^n$ for some $n \geq 1$ and $x \in V^{n+1}$ since $v^{-1} \in V^{-1} = V$) and $x \in xV$, which imply $G - G'$ is open. We conclude G' is clopen. □

Note, using a similar argument, one can show that the connected component of a Lie group containing the identity is an open normal subgroup and hence a Lie subgroup of the same dimension.

Lemma 2.10. *Let G be a connected Lie group, and $N \triangleleft G$ any discrete normal subgroup. Then $N \leq Z(G)$.*

Proof. Fix $n \in N$. Consider the following diagram:

$$G \xrightarrow{\Delta^*} G \times G \xrightarrow{m} G \times G \xrightarrow{\mu} G$$

where

$$\Delta^*(g) = (g, g^{-1}), m(g_1, g_2) = (g_1 n, g_2 n^{-1}), \mu(g_1, g_2) = g_1 g_2$$

Hence, under the composition of these maps, we have

$$g \mapsto (g, g^{-1}) \mapsto (gn, g^{-1}n^{-1}) \mapsto gng^{-1}n^{-1}$$

Note that all maps here are continuous.

Since N is discrete, there is some neighborhood U of the identity, s.t. $U \cap N = \{e\}$. By continuity, there is some neighborhood V of the identity s.t. $gng^{-1}n^{-1} \in U$ for all $g \in V$. But $gng^{-1}n^{-1} \in N$ as well, for N is a normal subgroup. Thus $gng^{-1}n^{-1} = e$ for all $g \in V$, i.e. $V \subset \mathbf{C}(n)$ where $\mathbf{C}(n)$ is the centralizer of n. By Lemma 2.9, V generates G. Thus $G \subset \mathbf{C}(n)$. Since $n \in N$ is arbitrary, we conclude $N \leq Z(G)$. □

3. Covering Spaces and Isogeny Class

One helpful concept in studying Lie groups is the isogeny class. To introduce it, we need to know covering spaces, which is itself a topic rich enough to write a paper on. We will first review some main definitions and properties of covering spaces, but will not prove them here. A detailed discussion on covering spaces can be found in any standard text in algebraic topology. Then we will prove two theorems regarding coverings of Lie groups, and finally we introduce the isogeny class.

We will see fundamental groups a few times in this section. Readers can also find them in any text in algebraic topology if they are not familiar with this concept.

3.1. **Covering Spaces.** Let $p : E \to B$ be a map. For any $b \in B$, we call $p^{-1}(b)$ the **fiber** of b. If we have maps $p_1 : E_1 \to B, p_2 : E_2 \to B$, a map $\phi : E_1 \to E_2$ is **fiber-preserving** if the diagram

$$\begin{tikzcd} E_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

commutes. We usually call B the **base space** and E the **total space**.

Definition 3.1. A **covering space** or **covering** of the space B is a surjective map $p : E \to B$ s.t. there is a discrete space F and for each $b \in B$ a neighborhood U, s.t. $p^{-1}(U)$ is homeomorphic to $U \times F$ via fiber-preserving ϕ. That is to say the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi", "\cong"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

commutes, where π_U is the projection onto U. We say such U is **evenly covered**. **Maps between coverings** over B are fiber-preserving maps.

Sometimes we use the term "covering" to mean the total space as well.

In the above definition, since fibers are discrete, $p^{-1}(U) \cong U \times F = \sqcup_{x\in F} U \times \{x\}$, and $\phi^{-1}(U \times \{x\})$ is mapped homeomorphically to U under p. Hence a covering is locally a homeomorphism (and thus an open map). The summands $\phi^{-1}(U \times \{x\}) = U_x$ are the **sheets** of the covering over U.

Example 3.2. A classical example of covering spaces is $p : \mathbb{R} \to S^1, a \mapsto e^{2\pi i a}$ in which the fiber is $\mathbb{Z}$.

Many nice properties of covering spaces are about liftings. For example,

Proposition 3.3. *Let $p : E \to B$ be a covering. Let $F_0, F_1 : X \to E$ be liftings of $f : X \to B$. Suppose F_0 and F_1 agree somewhere. If X is connected, then $F_0 = F_1$.*

Proposition 3.4. *(Path Lifting Property) Let $p : E \to B$ be a covering. Let $w : I \to B$ be a path starting at $p(e) = w(0)$. Then there exists a lifting of w which begins at e, i.e. a path $\tilde{w} : I \to E$ s.t. $p\tilde{w} = w$ and $\tilde{w}(0) = e$.*

In fact, covering spaces are much nicer than this that they have the *homotopy lifting property* (HLP) for all spaces, and hence a covering is a *fibration*. The path lifting property is the same as to say a covering has the HLP for a single point.

Theorem 3.5. *Let $p : E \to B$ be a covering. Suppose Z is path-connected and locally path-connected. Let $f : (Z, z) \to (B, p(x))$ be a map. Then there exists a lifting $\Phi : Z \to E$ of f with $\Phi(z) = x$ iff $f_*\pi_1(Z, z) \subset p_*\pi_1(E, x)$.*

$$\begin{tikzcd} & (E,x) \arrow[d, "p"] \\ (Z,z) \arrow[ur, dotted, "\Phi"] \arrow[r, "f"] & (B,p(x)) \end{tikzcd}$$

Note liftings in Proposition 3.4 and Theorem 3.5 are unique by Proposition 3.3.

We call a covering $p : E \to B$ a **universal covering** if E is simply connected.

Suppose B is path-connected, locally path-connected and semi-locally simply connected. Then it is guaranteed that a universal covering of B exists. Suppose $p : (E, x) \to (B, b)$ is a universal cover over B. Since B is locally path-connected and a covering is a local homeomorphism, the total space of a covering over B is also locally path-connected. Therefore, for any covering $p' : (E', x') \to (B, b)$, we can apply Theorem 3.5 and lift p to a (unique) pointed map $f : (E, x) \to (E', x')$ s.t. $p' \circ f = p$. When E' is connected, f is also a covering map,

We list some important properties but will not prove them.

Proposition 3.6. *For any B path-connected, locally-path-connected and semi-locally simply connected,*

1. *The universal covering $p : E \to B$ is unique up to isomorphisms.*
2. *The fibers of the universal cover p over B are (set) isomorphic to $\pi_1(B)$.*
3. *The group $\mathrm{Aut}(p) = \{\alpha : E \to E : \alpha$ is an automorphism and $p \circ \alpha = p\}$, called deck transformations, is isomorphic to $\pi_1(B)$*
4. *Any map between connected coverings of B is also a covering map. In particular, any connected covering of B is also covered by the universal cover E.*

3.2. **Coverings of Lie Groups.** In this section, we will prove two theorems regarding coverings of Lie groups, and then introduce isogeny classes. We will first prove Theorem 3.9 that the covering of a Lie group is a Lie group. Before that, let us consider coverings of a manifold.

Recall a manifold is Hausdorff, second-countable and locally Euclidean. Any manifold is locally path-connected and locally simply connected (which implies semi-locally simply connected) as any open subset of $\mathbb{R}^n$ is. Thus a connected manifold M has a universal covering.

Let $p : E \to M$ be the universal covering of a connected manifold M. A natural question to ask is that is E also a manifold? The answer is yes. If $x, y \in E$ are distinct points with $p(x) \neq p(y)$, then since M is Hausdorff, we can take pullbacks of disjoint neighborhoods of $p(x)$ and $p(y)$, which results in disjoint neighborhoods of x and y; if $p(x) = p(y)$, then since p is a covering, x, y must lie in different sheets. Thus E is Hausdorff.

E is locally Euclidean because p is a local homeomorphism and M is locally Euclidean. To be precise, for any $x \in E$, there is a neighborhood U s.t. U is a sheet of an evenly covered open subset of M. Now $p(x)$ has a neighborhood contained in $p(U)$ that is homeomorphic to an open subset of $\mathbb{R}^n$. Hence we get a neighborhood of x (contained in U, so homeomorphic to its image under p) that is homeomorphic to an open subset of $\mathbb{R}^n$. These are the charts on E induced by the covering map p; they are of the form $(V, \phi \circ p)$ where V is a sheet of evenly covered $p(V)$ and $(p(V), \phi)$ is a chart for M.

Note the charts on E induced by p may not be the unique ones that make E locally Euclidean. For example, consider $p : \mathbb{R}' \to \mathbb{R}, x \mapsto x^3$ where $\mathbb{R}'$ is also the reals. The total space $\mathbb{R}'$ has the usual chart $(\mathbb{R}', x \mapsto x)$, but the chart induced by p is $(\mathbb{R}', x \mapsto x^3)$.

The last thing we want to prove is that E is second-countable. Let $\mathcal{U}'$ be a countable basis for M and let $\mathcal{U}$ be the subset of all evenly covered ones. We show $\mathcal{U}$ is also a basis. For any open $V \subset M$ and any $b \in V$, there is an evenly covered neighborhood of b, whose intersection with V, denoted by C, is also evenly covered. Now there exists $U \in \mathcal{U}'$ s.t. $b \in U \subset C \subset V$, and U is evenly covered as C is. Thus $U \in \mathcal{U}$, and $\mathcal{U}$ is a basis. We conclude M has a countable basis in which every element is evenly covered.

Note we can also assume elements in $\mathcal{U}$ to be path-connected[4]. This is because we can always take the collection of path-components (which are open) of elements in $\mathcal{U}$, and the number of path-components of any $U \in \mathcal{U}$ is countable by the secound-countability of M. Hence the whole collection is countable.

To proceed, we need Theorem 7.21 in [3] which states that the fundamental group of a manifold is countable. By Proposition 3.6, the fibers of the universal cover $p : E \to M$ are all countable. Let $\bar{\mathcal{U}}$ be the collection of all sheets over elements in $\mathcal{U}$. Then $\bar{\mathcal{U}}$ is countable as Cartesian product of two countable sets is countable. We show that $\bar{\mathcal{U}}$ is a basis for E, so E is second-countable. For any open set $V \subset E$ and any $x \in V$, the point $b = p(x)$ has an evenly covered neighborhood C and let C_x be a sheet over C containing x. We can assume $C_x \subset V$ as we can always take $V \cap C_x$, whose image under p is evenly covered as well. Now there exists $U \in \mathcal{U}$ s.t. $b \in U \subset C$, so the corresponding sheet $U_x \subset C_x$ (take $U_x = C_x \cap p^{-1}(U)$) is an element in $\bar{\mathcal{U}}$ containing x and contained in V. Thus $\bar{\mathcal{U}}$ is a basis for E. We conclude E is a manifold.

The only thing that takes us some time is the second-countability part, which is deduced from the fact that the fibers of the universal cover are countable. By property 4 of Proposition 3.6, any connected covering is covered by the universal cover. Hence any connected covering must also have countable fibers. We have proved:

Proposition 3.7. *Any connected covering of a manifold is also a manifold.*

Furthermore, if M is a smooth manifold, then the differentiable structure on M induces via p (see here) a differentiable structure on E which makes $p : E \to M$ smooth and a local diffeomorphism (and this is the unique differentiable structure s.t. p is a *smooth covering map*[5]).

---

[4]The reason we want path-connectedness is that we want the decomposition of $p^{-1}(U)$ into sheets to be unique.

[5]See Chapter 4 of [4]. The definition of a smooth covering map states that any $b \in M$ has an evenly covered open U whose sheets are mapped diffeomorphically to U under p. Our discussion here provides an example where under the usual differentiable structure $(\mathbb{R}', x \mapsto x)$, p is smooth but not locally a diffeomorphism and hence not a smooth covering map.

Remark 3.8. In the remaining of this section, we will see the term "smooth covering map" often. By using this term we are just specifying that the differentiable structure on the total space is the one induced by the covering map - what we naturally assume.

Dropping "smooth" and simply using "covering map" are actually harmless in our case. This is because given the topology and the group structure, the differentiable structure that will make the group a Lie group is unique. For example, in the next theorem, H is of course given a topology, and the uniqueness of the group structure will be shown at the very beginning, so the Lie group structure (if exists) must be unique.

We have the following result for Lie groups, analogous to Proposition 3.7:

Theorem 3.9. *Let G be a Lie group, H a connected manifold, and $\phi : H \to G$ a covering map. Let $e' \in H$ be an element s.t $\phi(e') = e$ where e is the identity in G. Then there is a unique Lie group structure on H s.t. e' is the identity and ϕ is a map of Lie groups and a smooth covering map; and the kernel of ϕ lies in the center of H.*

Proof. To construct a group structure, consider the diagram

$$\begin{tikzcd} & & (H, e') \arrow[d, "\phi"] \\ (H \times H, (e', e')) \arrow[r, "\phi\times\phi"] \arrow[urr, dotted, "\mu"] & (G \times G, (e, e)) \arrow[r, "m"] & (G, e) \end{tikzcd}$$

where m is the multiplication of G and $H \times H$ is path-connected and locally path-connected. We show that $m_*(\phi \times \phi)_*\pi_1(H \times H, (e', e')) \subset \phi_*\pi_1(H, e')$. Then by Theorem 3.5 we conclude that there is a unique pointed lifting $\mu : (H \times H, (e', e')) \to (H, e')$ of $m \circ (\phi \times \phi)$.

Recall that $\pi_1(H \times H, (e', e')) \cong \pi_1(H, e') \times \pi_1(H, e')$ via identifying a loop with its projections. Hence

$$(\phi \times \phi)_*\pi_1(H \times H, (e', e')) \cong \phi_*\pi_1(H, e') \times \phi_*\pi_1(H, e') \subset \pi_1(G, e) \times \pi_1(G, e)$$

And m_* maps $(f, g) \in \pi_1(G, e) \times \pi_1(G, e)$ to $f * g \in \pi_1(G, e)$. Thus $m_*(\phi_*\pi_1(H, e') \times \phi_*\pi_1(H, e')) = \phi_*\pi_1(H, e')$.

Therefore the (unique) lifting μ exists. We show that μ is a group operation.

(1) Associativity: This is another use of Theorem 3.5. The map we want to lift is $H \times H \times H \to G \times G \times G \to G$.
(2) Identity: Consider $\iota : H \to H \times H, h \mapsto (h, e')$. Note $m \circ (\phi \times \phi) \circ \iota$ has a lifting $\mu \circ \iota$, where $m \circ (\phi \times \phi) \circ \iota = \phi$. Thus $\mu \circ \iota$ must be the identity map by uniqueness of the lifting (as $\mu \circ \iota(e') = e'$). We conclude e' is a right identity. Similarly we can show it is a left identity.
(3) Inverse: Fix $x \in H$, let $b = \phi(x)$ and consider $\iota : H \to H \times H, h \mapsto (h, x)$. Note $m \circ (\phi \times \phi) \circ \iota$ which maps $h \mapsto \phi(h)b$ is also a covering over G. Meanwhile, $\mu \circ \iota$ is a lifting of $m \circ (\phi \times \phi) \circ \iota$ and hence a map of covering spaces. Thus by Proposition 3.6, $\mu \circ \iota$ is a covering map, so it is surjective. We conclude x has a left inverse. Similarly we can show it has a right inverse. They must be the same by associativity.

Thus μ is a group operation, and by construction ϕ is a group homomorphism.

Let the differentiable structure on H be the one induced by the covering map ϕ. Next we want to show that μ and the inversion map on H are smooth. Since $m \circ (\phi \times \phi)$ is smooth and ϕ a local diffeomorphism, μ is smooth. Let $Iv_G : G \to G$ be the inversion map $g \mapsto g^{-1}$. Note $Iv_G \circ \phi$ is also a covering over G, so it lifts to a unique map $Iv_H : H \to H$ that maps $e' \mapsto e'$.

$$\begin{tikzcd}
 & & (H,e') \arrow[d, "\phi"] \\
(H,e') \arrow[r, "\phi"] \arrow[urr, dotted, "Iv_H"] & (G,e) \arrow[r, "Iv_G"] & (G,e)
\end{tikzcd}$$

Then the map $m \circ (\phi \times \phi) \circ (Id \times Iv_H) : H \to G$ is trivial. Its unique lifting, $\mu \circ (Id \times Iv_H)$, maps $e' \mapsto e'$ and maps everything into the (discrete) fiber over e. Hence $\mu \circ (Id \times Iv_H)$ must be trivial, meaning Iv_H is indeed the inversion map. Since $Iv_G \circ \phi$ is smooth and ϕ a local diffeomorphism, Iv_H is smooth.

We have proven H is a Lie group and ϕ a Lie group map (and a smooth covering map). For the last statement, note that $\ker \phi$ is the fiber of e, which is a discrete subspace. By Lemma 2.10, $\ker \phi$ lies in the center of H. □

Theorem 3.9 shows that connected coverings of a Lie group are also Lie groups. Furthermore, the universal cover of a Lie group is unique up to Lie group isomorphisms:

Proposition 3.10. *Let G be a connected Lie group. Let $\phi : H \to G, \tilde{\phi} : \tilde{H} \to G$ be connected coverings, where $\phi, \tilde{\phi}$ are Lie group maps and smooth covering maps. Suppose $\tilde{H}$ simply connected. Then there is a unique lifting $\sigma : \tilde{H} \to H$ mapping identity $\tilde{e}'$ to identity e' that is a Lie group map and a smooth covering map.*

In particular, this implies: The universal covering of a Lie group is unique in the sense that if G has universal coverings $\phi : H \to G, \tilde{\phi} : \tilde{H} \to G$ where $\phi, \tilde{\phi}$ are Lie group maps and smooth covering maps, then there is a Lie group isomorphism $\sigma : \tilde{H} \to H$ s.t. $\phi \circ \sigma = \tilde{\phi}$.

Proof. Let $(H, e'), (\tilde{H}, \tilde{e}')$ be two connected coverings of connected Lie group (G, e), with $\phi, \tilde{\phi}$ being Lie group maps and smooth covering maps. Suppose $\tilde{H}$ simply connected. By Theorem 3.5, there is a unique map $\sigma : (\tilde{H}, \tilde{e}') \to (H, e')$ s.t.

$$\begin{tikzcd}
 & (H,e') \arrow[d, "\phi"] \\
(\tilde{H},\tilde{e}') \arrow[r, "\tilde{\phi}"] \arrow[ur, "\sigma"] & (G,e)
\end{tikzcd}$$

commutes. By Proposition 3.6, σ is a covering map. We want to show σ is a smooth covering map and a group homomorphism.

We can take an open set U of G that is evenly covered by both ϕ and $\tilde{\phi}$ where each sheets are mapped diffeomorphically to U under $\phi, \tilde{\phi}$. Taking intersections if necessary, we can assume U is a chart of G and is path-connected. Note σ maps a sheet homeomorphically to another sheet. Since ϕ and $\tilde{\phi}$ are diffeomorphisms on these sheets, we conclude σ is diffeomorphic when restricted to one sheet. This shows that σ is a smooth covering map.

To see σ is a group homomorphism, we consider the diagram

$$\begin{tikzcd}
 & & & (\tilde{H},\tilde{e}') \arrow[d, "\sigma"] \\
 & & & (H,e') \arrow[d, "\phi"] \\
\tilde{H}\times\tilde{H} \arrow[r, "\sigma\times\sigma"] \arrow[urrr, "\tilde{\mu}"] & H\times H \arrow[r, "\phi\times\phi"] \arrow[urr, "\mu"] & G\times G \arrow[r, "m"] & (G,e)
\end{tikzcd}$$

where $m, \mu, \tilde{\mu}$ are multiplication maps. Note the inner and outer triangles both commute as $\phi, \tilde{\phi} = \phi \circ \sigma$ are both homomorphisms. Hence $\sigma \circ \tilde{\mu}, \mu \circ (\sigma \times \sigma)$ are both lifts of the map $m \circ (\phi \times \phi) \circ (\sigma \times \sigma)$. By Proposition 3.3, $\sigma \circ \tilde{\mu} = \mu \circ (\sigma \times \sigma)$. Thus σ is a Lie group map. $\square$

Note the above proof works for the case $\tilde{\phi}_* \pi_1(\tilde{H}, \tilde{e}') \subset \phi_* \pi_1(H, e')$ (cf. Theorem 3.5).

We also have the following result, which is in some sense the inverse of Theorem 3.9.

Theorem 3.11. *Let H be a Lie group, and $\Gamma \subset Z(H)$ a discrete subgroup of its center. Then there is a unique Lie group structure on the quotient group $G = H/\Gamma$ s.t. the quotient map $\pi : H \to G$ is a Lie group map and a smooth covering map.*

We need the following lemma:

Lemma 3.12. *Let H be a Lie group, and $\Gamma \subset Z(H)$ a discrete subgroup of its center. For any $h \in H$, there is a neighborhood V of h s.t. $V \cap gV = \emptyset$ for all $e \neq g \in \Gamma$.*

In technical terms, this is equivalent to say the action of Γ on H is properly discontinuous.

Proof. We first consider the identity e. Since Γ is discrete, there is a neighborhood U of e s.t. $U \cap \Gamma = e$. Note the composition of maps

$$H \times H \xrightarrow{Id \times Iv_H} H \times H \xrightarrow{\mu} H$$

is continuous. Hence there exists a neighborhood V of e s.t. $VV^{-1} \subset U$. Now for any $g \in \Gamma$, if $V \cap gV \neq \emptyset$, then $v_1 = gv_2$ for some $v_1, v_2 \in V$, so $g = v_1 v_2^{-1} \in VV^{-1} \cap \Gamma = e$. Hence $g = e$.

For any $h \in H$, hV is a neighborhood containing h. If $hV \cap ghV \neq \emptyset$ for some $g \in \Gamma$, then $hv_1 = ghv_2$ for some $v_1, v_2 \in V$. As g is in the center, we again get $g = v_1 v_2^{-1}$, so $g = e$. $\square$

Proof of Theorem 3.11. By Lemma 3.12, each $h \in H$ has a neighborhood V s.t. $V \cap gV = \emptyset$ for all $e \neq g \in \Gamma$. It follows immediately that for any distinct $x, y \in \Gamma$, $xV \cap yV = \emptyset$. Hence under the qoutient map π, V is mapped homeomorphically to its image $\bar{V}$, and $\pi^{-1}(\bar{V}) = \sqcup_{g \in \Gamma} gV \cong V \times \Gamma$. In particular, π is a covering map. Note we can take V connected, so $\bar{V}$ will be connected.

Hausdorff and second-countable are not hard to check. Under the covering π, G is naturally locally Euclidean. We have charts of the form $(U, \phi \circ \pi^{-1})$ where U is connected, evenly covered with a sheet U', and (U', ϕ) is a chart of H. Note the choice of sheet U' does not matter, as other sheets are $gU', g \in \Gamma$, which is diffeomorphic to U'. Under such charts, π maps U' diffeomorphically to U.

Hence G is a smooth manifold and a Lie group[6], and π a smooth covering map. Uniqueness is not hard to check either. □

3.3. **Isogeny Class.** We are now ready to introduce isogeny class.

Definition 3.13. A map between connected Lie groups G and H is an **isogeny** if it is a Lie group map and a *smooth covering map* of the underlying manifolds; we say G and H are **isogenous** if there is an isogeny between them.

Isogeny is not an equivalence class, but generates one. For any connected Lie group G, by the existence of the universal covering and Theorem 3.9, it is isogenous to a simply connected Lie group. We define the isogeny class of G as the class of all connected Lie groups isogenous to the universal cover of G. Note if a simply connected Lie group is in the isogeny class of G, by Proposition 3.10 it must be Lie group isomorphic to the universal cover of G. Hence every isogenous class contains a unique initial member - the simply connected one. Again by Proposition 3.10, two distinct isogenous classes cannot have a common member; and if H is isogenous to G, then H lies in the isogenous class of G.

The main result we want to keep in mind is that isogeny classes correspond to (isomorphism classes of) simply connected Lie groups.

4. Lie Algebras

We can now finally consider our main topic: the representations of Lie groups. How do we start? Recall that in Lemma 2.9 we showed a connected Lie group is generated by any neighborhood of the identity. Hence any Lie group map $\rho : G \to H$, with G being connected, must be determined by what it does on any open set containing the identity in G. This can be extended further: we will show later in this section that

Theorem 4.1. *(First Principle) Let G and H be Lie groups, with G connected. A Lie group map $\rho : G \to H$ is uniquely determined by its differentiable $d\rho_e : T_eG \to T_eH$ at the identity.*

Thus by the First Principle we can completely describe a homomorphism by a linear map between two vector spaces. Given this, a natural question to ask is that which maps between these two vector spaces actually arise as differentials of Lie group maps? This is answered by the *Second Principle*:

Theorem 4.2. *(Second Principle) Let G and H be Lie groups, with G simply connected. A linear map $T_eG \to T_eH$ is the differential of a Lie group map $\rho : G \to H$ iff it preserves the **bracket operation**, which we will soon introduce.*

Thus by the First and Second Principle, when G is simply connected, there is a 1-1 correspondence between Lie group maps $G \to H$ and linear maps $T_eG \to T_eH$ that preserves the *bracket operation*.

[6] To rigorously check this, use the fact that $\pi, \pi \times \pi$ are both local diffeomorphisms.

4.1. **Definition of Lie Algebras.** In this section, we will start by introducing the *bracket operation*, and then introduce the abstract definition of Lie algebras.

Let G be a Lie group. For any $g \in G$, we define the conjugation map

$$\Psi_g : G \to G,\ h \mapsto ghg^{-1}$$

which is a Lie group automorphism. Note Ψ_g fixes the identity and $\Psi_{gg'} = \Psi_g \circ \Psi_{g'}$. We set

$$\mathrm{Ad}(g) = (d\Psi_g)_e : T_eG \to T_eG$$

where by the chain rule we have $\mathrm{Ad}(gg') = \mathrm{Ad}(g) \circ \mathrm{Ad}(g')$. This gives us a group representation

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG) = GL(T_eG)$$

of the group G on its own tangent space, called the **adjoint representation** of G.

The Ad map is in fact smooth. To see this, consider

$$F : G \times G \to G,\ (g, h) \mapsto ghg^{-1}$$

$$\iota_g : G \to G \times G,\ h \mapsto (g, h)$$

where $\Psi_g = F \circ \iota_g$. Taking differentials at the identity, we get

$$\begin{array}{ccccc} T_eG & \xrightarrow{(d\iota_g)_e} & T_gG \times T_eG & \xrightarrow{(dF)_{(g,e)}} & T_eG \\ X & \longmapsto & (0, X) & \longmapsto & (dF)_{(g,e)}(0, X) \end{array}$$

where $(dF)_{(g,e)}(0, X) = (d\Psi_g)_e(X)$ for all $X \in T_eG$. Under a fixed appropriate chart (U, ϕ) about the identity $e \in G$, $(d\Psi_g)_e$ is represented as a matrix. To show Ad is smooth, we want to prove that the entries of the matrix $(d\Psi_g)_e$ depend smoothly on g. Fix a chart (V, ψ) about g. Then $(dF)_{(g,e)}$ is represented by a matrix whose entries are smooth functions on $V \times U \ni (g, e)$. Hence, when e is fixed, entries of $(dF)_{(g,e)}$ depend smoothly on g. Now $(d\Psi_g)_e(-) = (dF)_{(g,e)}(0, -)$, so entries of $(d\Psi_g)_e$ depend smoothly on g.

Next we take the differential of the map Ad. Since $\mathrm{Aut}(T_eG) = GL(T_eG)$ is just an open subset of $\mathrm{End}(T_eG)$, its tangent space at the identity is naturally identified with $\mathrm{End}(T_eG)$. The differential of Ad at the identity yields a map

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$$

which can be identified with a bilinear map

$$[\,,\,] : T_eG \times T_eG \to T_eG$$

by defining

$$[X, Y] := \mathrm{ad}(X)(Y)$$

This is called the **bracket operation**.

We observe that the bracket operation behaves well with Lie group maps. Let $\rho : G \to H$ be a Lie group map. We have $\rho \circ \Psi_g = \Psi_{\rho(g)} \circ \rho$. Hence, taking differentials at the identity, we get

$$d\rho_e \circ \mathrm{Ad}(g) = \mathrm{Ad}(\rho(g)) \circ d\rho_e$$

Note the above compositions can be regarded as matrix multiplications. Taking differentials on both sides w.r.t. g at the identity, we get

$$d\rho_e \circ \mathrm{ad}(X) = (\mathrm{ad}(d\rho_e(X))) \circ (d\rho)_e$$

for all $X \in T_eG$, i.e.

$$d\rho_e(\mathrm{ad}(X)(Y)) = \mathrm{ad}(d\rho_e(X))(d\rho_e(Y))$$

or, equivalently,

$$d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)]$$

This proves one direction of the Second Principle.

All above could be fairly confusing. In Example 4.5 we will see why we define the bracket in this way more explicitly. But now let us look at two important properties of the bracket operation.

Proposition 4.3. *With the notations as in the above discussion, we have*

(1) The bracket operation is skew-commutative, meaning $[X, X] = 0$ for all $X \in T_eG$.
(2) The bracket operation satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof. For the first one, we use the fact that any $X \in T_eG$ is the derivative of a Lie group map $\rho : \mathbb{R} \to G$ at 0 (we will prove this fact later). Note the Ad map on $\mathbb{R}$ is constant, so ad is the zero map for $\mathbb{R}$. Hence $[X, X] = [d\rho_0(1), d\rho_0(1)] = d\rho_0[1, 1] = 0$.

For the second one, note that $\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$, as $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ is the differential of Ad at the identity. Using the fact that bracket operation for $GL(V)$ is $[M_1, M_2] = M_1M_2 - M_2M_1$ (see Example 4.5), we get

$$\begin{aligned}
[[X, Y], Z] &= \mathrm{ad}([X, Y])(Z) \\
&= [\mathrm{ad}(X), \mathrm{ad}(Y)](Z) \\
&= \mathrm{ad}(X)(\mathrm{ad}(Y)(Z)) - \mathrm{ad}(Y)(\mathrm{ad}(X)(Z)) \\
&= [X, [Y, Z]] + [Y, [Z, X]]
\end{aligned}$$

□

Definition 4.4. A **Lie algebra** $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity.

A map of Lie algebras is a linear map $\rho : \mathfrak{g} \to \mathfrak{h}$ that preserves the bracket, i.e.

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$

For a Lie group G, its Lie algebra $\mathfrak{g}$ is the tangent space at the identity with the bracket operation we just defined. By the above discussion, the differential of a Lie group map at the identity is a Lie algebra map.

4.2. **Examples.** In this section, we will see some interesting examples and introduce the definition of representations of Lie algebras.

Example 4.5. Let $G = GL_n\mathbb{R}$. For any $g \in G$, the map $\Psi_g(x) = gxg^{-1}$ is, after all, a linear map. Hence Ψ_g is represented by a matrix, and the differential $(d\Psi_g)_e$ is then just the same matrix. Thus $\mathrm{Ad}(g) = (d\Psi_g)_e$, when acting on $\mathfrak{g} = \mathfrak{gl}_n\mathbb{R} = \mathrm{End}(\mathbb{R}^n) = M_n\mathbb{R}$, is still the same conjugation, i.e. $\mathrm{Ad}(g)(M) = gMg^{-1}$.

For any $X, Y \in \mathfrak{g}$, let $\gamma : I \to G$ be an arc starting at e with tangent vector $\gamma'(0) = X$. Then our definition of $[X, Y]$ is that

$$[X,Y] = \mathrm{ad}(X)(Y) = \frac{d}{dt}|_{t=0}(\mathrm{Ad}(\gamma(t))(Y))$$

Applying the product rule to $\mathrm{Ad}(\gamma(t))(Y) = \gamma(t)Y\gamma(t)^{-1}$[7], this yields

$$\begin{aligned} &= \gamma'(0)Y\gamma(0) + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}) \\ &= XY - YX \end{aligned}$$

which explains the bracket notation.

In general, when a Lie group is a subgroup of $GL_n\mathbb{R}$, its Lie algebra is naturally embedded in $\mathfrak{gl}_n\mathbb{R}$ via the differential of the inclusion $\iota : G \hookrightarrow GL_n\mathbb{R}$; since bracket is preserved by this differential, the bracket operation on $\mathfrak{g} = T_eG$ coincides with the matrix bracket, i.e. the commutator.

Definition 4.6. A **representation of Lie algebra** $\mathfrak{g}$ on a vector space V is a map of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$$

i.e., a linear map s.t. $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.

Viewing ρ as an action of $\mathfrak{g}$ on V, we have

$$[X,Y]v = X(Yv) - Y(Xv)$$

for all $v \in V$.

By the First Principle a representation $\rho : G \to GL_n\mathbb{R}$ of a connected Lie group G is completely determined by the representation of its Lie algebra $\alpha : \mathfrak{g} \to \mathfrak{gl}_n\mathbb{R}$ given by the differential of ρ. By the Second Principle the representations of a simply connected Lie group are in 1-1 correspondence with the representations of its Lie algebra.

Example 4.7. Consider the special linear group $SL_n\mathbb{R}$. Let $A(t)$ be an arc in $SL_n\mathbb{R}$ starting at $A(0) = I$ and with tangent vector $A'(0) = X$ at $t = 0$. Then

$$\det A(t) = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma \prod_{i=1}^{n} A(t)_{(i,\sigma(i))} = 1$$

for all t, where $A(t)_{(i,j)}$ is the (i,j)-entry of $A(t)$. Taking derivative and evaluating at $t = 0$, we have by the product rule

$$0 = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma (\sum_{j=1}^{n} A'(0)_{(j,\sigma(j))} \prod_{1 \le i \le n, i \ne j} A(0)_{(i,\sigma(i))})$$

where, since $A(0) = I$, $A(0)_{(i,\sigma(i))} = 0$ for some i when $\sigma \neq id$. Hence we get

$$0 = \sum_{j=1}^{n} A'(0)_{(j,j)} = \mathrm{Tr}(X)$$

The elements in $\mathfrak{sl}_n\mathbb{R}$ thus all have trace 0. Comparing dimensions, we see that $\mathfrak{sl}_n\mathbb{R}$ is exactly the vector space of traceless $n \times n$ matrices.

[7] $\gamma(t)\gamma(t)^{-1} = I$, differentiating on both sides to get $(\gamma(t)^{-1})'$.

Example 4.8. Consider the orthogonal group $O_n\mathbb{R}$, as the group of automorphisms on $V = \mathbb{R}^n$ preserving the inner product. Let $A(t)$ be an arc in $O_n\mathbb{R}$ starting at $A(0) = I$ and with tangent vector $A'(0) = X$ at $t = 0$. We know $A(t)^T A(t) = I$ for all t. Hence, taking derivatives and evaluating at $t = 0$, we get

$$X^T + X = 0$$

i.e. elements in $\mathfrak{o}_n\mathbb{R}$ are skew-symmetric matrices.

We will show later using *exponential map* that any skew-symmetric matrix lies in $\mathfrak{o}_n\mathbb{R}$. Thus $\mathfrak{o}_n\mathbb{R}$ is precisely the space of skew-symmetric matrices, which clearly has dimension $\frac{n(n-1)}{2}$. Since $SO_n\mathbb{R}$ is an open subset of $O_n\mathbb{R}$, $\mathfrak{so}_n\mathbb{R} = \mathfrak{o}_n\mathbb{R}$.

We close this section by stating a deep result about Lie algebras, which we will use later. Interested readers can find the proofs in Chapter 3.17 of [6] and Appendix E of [1].

Theorem 4.9. *(Ado's Theorem) Every finite-dimensional real Lie algebra admits a faithful finite-dimensional representation.*

Hence any finite-dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n\mathbb{R}$ for some n.

4.3. **The Exponential Map.** The exponential map is an essential topic in studying the relationship between a Lie group and its Lie algebra. Precisely, it gives a map from a Lie algebra to its Lie group.

Let G be a Lie group. We will first associate 1-1 correspondences between the following sets:

(1) The Lie algebra $\mathfrak{g}$ of Lie group G, i.e. the tangent space at the identity.
(2) The *left invariant vector fields* on G.
(3) The *one-parameter subgroups* of G.

Definition 4.10. Let X be a (smooth) vector field on G. We say X is **left invariant** if,

$$(DL_x)_g X_g = X_{xg}, \text{ for all } x, g \in G$$

where $L_x : g \mapsto xg$ is the left-multiplication-by-x map.

One can see from the definition that a left invariant vector field is uniquely determined by its value at any point. Let $\mathrm{Lie}(G)$ denote the set of all left invariant vector fields on G. Then the evaluation map $\epsilon : \mathrm{Lie}(G) \to \mathfrak{g}, X \mapsto X_e$ is an linear monomorphism. It is also surjective: For any $v \in \mathfrak{g}$, we can define a vector field X by

$$X_g := (DL_g)_e v$$

which is clearly left invariant. For smoothness of X, a detailed proof can be found in Theorem 8.37 of [4]. Thus ϵ is a linear isomorphism.

Remark 4.11. It is worth noting that in many texts (e.g. [4]), authors start by defining the Lie algebra of a Lie group G as the set of left invariant vector fields. Then the Lie bracket operation on vector fields carries to a bracket operation on the Lie algebra. Via the linear isomorphism we just introduced, they identify the Lie algebra with the tangent space at the identity, and the Lie bracket induces a bracket operation on the tangent space. The adjoint representation of G is defined in the same way. Then they show that for $X, Y \in T_eG = \mathfrak{g}$, we have $\mathrm{ad}(X)(Y) = [X, Y]$, where the bracket is induced by the Lie bracket on the vector fields (see Theorem 20.27 in [4]).

Definition 4.12. A **one-parameter subgroup** of G is a Lie group homomorphism $\gamma : \mathbb{R} \to G$ where $\mathbb{R}$ is considered as a Lie group under addition.

First we show that a one-parameter subgroup γ is uniquely determined by the tangent vector $\gamma'(0)$ at the identity. Let $X = \gamma'(0)$ and let X also denote the left invariant vector field generated by $\gamma'(0)$ by abusing the use of notations. Since $\gamma(s+t) = \gamma(s)\gamma(t)$, differentiating on both sides w.r.t. t and evaluating at $t = 0$ results in $\gamma'(s) = (DL_{\gamma(s)})_e\gamma'(0) = (DL_{\gamma(s)})_e X_e = X_{\gamma(s)}$. Thus γ is an integral curve of X starting at the identity (and a maximal one), and γ is uniquely determined by $\gamma'(0)$.

Next we show that given a left invariant vector field X on G, we can generate a unique one-parameter subgroup. Together with the previous paragraph, we conclude one-parameter subgroups are isomorphic to left invariant vector fields.

By Theorem 1.7, there is a unique maximal integral curve ϕ of X starting at the identity that is defined on all $t \in \mathbb{R}$, i.e. a smooth curve $\phi : \mathbb{R} \to G$, with $\phi(0) = e$ and $\phi'(t) = X_{\phi(t)}$. We will show that ϕ is a one-parameter subgroup. That means we need to show ϕ is a homomorphism, i.e. $\phi(s + t) = \phi(s)\phi(t)$.

Let s be fixed and define two arcs α, β by

$$\alpha(t) = \phi(s)\phi(t), \beta(t) = \phi(s + t), t \in \mathbb{R}$$

Then $\alpha(t) = (L_{\phi(s)} \circ \phi)(t)$, so $\alpha'(t) = (DL_{\phi(s)})_{\phi(t)}\phi'(t) = X_{\phi(s)\phi(t)} = X_{\alpha(t)}$ as X is left invariant. On the other hand, $\beta'(t) = \phi'(s + t) = X_{\phi(s+t)} = X_{\beta(t)}$. Hence α, β are integral curves of X with the same starting point. Thus $\alpha(t) = \beta(t)$ for all t, and $\phi(s+t) = \phi(s)\phi(t)$. We conclude ϕ is a one-parameter subgroup and denote it by ϕ_X.

To sum up, the following are identified with each other:

(1) The tangent vector $X \in \mathfrak{g}$;
(2) The left invariant vector field generated by $X \in \mathfrak{g}$;
(3) The one-parameter subgroup with tangent vector X at 0 (or generated by X), which is the same as
(4) The maximal integral curve of vector field X starting at e.

We are now ready to define the exponential map.

Definition 4.13. We define the **exponential map**

$$\exp : \mathfrak{g} \to G$$

by

$$\exp(X) = \phi_X(1)$$

where ϕ_X is the unique one-parameter subgroup with tangent vector X at 0.

Proposition 4.14. *Let G be a Lie group. For any $X \in \mathfrak{g}$, $\gamma(s) = \exp(sX)$ is the one-parameter subgroup of G generated by X.*

Proof. Let $\gamma : \mathbb{R} \to G$ be the one-parameter subgroup of G generated by X. For any $s \in \mathbb{R}$, consider the rescaled one-parameter subgroup $\tilde{\gamma}(t) = \gamma(st)$. Note $\tilde{\gamma}'(0) = sX$. Hence $\tilde{\gamma}$ is the one-parameter subgroup generated by sX. Thus $\exp(sX) = \tilde{\gamma}(1) = \gamma(s)$. □

Proposition 4.15. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.*

(1) The exponential map is smooth.
(2) For any $X \in \mathfrak{g}, s, t \in \mathbb{R}$, $\exp(s+t)X = \exp sX \exp tX$.
(3) For any $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.
(4) The differential $(D\exp)_0$ is the identity map[8].
(5) The exponential map restricts to a diffeomorphism from some neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G$.
(6) If $f : G \to H$ is a Lie group map, then the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

Proof. For any $X \in \mathfrak{g}$, let θ_X denote the (global) flow generated by (the left invariant vector field generated by) X. To show that $\exp : \mathfrak{g} \to G$ is smooth, we need to show that $\theta_X^e(1) = \exp X$ depends smoothly on X.

Define a vector field Ξ on the product manifold $G \times \mathfrak{g}$ by

$$\Xi_{(g,X)} = (X_g, 0) \in T_gG \oplus T_X\mathfrak{g} \cong T_{(g,X)}(G \times \mathfrak{g})$$

To see that Ξ is a smooth vector field, we show that Ξf is smooth for any $f \in C^\infty(G \times \mathfrak{g})$.

Let $(X_1, ..., X_k)$ be a basis for $\mathfrak{g}$ ($X_{i,g}$ denotes X_i being pushed into T_gG). Let $(x^1, ..., x^k)$ be the corresponding global coordinates for $\mathfrak{g}$ (i.e. $x^j : \sum c^i X_i \mapsto c^j$). Let $(w^1, ..., w^k)$ be a local coordinates for G. Abusing the use of notations, we think of x^i, w^i as the input variables (in some sense $g = (w^1, ..., w^k), X = \sum x^i X_i = (x^1, ..., x^k)$) and show that $\Xi f(w^1, ..., w^k, x^1, ..., x^k)$ depends smoothly on $(w^1, ..., w^k, x^1, ..., x^k)$. Now we have

$$\begin{aligned} \Xi f(w^1, ..., w^k, x^1, ..., x^k) &= \Xi_{(w^1,...,w^k,x^1,...,x^k)} f \\ &= (\sum_j x^j X_{j,(w^1,...,w^k)}, 0) f \\ &= \sum_j x^j (X_j, 0) f(w^1, ..., w^k, x^1, ..., x^k) \end{aligned}$$

where $(X_j, 0) : (g, X) \mapsto (X_{j,g}, 0)$ is a smooth vector field (this is the left invariant vector field on $G \times \mathfrak{g}$ generated by $(X_{j,e}, 0)$) that differentiates f only in w^i-directions, and hence $(X_j, 0)f$ depends smoothly on $(w^1, ..., w^k, x^1, ..., x^k)$. We conclude the last expression depends smoothly on $(w^1, ..., w^k, x^1, ..., x^k)$.

Thus Ξ is a smooth vector field. Let Θ be the flow generated by Ξ. We note that $\Theta(t, (g, X)) = (\theta_X(t, g), X)$, as $(\theta_X(0, g), X) = (g, X)$ and $(\theta_X(t, g), X)'(t) = ({\theta_X^g}'(t), 0) = (X_{\theta_X(t,g)}, 0) = \Xi_{(\theta_X(t,g),X)}$. By Theorem 1.6, Θ is smooth. But $\exp X = \pi_G(\Theta_1(e, X))$ where $\pi_G : G \times \mathfrak{g} \to G$ is the projection. It follows that exp is smooth.

(2) and (3) follow directly from Proposition 4.14.

(4): For any $X \in \mathfrak{g}$, let $\sigma : \mathbb{R} \to \mathfrak{g}$ be $\sigma(t) = tX$. Then $\sigma'(0) = X$. Hence

$$(D\exp)_0(X) = (\exp \circ \sigma)'(0) = (\exp tX)'(0) = X$$

[8] To be precise, we identify $T_0\mathfrak{g}$ with $\mathfrak{g}$ in the following way. Let $(X_1, ..., X_k)$ be a basis for $\mathfrak{g}$. Then $(x^1, ..., x^k)$ where $x^j : \sum c^i X_i \mapsto c^j$ is a global chart on $\mathfrak{g}$. We identify $X_j \leftrightarrow \frac{\partial}{\partial x^j}|_0$.

by Proposition 4.14.

(5) follows immediately from (4) and Theorem 1.4.

(6): To show the diagram commutes, i.e. $f(\exp X) = \exp(f_* X)$ for all $X \in \mathfrak{g}$, we note that $f(\exp tX)$ is a one-parameter subgroup of H whose tangent vector at 0 is

$$\frac{d}{dt}|_{t=0}(f \exp tX) = f_*((\exp tX)'(0)) = f_* X$$

Hence $f(\exp X) = \exp(f_* X)$. □

Example 4.16. Recall the matrix exponential. For a matrix $A \in M_n\mathbb{R}$, we define

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2} A^2 + ...$$

which always converges and the result e^A is invertible with inverse e^{-A}. Let $\gamma : \mathbb{R} \to GL_n\mathbb{R}$ be the map $t \mapsto e^{tA}$. Then $\gamma'(0) = A$; i.e. γ is the one-parameter subgroup of $GL_n\mathbb{R}$ with tangent vector A at 0. We now see that the exp map we define previously coincides with the matrix exponential, which explains the name "exponential map".

Example 4.17. Let A be a skew-symmetric matrix, i.e. $A^T = -A$. Then

$$\exp(A)\exp(A)^T = \exp(A)\exp(A^T) = \exp(A)\exp(-A) = I$$

meaning that $\exp(A)$ is orthogonal. Hence $\exp(tA)$ is a one-parameter subgroup of $O(n)$, so $A = \exp(tA)'|_{t=0} \in \mathfrak{o}(n)$. This completes Example 4.8.

There are a lot of interesting results we can derive from the exponential map. For example, the fact that any continuous group homomorphism between Lie groups is smooth, from which we deduce that given the topology and the group structure on G, the smooth structure that makes G a Lie group is unique. I would really like to elaborate here, but it is going to be too long. Interested readers can take a look at Exercise 20-11 in [4].

Recall that exp is a diffeomorphism around the identities. However, in general exp is not a group homomorphism i.e. $\exp(X)\exp(Y) \neq \exp(X+Y)$. Then a natural question one can ask is that, supposing $\exp(X)\exp(Y) = \exp(Z)$ for some $Z \in \mathfrak{g}$, what is the relation of Z to X and Y? The answer to this question is provided in the Baker–Campbell–Hausdorff formula (BCH formula), which states

Theorem 4.18. *(BCH Formula) Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For sufficiently small $X, Y \in \mathfrak{g}$, we have $\exp(X)\exp(Y) = \exp(Z)$ where*

$$Z = (X+Y) + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) + ...$$

and the RHS is a convergent series.

It is not meant to be evident what the omitted terms are. The important point is that all of the terms are given by iterated brackets of X and Y. Of all the references I could find, the proofs of the BCH formula are very analytical. Interested readers can read Section 2.15 of [6] or Chapter 3 of [2] for a complete formula and detailed proofs.

With the BCH formula, we now see some hope in proving Second Principle. Since exp is locally a diffeomorphism, given a Lie algebra map $f : \mathfrak{g} \to \mathfrak{h}$, if the corresponding Lie group map $F : G \to H$ exists, then near the identity it must satisfy $F(\exp(X)) = \exp(f(X))$. If everything is nice enough (e.g. supposing $\exp(X)\exp(Y) = \exp(Z)$ and $\exp(f(X))\exp(f(Y)) = \exp(f(Z))$), we would have

$$\begin{aligned} F(\exp(X)\exp(Y)) &= F(\exp(Z)) \\ &= \exp(f(Z)) \\ &= \exp(f(X))\exp(f(Z)) \\ &= F(\exp(X))F(\exp(Y)) \end{aligned}$$

which is what we want to make F a group homomorphism.

5. Lie Group-Lie Algebra Correspondence

With all things we have introduced, we will now prove our main results - the correspondence between Lie groups and Lie algebras. This correspondence consists of several crucial results, including the forementioned First Principle and Second Principle.

The First Principle follows easily from Property (5) and (6) of Proposition 4.15 and Lemma 2.9. If we are given the differential of a Lie group map $f : G \to H$, then, as exp is a local diffeomorphism at the identity and G is connected, f is completely determined by its differential.

Next we prove the following **correspondence between Lie subgroups and Lie subalgebras**. [9]

Theorem 5.1. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of G generated by $\exp(\mathfrak{h})$ is an immersed subgroup H with tangent space $T_eH = \mathfrak{h}$.*

Proof. Note that the subgroup generated by $\exp(\mathfrak{h})$ is the same as the subgroup generated by $\exp(U)$ for any neighborhood U of the origin in $\mathfrak{h}$.

Let Δ be a disk centered at the origin in $\mathfrak{g}$ on which exp is a diffeomorphism and the BCH formula holds. I.e., for $X, Y \in \Delta$, we have $\exp(X)\exp(Y) = \exp(Z)$ where Z can be represented by the convergent series:

$$Z = (X + Y) + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + ...$$

Let $G_0 = \exp(\Delta), H_0 = \exp(\Delta \cap \mathfrak{h})$. We show that the subgroup H of G generated by H_0 is an immersed subgroup of G with tangent space $T_eH = \mathfrak{h}$.

Observe that, as Δ is a disk, we have $G_0^{-1} = G_0$ and $H_0^{-1} = H_0$. Hence the subgroup H generated by H_0 is precisely

$$H = \cup_{n\geq 1} H_0^n$$

We put a topology on H. Let H_0 be open in H, which is naturally homeomorphic to $\Delta \cap \mathfrak{h}$ under exp. Any coset hH_0 is naturally homeomorphic to H_0. In such a way we generate a topology on H which at the same time gives a differentiable structure via exp[10]. H is a Lie group as locally it behaves like a regular submanifold (H_0 is a regular submanifold of G_0). Then the smooth inclusion $H \hookrightarrow G$ is an immersion: It is clearly an immersion at the identity; then we use the fact that

[9]The following proof is slightly adjusted from [1]'s proof of Proposition 8.41. There is another proof using *distributions*, which is more commonly seen, done in Theorem 19.26 of [4].

[10]There are some nontrivil details that I am skipping here.

a Lie group homomorphism is always of constant rank. Thus H is our desired immersed subgroup whose Lie algebra is (isomorphic to) $\mathfrak{h}$. [11] $\square$

Note that the connected component of H in Theorem 5.1 is the unique connected immersed subgroup of G with Lie algebra $\mathfrak{h}$. This is because any connected immersed subgroup with Lie algebra $\mathfrak{h}$ must contain $\exp(\mathfrak{h})$.

Recall that by Ado's Theorem, every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n\mathbb{R}$ for some n. Thus Theorem 5.1 tells us that every finite-dimensional Lie algebra is (isomorphic to) the Lie algebra of a Lie group and, by Theorem 3.9, of a simply connected Lie group. This is nowadays known as the **Lie's Third Theorem**:

Theorem 5.2. *Every finite-dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.*

As a another consequence of Theorem 5.1, we are now ready to prove the Second Principle, also called **the homomorphism theorem**:

Theorem 5.3. *Let G and H be Lie groups, with G simply connected. A linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ is the differential of a Lie group map $\rho : G \to H$ iff it preserves brackets, i.e. a Lie algebra map.*

Proof. Consider the product $G \times H$, whose Lie algebra is (isomorphic to) $\mathfrak{g} \oplus \mathfrak{h}$. Let $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of the map α; i.e. $\mathfrak{j} = \{(v, \alpha(v)) : v \in \mathfrak{g}\}$. First we show that $\mathfrak{j}$ is a Lie subalgebra. Clearly it is a subspace. We need to show that it is closed under bracket.

Note that the left invariant vector field on $G \times H$ generated by $(v, 0) \in \mathfrak{g} \oplus \mathfrak{h}$ is $X_{(g,h)} = (X_g, 0)$ where $X_g = (DL_g)_e v$, the left invariant vector field on G generated by v. This is because left multiplication by (g, h) is the same as multiplied by (e, h) then (g, e); and $(DL_{(e,h)})_{(e,e)}$ does not change $(v, 0)$. Similarly $(0, w)$ would generated a vector field of the form $Y_{(g,h)} = (0, Y_h)$. Thus, for any $v \in \mathfrak{g}, w \in \mathfrak{h}$ and any $f \in C^\infty(G \times H)$,

$$[(v, 0), (0, w)]f = (X_{(e,e)}Y - Y_{(e,e)}X)f = 0$$

where X, Y denote the left invariant vector fields generated by $(v, 0), (0, w)$ respectively. We conclude $[(v, 0), (0, w)] = 0$ for any $v \in \mathfrak{g}, w \in \mathfrak{h}$. It follows that $\mathfrak{j}$ is closed under bracket, as readers can verify.

By Theorem 5.1, there is an immersed (connected) subgroup $J \subset G \times H$ with tangent space $T_{(e,e)}J = \mathfrak{j}$. Consider the projection $\pi : J \to G$ to the first factor. Note the differential $(D\pi)_{(e,e)}$ is an isomorphism. Thus the map $\pi : J \to G$ is a local diffeomorphism at the identity, which implies π is surjective by Lemma 2.9. Since G is simply connected, π is also a (smooth) covering map, and therefore an isogeny. It follows that π is an isomorphism[12].

Finally, the projection $\eta : G \cong J \to H$ is a Lie group map whose differential at the identity is precisely α. $\square$

An immediate corollary of Theorem 5.3 is:

[11] Readers may be aware that we did not prove secound-countable (Hausdorff part isn't hard from the construction). It is a fact that a non-second-countable connected manifold M cannot be immersed into a second countable manifold N. See this post for details.

[12] π is injective by Proposition 3.10.

Corollary 5.4. *If G, H are simply connected Lie groups with isomorphic Lie algebras, then G, H are isomorphic.*

Proof. By Theorem 5.3 we have Lie group maps in both directions, and the differentials of their compositions are the identity maps on the Lie algebras. Thus their compositions must be the identity maps on the Lie groups by the First Principle. □

The three theorems we introduced in this section are the main results of the **Lie group-Lie algebra correspondence**. We collect them as follows:

- (The Subgroup-Subalgebra Correspondence, done in Theorem 5.1) Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then there is a unique immersed subgroup H of G with Lie algebra $T_eH = \mathfrak{h}$.
- (Lie's Third Theorem, done in Theorem 5.2) Every finite-dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.
- (The Homomorphism Theorem, done in Theorem 5.3) Let G and H be Lie groups, with G simply connected. Any Lie algebra map $\alpha : \mathfrak{g} \to \mathfrak{h}$ is the differential of a unique Lie group map $\rho : G \to H$.

Together they give **the Lie Correspondence**:

Theorem 5.5. *(The Lie Correspondence) There is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups, given by associating each simply connected Lie group with its Lie algebra.*

Proof. Injectivity is given by Corollary 5.4 and surjectivity by Theorem 5.2. □

6. A Word on Representation of Lie Groups

At the end of this paper, let's take a quick look at how representation theory of Lie groups develop.

Recall that by the First Principle, a representation $\rho : G \to GL_n\mathbb{R}$ of a connected Lie group is uniquely determined by the corresponding Lie algebra representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}_n\mathbb{R}$. Together with the Lie group-Lie algebra correspondence, we reduce the representation theory of Lie groups to representation theory of Lie algebras, which is much easier to study as they are linear spaces. Let's now change our focus to complex Lie algebras. Real Lie algebras can be complexified, so if we understand complex ones, real ones are almost equally understood, and complex Lie algebras are simpler to analyse.

By *Levi's Theorem*, any (real or complex) Lie algebra is isomorphic to a direct sum of two Lie subalgebras, in which one is *solvable* and the other *semisimple*. The Ado's Theorem also holds for complex Lie algebras. It turns out that complex solvable Lie algebras can be (faithfully) represented as upper triangular matrices. For (real or complex) semisimple Lie algebras, they are direct sums of *simple* Lie algebras (sometimes this is taken as the definition for "semisimple"). Finally, complex simple Lie algebras is completely classified - every complex simple Lie algebra is isomorphic to either $\mathfrak{sl}_n\mathbb{C}, \mathfrak{so}_n\mathbb{C},$ or $\mathfrak{sp}_n\mathbb{C}$ for some n except for five strange ones, denoted $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7$ and $\mathfrak{e}_8$.

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